A fractional order network model for ZIKA

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Abstract

Zika is a fast spreading epidemic. So far it is known to have two transmission routes one via mosquito and the other is via sexual contact. It is dangerous on pregnant women otherwise it is mild or asymptomatic. Therefore we present a fractional order network model for it.

Keywords: Fractional-order, Zika epidemic, homogenous network, stability, numerical solutions.

1 Introduction

Zika is a fast spreading epidemic. Within one year it has spread to more than 32 countries including south, central and north America. One case appeared in Europe and in Australia. Therefore it should be studied using networks [22]. So far it is known to have two transmission routes one via mosquito and the other is via sexual contact. It is dangerous on pregnant women otherwise it is mild or asymptomatic. Therefore we expect that in the future the second route will be more difficult to control. Fractional order (FO) models [14-18] are quite useful in epidemic models to predict the spread of diseases, how to prevent epidemics and so much more. Therefore we present a fractional order network model for ZIKA. The benefit of simple models is that we can average out some of this complexity and try to understand the big picture. Our model will be useful as a conceptual tool for modeling the impact of interventions aiming to control the disease.

In sec. 2 a brief introduction to FO calculus is given. In sec. 3 the model is given and studied. Sec.4 contains our conclusions.

2 Fractional order calculus

Definition 1 The fractional integral of order $\beta \in R^+$ of the function $f(t)$, $t > 0$ is defined by

$$I^\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds \tag{1}$$

and the fractional derivative of order $\alpha \in (n-1, n)$ of $f(t)$, $t > 0$ is defined by

$$D_*^\alpha \ f(t) = I^{n-\alpha} D^n f(t), \quad D_* = \frac{d}{dt}. \tag{2}$$

The following properties are some of the main ones of the fractional derivatives and integrals (see [6]-[8], [10], [12], [20], [21]).

Let $\beta, \gamma \in R^+$ and $\alpha \in (0, 1)$. Then

(i) $I_a^\beta : L^1 \to L^1$, and if $f(y) \in L^1$, then $I_a^\gamma \ I_a^\beta \ f(y) = I_a^{\gamma+\beta} \ f(y)$.

(ii) $\lim_{\beta \to n} I_a^\beta \ f(y) = I_a^n \ f(y)$ uniformly on [a,b], $n = 1, 2, 3, \cdots$,

where $I_a^1 \ f(y) = \int_a^y f(s) \ ds$.

(iii) $\lim_{\beta \to 0} I_a^\beta \ f(y) = f(y)$ weakly.

(iv) If $f(y)$ is absolutely continuous on $[a, b]$, then $\lim_{\alpha \to 1} D_*^\alpha \ f(y) = \frac{df(y)}{dy}$.

(v) If $f(y) = k \neq 0$, k is a constant, then $D_*^\alpha \ k = 0$.

The following lemma can be easily proved (see [10]).

Lemma 1 Let $\beta \in (0, 1)$ if $f \in C[0, T]$, then $I^\beta f(t)|_{t=0} = 0$.

2.1 Equilibrium points and their asymptotic stability

Let $\alpha \in (0, 1]$ and consider the system ([1]-[3], [11], [13])

$$\begin{aligned}
D_*^\alpha y_1(t) &= f_1(y_1, y_2, y_3), \\
D_*^\alpha y_2(t) &= f_2(y_1, y_2, y_3), \\
D_*^\alpha y_3(t) &= f_3(y_1, y_2, y_3),
\end{aligned} \tag{3}$$

with the initial values

$$y_1(0) = y_{o1} \quad \text{and} \quad y_2(0) = y_{o2} \quad \text{and} \quad y_3(0) = y_{o3}. \tag{4}$$

To evaluate the equilibrium points, let

$$D_*^\alpha y_i(t) = 0 \Rightarrow f_i(y_1^{eq}, y_2^{eq}, y_3^{eq}) = 0, \quad i = 1, 2, 3$$

from which we can get the equilibrium points $y_1^{eq}, \ y_2^{eq}, y_3^{eq}$.

To evaluate the asymptotic stability, let

$$y_i(t) = y_i^{eq} + \varepsilon_i(t),$$

So the the equilibrium point $(y_1^{eq}, y_2^{eq}, y_3^{eq})$ is locally asymptotically stable if the eigenvalues of the Jacobian matrix A

$$\begin{bmatrix}
\frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} \\
\frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} \\
\frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial y_2} & \frac{\partial f_3}{\partial y_3}
\end{bmatrix}$$

evaluated at the equilibrium point satisfiesis ($|\arg(\lambda_1)| > \alpha\pi/2, |\arg(\lambda_2)| > \alpha\pi/2, |\arg(\lambda_3)| > \alpha\pi/2$) ([2], [3], [13], [19]).The stability region of the fractional-order system with order α is illustrated in Fig. 1 (in which σ, ω refer to the real and imaginary parts of the eigenvalues, respectively, and $j = \sqrt{-1}$). From Fig. 1, it is easy to show that the stability region of the fractional-order case is greater than the stability region of the integer-order case.
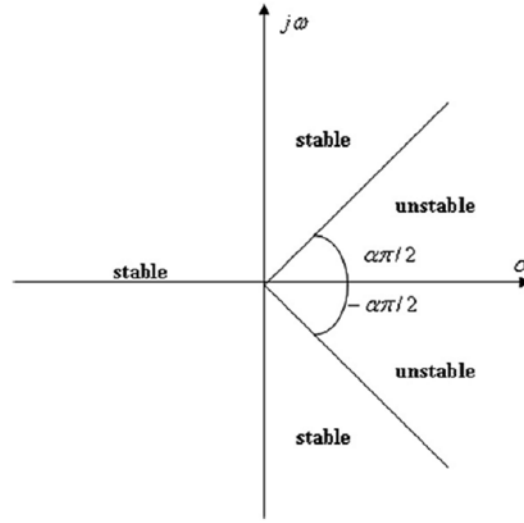
Fig. 1. Stability region of the fractional-order system.

The eigenvalues equation of the equilibrium point $(y_1^{eq}, y_2^{eq}, y_3^{eq})$ is given by the following polynomial:

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{5}$$

and its discriminant $D(P)$ is given as:

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3(a_1)^3 - 4(a_2)^3 - 27(a_3)^2, \tag{6}$$

using the results of Ref. [2], we have the following fractional Routh-Hurwitz conditions:

(i) If $D(P) > 0$, then the necessary and sufficient condition for the equilibrium point $(y_1^{eq}, y_2^{eq}, y_3^{eq})$, to be locally asymptotically stable, is $a_1 > 0, a_3 > 0, a_1a_2 - a_3 > 0$.

(ii) If $D(P) < 0$, $a_1 \geq 0, a_2 \geq 0, a_3 > 0$,then $(y_1^{eq}, y_2^{eq}, y_3^{eq})$ is locally asymptotically stable for $\alpha < 2/3$. However, if $D(P) < 0, a_1 < 0, a_2 < 0, \alpha > 2/3$, then all roots of Eq. (5) satisfy the condition $|\arg(\lambda)| < \alpha\pi/2$.

(iii) If $D(P) < 0, a_1 > 0, a_2 > 0, a_1a_2 - a_3 = 0$, then $(y_1^{eq}, y_2^{eq}, y_3^{eq})$ is locally asymptotically stable for all $\alpha \in (0,1)$.

(iv) The necessary condition for the equilibrium point $(y_1^{eq}, y_2^{eq}, y_3^{eq})$, to be locally asymptotically stable, is $a_3 > 0$.

3   Fractional-order SIRS epidemic model on homogenous networks.

Let $S(t)$ be the number of individuals in the susceptible class at time t, $I(t)$ be the number of individuals who are infectious at time t and $R(t)$ be the recovered or vaccinated individuals at time t [22].

3

The fractional-order SIRS epidemic model on homogenous networks is given by

$$
\begin{aligned}
D_*^{\alpha_1} S(t) &= \lambda - \frac{\beta\langle k\rangle SI}{S+I+R} + \gamma R - (\upsilon+\mu)S, \\
D_*^{\alpha_1} I(t) &= \frac{\beta\langle k\rangle SI}{S+I+R} - (\kappa+\mu+\alpha)I, \\
D_*^{\alpha_1} R(t) &= \kappa I - (\mu+\gamma)R + \upsilon S,
\end{aligned}
\tag{7}
$$

where $0 < \alpha_1 \le 1$ and the parameters $\lambda, \mu, \beta, \kappa, \upsilon, \gamma$ and α are positive constants, and $\langle k\rangle$ is the average connectivity in the network neglecting the heterogeneity of the node degrees [22].

Which, together with $N = S + I + R$, implies

$$
D_*^{\alpha_1} N = \lambda - \mu N - \alpha I.
$$

Thus the total population size N may vary in time.

To evaluate the equilibrium points, let

$$
\begin{aligned}
D_*^{\alpha_1} S &= 0, \\
D_*^{\alpha_1} I &= 0, \\
D_*^{\alpha_1} R &= 0,
\end{aligned}
$$

then $(S_{eq}, I_{eq}, R_{eq}) = (\frac{\lambda(\gamma+\mu)}{\mu(\gamma+\mu+\upsilon)}, 0, \frac{\lambda\upsilon}{\mu(\gamma+\mu+\upsilon)})$, (S_*, I_*, R_*), are the equilibrium points where,

$$
\begin{aligned}
S_* &= \frac{\lambda A(\kappa+\gamma+\mu)}{AB[\mu R_0 + \alpha(R_0-1)] + \beta\langle k\rangle\mu\kappa}, \\
I_* &= \frac{\lambda AB(R_0-1)}{AB[\mu R_0 + \alpha(R_0-1)] + \beta\langle k\rangle\mu\kappa}, \\
R_* &= \frac{\kappa I_* + \upsilon S_*}{\mu+\gamma},
\end{aligned}
$$

$$
R_0 = \frac{\beta\langle k\rangle(\mu+\gamma)}{(\kappa+\mu+\alpha)(\mu+\gamma+\upsilon)}, \quad A = (\kappa+\mu+\alpha), \quad B = (\mu+\gamma+\upsilon).
$$

For a disease-free equilibrium point $(S_{eq}, I_{eq}, R_{eq}) = (\frac{\lambda(\gamma+\mu)}{\mu(\gamma+\mu+\upsilon)}, 0, \frac{\lambda\upsilon}{\mu(\gamma+\mu+\upsilon)})$ we find that

$$
A = \begin{bmatrix}
-\upsilon-\mu & \frac{\beta\langle k\rangle(\mu+\gamma)}{\mu+\gamma+\upsilon} & \gamma \\
0 & \frac{\beta\langle k\rangle(\mu+\gamma)}{\mu+\gamma+\upsilon} - (\kappa+\mu+\alpha) & 0 \\
\upsilon & \kappa & -(\mu+\gamma)
\end{bmatrix},
$$

and its eigenvalues are

$$
x_1 = -\mu < 0,
$$

$$
x_2 = -(\mu+\gamma+\upsilon) < 0,
$$

$$
x_3 = -(\kappa+\mu+\alpha)(1-R_0) < 0 \quad \text{if } R_0 < 1.
$$

Hence a disease-free equilibrium point $(S_{eq}, I_{eq}, R_{eq}) = (\frac{\lambda(\gamma+\mu)}{\mu(\gamma+\mu+\upsilon)}, 0, \frac{\lambda\upsilon}{\mu(\gamma+\mu+\upsilon)})$ is locally asymptotically stable if $R_0 < 1$.

For a unique endemic equilibrium point $(S_{eq}, I_{eq}, R_{eq}) = (S_*, I_*, R_*)$ the characteristic polynomial is given by:

$$
x^3 + a_1 x^2 + a_2 x + a_3 = 0,
$$

where

$$
\begin{aligned}
a_1 &= \gamma + 2\mu + \upsilon + pw, \\
a_2 &= (p - \tau)w^2 + (\gamma p + \kappa\tau + p\mu + \tau\mu)w + \gamma\mu + \mu\upsilon + \mu^2, \\
a_3 &= [p(\gamma + \mu) - \tau(\mu + \upsilon + \gamma)]w^2 + [\tau(\mu + \upsilon + \gamma)(\mu + \kappa) - \gamma\kappa p]w, \\
p &= \frac{I_*}{N_*}, \tau = \frac{I_*}{N_*}, N_* = S_* + I_* + R_*, \\
w &= (\alpha + \mu + \kappa) = \frac{\beta\langle k\rangle S_*}{N_*}.
\end{aligned}
$$

A sufficient condition for the local asymptotic stability of a unique endemic equilibrium point $(S_{eq}, I_{eq}, R_{eq}) = (S_*, I_*, R_*)$ is

$$
|\arg(x_1)| > \alpha_1\pi/2, |\arg(x_2)| > \alpha_1\pi/2, |\arg(x_3)| > \alpha_1\pi/2. \tag{8}
$$

3.1 Numerical methods and results

An Adams-type predictor-corrector method has been introduced and investigated further in ([1]-[3], [4], [5], [9]). In this paper we use an Adams-type predictor-corrector method for the numerical solution of fractional integral equations.

The key to the derivation of the method is to replace the original problem (7) by an equivalent fractional integral equations

$$
\begin{aligned}
S(t) &= S(0) + I^{\alpha_1}[\lambda - \frac{\beta\langle k\rangle SI}{S + I + R} + \gamma R - (\upsilon + \mu)S], \\
I(t) &= I(0) + I^{\alpha_1}[\frac{\beta\langle k\rangle SI}{S + I + R} - (\kappa + \mu + \alpha)I], \\
R(t) &= R(0) + I^{\alpha_1}[\kappa I - (\mu + \gamma)R + \upsilon S], \tag{9}
\end{aligned}
$$

and then apply the **PECE** (Predict, Evaluate, Correct, Evaluate) method.

The approximate solutions displayed in Figs. 2-7 for $\langle k\rangle = 6, \lambda = 1, \beta = 0.2, \mu = 0.001, \gamma = 0.1, \alpha = 0.00087, \kappa = 0.5$ and different $0 < \alpha_1 \le 1$. In Figs. 2-4 we take $\upsilon = 0.3, S(0) = 450, I(0) = 550, R(0) = 0$ and found that a disease-free equilibrium point $(\frac{\lambda(\gamma + \mu)}{\mu(\gamma + \mu + \upsilon)}, 0, \frac{\lambda\upsilon}{\mu(\gamma + \mu + \upsilon)}) = (251.87, 0, 748.13)$ is locally asymptotically stable even if for a large fraction of the infected individuals at the initial time, the disease will eventually disappear where $R_0 = 0.602236 < 1$.

In Figs. 5-7 we take $\upsilon = 0.005, S(0) = 800, I(0) = 200, R(0) = 0$ and found that a disease-free equilibrium point $(\frac{\lambda(\gamma + \mu)}{\mu(\gamma + \mu + \upsilon)}, 0, \frac{\lambda\upsilon}{\mu(\gamma + \mu + \upsilon)}) = (952.83, 0, 47.1698)$ is unstable where $R_0 = 2.27827 > 1$ and a unique endemic equilibrium point $(S_*, I_*, R_*) = (386.518, 87.1414, 450.528)$ is locally asymptotically stable even if for a small fraction of the infected individuals at the initial time where the condition (8) is satisfied and the eigenvalues are given as:

$x_1 = -0.0010814$,

$x_{2,3} = -0.109533 \pm 0.236736i$,

$|\arg(x_1)| = \pi > \alpha_1\pi/2, |\arg(x_{2,3})| = 2.00415 > \alpha_1\pi/2$.

5

Fig. 2.

Fig. 3.

6

Fig. 4.

Fig. 5.

Fig. 6.

Fig. 7.
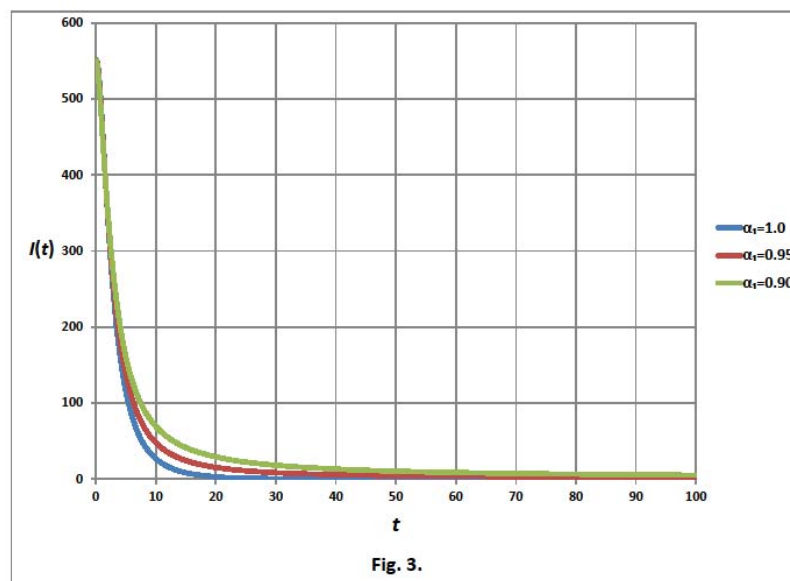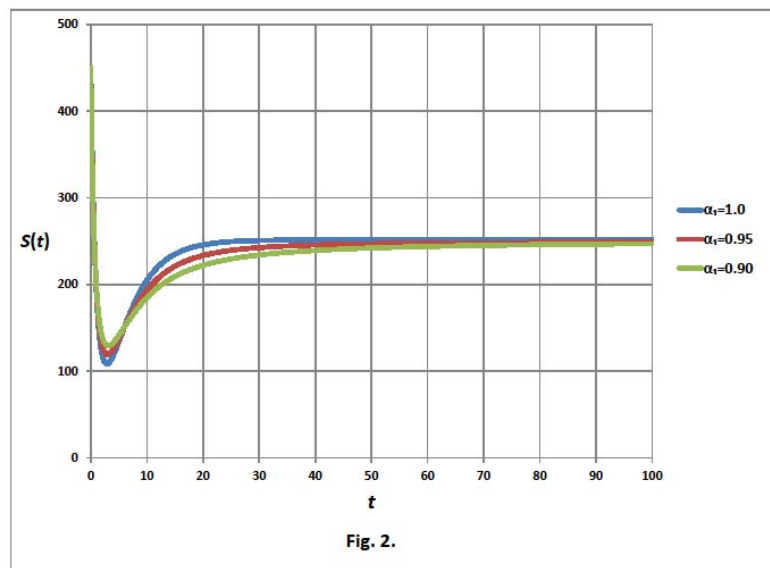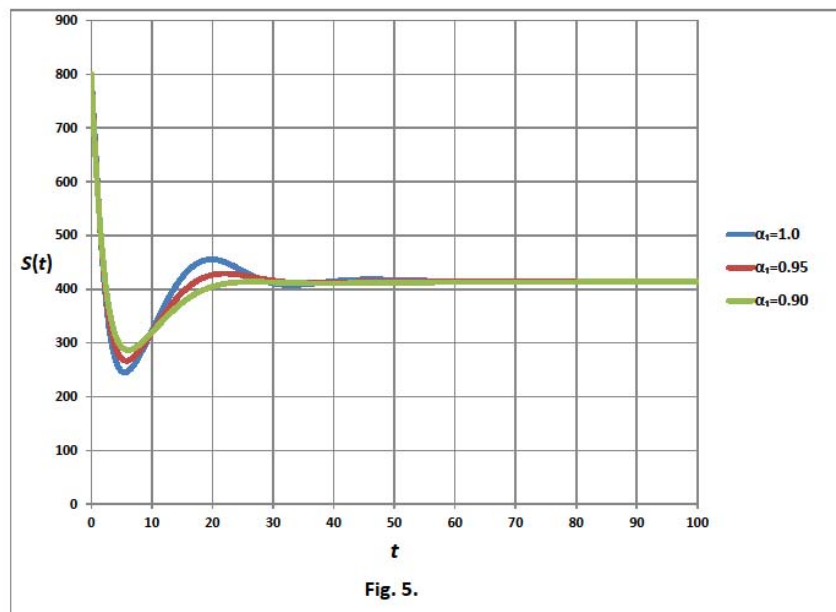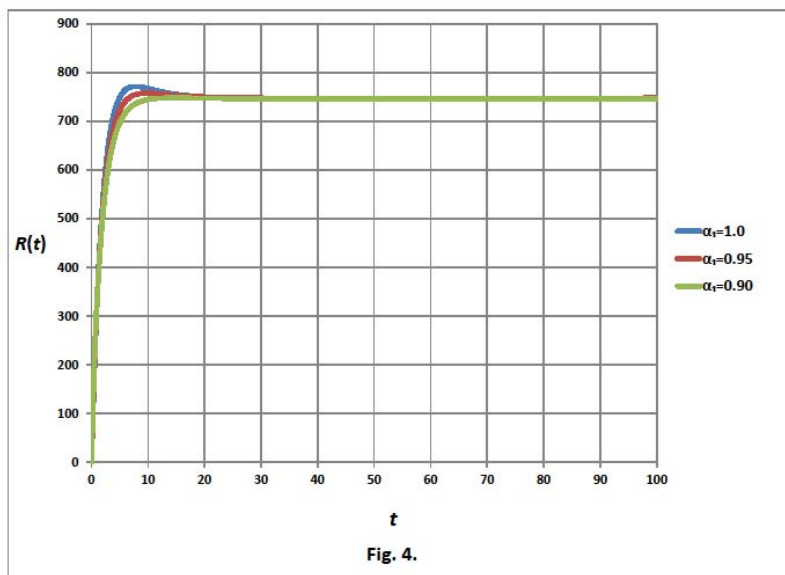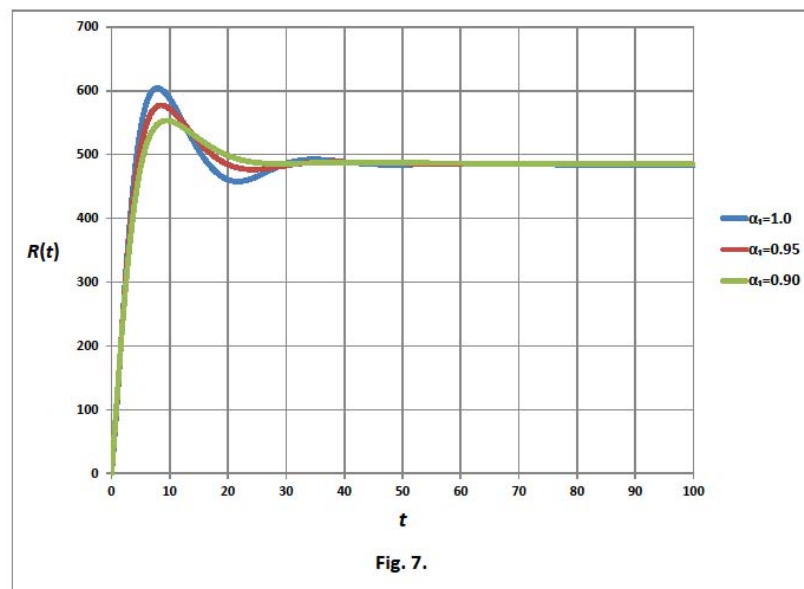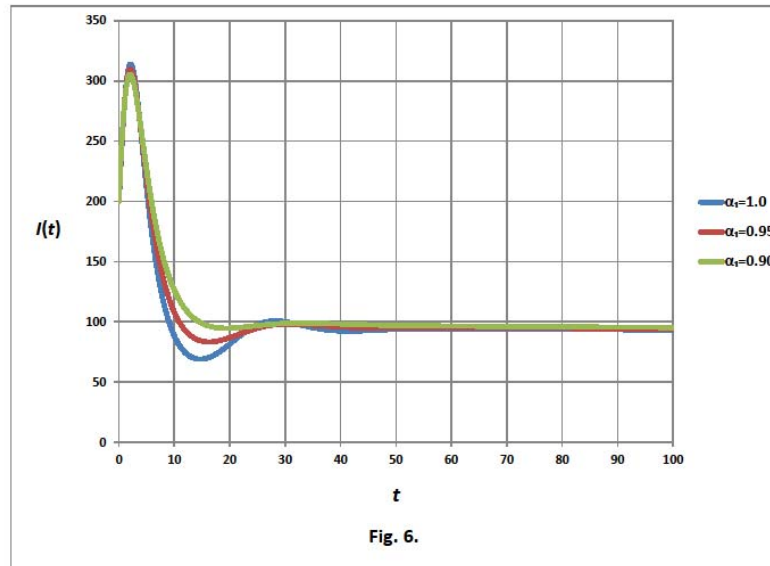
4   Conclusions

i- Zika is a fast spreading epidemic.

ii- It is dangerous on pregnant women otherwise it is mild or asymptomatic.

iii- So far it is known to have two transmission routes one via mosquito and the other is via sexual contact.

Therefore we expect that in the future the second route will be more difficult to control. Moreover the fact that Summer Olympics is expected in Brazil in 2016 makes it important to study the epidemic quickly.

Our model will be useful as a conceptual tool for modeling the impact of interventions aiming to control the disease.

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