

Race Model for the Stop Signal Paradigm Revisited:
Perfect Negative Dependence

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Introduction and motivation

In the stop-signal paradigm, participants perform a response time task (*go task*) and, occasionally, the go stimulus is followed by a stop signal after a variable delay, indicating subjects to withhold their response (*stop task*). The main interest is in estimating the unobservable stop-signal reaction time (SSRT), that is, the latency of the stopping process, as a characterization of the response inhibition mechanism. In the **Independent Race Model** ([1]) the stop-signal task is represented as a race with stochastically **independent** GO and STOP processes. Under certain simplifying assumptions, some statistics of SSRT can be estimated efficiently without making any distributional assumptions on processing times. Neurophysiological studies, however, have shown that the neural correlates of the GO and STOP processes produce saccadic eye movements through a network of **interacting** neurons ([2]). Here we propose a **Dependent Race Model** that assumes perfect negative stochastic dependence between GO and STOP processes. It resolves the apparent paradox between behavioral and neural data but nonetheless retains the distribution-free properties of the Independent Race Model.

The general race model

We distinguish two different experimental conditions termed context \mathcal{GO} , where only a go signal is presented, and context \mathcal{STOP} , where a stop signal is presented. In \mathcal{STOP} , let T_{go} and T_{stop} denote the random processing time for the go and the stop signal, respectively, with bivariate distribution function

$$H(s, t) = \Pr(T_{go} \leq s, T_{stop} \leq t), \quad (1)$$

where H has its probability mass concentrated on $[0, \infty) \times [0, \infty)$. The marginal distributions of $H(s, t)$ are denoted as

$$F_{go}(s) = \Pr(T_{go} \leq s, T_{stop} < \infty) \text{ and } F_{stop}(t) = \Pr(T_{go} < \infty, T_{stop} \leq t). \quad (2)$$

In any given trial, a go signal triggering the realization of random processing time T_{go} is presented either in context \mathcal{GO} or in context \mathcal{STOP} , but not in both at the same time. Thus, the distribution of T_{go} could differ depending on context. However, the general race model rules this out by adding the important **context invariance assumption**: For context \mathcal{GO} , the distribution of go signal processing time is assumed to be

$$F_{go}(s) = \Pr(T_{go} \leq s) = \Pr(T_{go} \leq s, T_{stop} < \infty), \quad (3)$$

identical to the marginal distribution $F_{go}(s)$ in the \mathcal{STOP} context.

In order to simplify calculations, it is further assumed that $H(s, t)$ is absolutely continuous, so that density functions for the marginals exist, denoted as $f_{go}(s)$ and $f_{stop}(t)$, respectively. Moreover, the partial derivatives of H are

$$H_1(s, t) = \frac{\partial H}{\partial s}(s, t) \text{ and } H_2(s, t) = \frac{\partial H}{\partial t}(s, t). \quad (4)$$

From these assumptions, the probability $p_r(t_d)$ of observing a response to the go signal given a stop signal presented at delay t_d ($t_d \geq 0$) after the go signal, is determined by

$$p_r(t_d) = \Pr(T_{go} < T_{stop} + t_d). \quad (5)$$

To help interpretation, by abuse of notation let us write, for any t ,

$$\begin{aligned} \Pr(T_{go} = t \cap T_{stop} + t_d > t) &= \Pr(T_{go} = t) - \Pr(T_{go} = t \cap T_{stop} + t_d \leq t) \\ &= f_{go}(t) - H_1(t, t - t_d). \end{aligned}$$

Then,

$$\begin{aligned}
 p_r(t_d) &= \int_0^\infty \Pr(T_{go} = t \cap T_{stop} + t_d > t) dt, \\
 &= \int_0^{t_d} \Pr(T_{go} = t) dt + \int_{t_d}^\infty \Pr(T_{go} = t \cap T_{stop} + t_d > t) dt, \\
 &= F_{go}(t_d) + \int_{t_d}^\infty \Pr(T_{go} = t) dt - \int_{t_d}^\infty \Pr(T_{go} = t \cap T_{stop} + t_d \leq t) dt, \\
 &= 1 - \int_{t_d}^\infty H_1(t, t - t_d) dt.
 \end{aligned} \tag{6}$$

According to the model, the probability of observing a response to the go signal before time t , given the stop signal was presented t_d msec later, equals

$$F_{sr}(t | t_d) = \Pr(T_{go} \leq t | T_{go} < T_{stop} + t_d) \tag{7}$$

This response time has sometimes been called *signal-response RT*. Its density is

$$\begin{aligned}
 f_{sr}(t | t_d) &= \Pr(T_{go} = t \cap T_{stop} + t_d > t) / p_r(t_d) \\
 &= f_{go}(t) - H_1(t, t - t_d) / p_r(t_d) \\
 &= \begin{cases} f_{go}(t) / p_r(t_d) & \text{if } t < t_d, \\ \frac{f_{go}(t) - H_1(t, t - t_d)}{p_r(t_d)} & \text{if } t \geq t_d. \end{cases}
 \end{aligned} \tag{8}$$

Thus, the distribution function can be written as

$$\begin{aligned}
 F_{sr}(t | t_d) &= \frac{1}{p_r(t_d)} \int_0^t f_{go}(t') - H_1(t', t' - t_d) dt' \\
 &= \frac{F_{go}(t)}{p_r(t_d)} \quad \text{if } t \leq t_d.
 \end{aligned} \tag{9}$$

For $t > t_d$, transformations analogous to those leading to (6) yield

$$\begin{aligned}
 F_{sr}(t | t_d) &= \frac{1}{p_r(t_d)} \int_0^t f_{go}(t') - H_1(t', t' - t_d) dt' \\
 &= \frac{1 - \int_{t_d}^t H_1(t', t' - t_d) dt'}{1 - \int_{t_d}^\infty H_1(t', t' - t_d) dt'}.
 \end{aligned} \tag{10}$$

Observable, or at least estimable from data, are the following components of the general stop signal race model: $F_{go}(t)$, $F_{sr}(t | t_d)$, and $p_r(t_d)$. The main interest in modeling is to obtain information about the distribution of unobservable stop signal processing time, T_{stop} .

Independent race model

[1] suggested the *independent race model* by assuming stochastic independence between T_{go} and T_{stop} :

Stochastic independence assumption.: for all real-valued s, t

$$H(s, t) = \Pr(T_{go} \leq s) \Pr(T_{stop} \leq t) = F_{go}(s) F_{stop}(t).$$

Therefore, under stochastic independence

$$\begin{aligned} p_r(t_d) &= \int_0^\infty \Pr(T_{go} = t \cap T_{stop} + t_d > t) dt \\ &= \int_0^\infty f_{go}(t) [1 - F_{stop}(t - t_d)] dt. \end{aligned} \quad (11)$$

For the density of the signal-response time distribution $F_{sr}(t|t_d)$, we have

$$f_{sr}(t | t_d) = f_{go}(t) [1 - F_{stop}(t - t_d)] / p_r(t_d). \quad (12)$$

As observed in [3], rearranging Equation (12) yields an explicit expression of the distribution of the unobservable stop signal processing time T_{stop} :

$$F_{stop}(t - t_d) = 1 - \frac{f_{sr}(t | t_d) p_r(t_d)}{f_{go}(t)}. \quad (13)$$

However, as investigated in [4, 5], obtaining reliable estimates for the stop signal distribution using Equation (13) requires unrealistically large numbers of observations in practice. The most common alternative estimation method, called *integration method*, assumes random variable T_{stop} to be equal to a constant, SSRT, say. Then

$$F_{stop}(t) = \begin{cases} 0, & \text{if } t < \text{SSRT} + t_d; \\ 1 & \text{if } t \geq \text{SSRT} + t_d. \end{cases}$$

Inserting into Equation (11) yields

$$p_r(t_d) = \int_0^{\text{SSRT}+t_d} f_{go}(t) dt. \quad (14)$$

Because estimates of both $p_r(t_d)$ and $f_{go}(t)$ are available, this allows estimation of stop signal processing mean SSRT.

Race model with negative dependence

Motivation: interactive race model based on neural data...Schall etal

Fréchet-Hoeffding bounds. Let $G(x, y)$ be the bivariate distribution function of a pair of random variables (X, Y) :

$$G(x, y) = P(X \leq x, Y \leq y)$$

with marginal distributions F_X and F_Y . Then, it always holds that

$$G^-(x, y) = \max\{F_X(x) + F_Y(y) - 1, 0\} \leq G(x, y) \leq \min\{F_X(x), F_Y(y)\},$$

for all x, y in the support of G . Both $G^-(x, y)$ and $\min\{F_X(x), F_Y(y)\}$ are known as

Fréchet bounds and are distribution functions for (X, Y) as well. Specifically, G^-

correspond to perfect negative dependence between X and Y , while

$G^+(x, y) = \min\{F_X(x), F_Y(y)\}$ corresponds to perfect positive dependence (Nelson20xx).

The upper bound G^+ will not play a role in the following.

Race model: Negative dependence. Replacing stochastic independence in the race model by perfect negative dependence between go and stop signal processing time (lower Fréchet bound G^-) results in the following joint distribution for T_{go} and T_{stop} :

$$H^-(s, t) = \max\{F_{go}(s) + F_{stop}(t) - 1, 0\}. \quad (15)$$

Then,

$$H_2^-(t, t - t_d) = \begin{cases} f_{stop}(t - t_d), & \text{if } F_{go}(t) + F_{stop}(t - t_d) - 1 > 0; \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

Let

$$A = \{t \mid F_{go}(t) + F_{stop}(t - t_d) - 1 > 0\}. \quad (17)$$

We introduce an indicator function $\mathbb{1}_{\{A\}}(t)$ with set A by

$$\mathbb{1}_{\{A\}}(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{else,} \end{cases}$$

Then Equation (16) can be rewritten more compactly as

$$H_2^-(t, t - t_d) = f_{stop}(t - t_d) \mathbb{1}_{\{A\}}(t). \quad (18)$$

Clearly, $H_2^-(t, t - t_d)$ is increasing in t , so we define the infimum (greatest lower bound) of A ,

$$t^* = \inf A, \quad (19)$$

and observe that $t^* > t_d$.

Next, we consider the signal-response RT distribution. It is more convenient to write it in terms of H_2 rather than H_1 :

$$\begin{aligned} F_{sr}(t \mid t_d) &= \Pr(T_{go} \leq t \mid T_{go} < T_{stop} + t_d) \\ &= \int_0^t H_2(t', t' - t_d) dt' / p_r(t_d) \\ &= \int_0^t f_{stop}(t - t_d) \mathbb{1}_{\{A\}}(t) dt' / p_r(t_d) \\ &= \int_{t^*}^t f_{stop}(t - t_d) dt' / p_r(t_d). \end{aligned} \quad (20)$$

Multiplying by $p_r(t_d)$ and evaluating the integral,

$$p_r(t_d) F_{sr}(t \mid t_d) = F_{stop}(t - t_d) - F_{stop}(t^* - t_d). \quad (21)$$

Letting $t \rightarrow \infty$ yields

$$p_r(t_d) = 1 - F_{stop}(t^* - t_d). \quad (22)$$

Finally, using (22) we can rewrite Equation (21),

$$p_r(t_d) F_{sr}(t \mid t_d) = F_{stop}(t - t_d) - [1 - p_r(t_d)]$$

and solve for the unobservable stop signal distribution,

$$F_{stop}(t - t_d) = 1 - p_r(t_d)[1 - F_{sr}(t | t_d)] \quad (23)$$

for all $t \geq t^*$.

Comparing this with the case of independence (Equation 13)

$$F_{stop}^{IND}(t - t_d) = 1 - \frac{f_{sr}(t | t_d)p_r(t_d)}{f_{go}(t)}, \quad (24)$$

we see that perfect negative dependence replaces the ratio $f_{sr}(t | t_d)/f_{go}(t)$ by the much simpler expression $1 - F_{sr}(t | t_d)$. Moreover, the expected value of T_{stop} is easy to compute:

$$\begin{aligned} E[T_{stop}] &= \int_{t_d}^{\infty} [1 - F_{stop}(t - t_d)] dt \\ &= p_r(t_d) \int_{t^*}^{\infty} [1 - F_{sr}(t | t_d)] dt \\ &= p_r(t_d) E[T_{go} | T_{go} < T_{stop} + t_d]. \end{aligned} \quad (25)$$

Example 1: Exponential Go and Stop distributions

While exponentially distributed go or stop signal reaction times lack empirical support, this first example serves to illustrate the difference between independent and negatively dependent race models. It also helps probing the derivations made so far.

Independent exponentially distributed T_{go} and T_{stop}

In addition to the assumptions of the general race model, we define independent, exponential distributions for T_{go} and T_{stop} with parameters $\lambda_{go} > 0$ and $\lambda_{stop} > 0$ for context *STOP* by

$$\begin{aligned} H(s, t) &= \Pr(T_{go} \leq s) \Pr(T_{stop} \leq t) \\ &= (1 - \exp[-\lambda_{go} s])(1 - \exp[-\lambda_{stop} t]), \end{aligned}$$

for all $s, t \geq 0$. Inserting into (11),

$$\begin{aligned}
 p_r(t_d) &= \Pr(T_{go} < T_{stop} + t_d) \\
 &= \int_0^\infty f_{go}(t) [1 - F_{stop}(t - t_d)] dt \\
 &= \int_0^{t_d} f_{go}(t) dt + \int_{t_d}^\infty f_{go}(t) [1 - F_{stop}(t - t_d)] \\
 &= 1 - \exp[-\lambda_{go} t_d] + \frac{\lambda_{go}}{\lambda_{stop} + \lambda_{go}} \exp[-\lambda_{go} t_d] \\
 &= 1 - \frac{\lambda_{stop}}{\lambda_{stop} + \lambda_{go}} \exp[-\lambda_{go} t_d].
 \end{aligned} \tag{26}$$

For $t > t_d$, the density of the signal-response distribution is given by,

$$\begin{aligned}
 f_{sr}(t | t_d) &= f_{go}(t) [1 - F_{stop}(t - t_d)] / p_r(t_d) \\
 &= \lambda_{go} \exp[-\lambda_{go} t] \exp[-\lambda_{stop}(t - t_d)] / \left(1 - \frac{\lambda_{stop}}{\lambda_{stop} + \lambda_{go}} \exp[-\lambda_{go} t_d] \right) \\
 &= \frac{1}{K} (\lambda_{go} + \lambda_{stop}) \exp[-(\lambda_{go} + \lambda_{stop})(t - t_d)],
 \end{aligned} \tag{27}$$

with $K = \exp[\lambda_{go} t_d] (1 + \lambda_{stop}/\lambda_{go}) - \lambda_{stop}/\lambda_{go}$. Note that for $t_d = 0$, we have $K = 1$ and the signal-response density is identical to an exponential density for an independent race between T_{stop} and T_{go} , with parameter $\lambda_{go} + \lambda_{stop}$ and $p_r(t_d) = \lambda_{go}/(\lambda_{go} + \lambda_{stop})$.

For $t \leq t_d$, the density simplifies to

$$\begin{aligned}
 f_{sr}(t | t_d) &= f_{go}(t) / p_r(t_d) \\
 &= \frac{\lambda_{go} \exp[-\lambda_{go} t]}{\frac{\lambda_{go}}{\lambda_{stop} + \lambda_{go}} \exp[-\lambda_{go} t_d]} \\
 &= \frac{1}{\lambda_{stop} + \lambda_{go}} \exp[-\lambda_{go}(t - t_d)].
 \end{aligned} \tag{28}$$

Comparing the distribution for go-signal response times, $F_{go}(t)$, with the signal-response distribution, $F_{sr}(t | t_d)$, yields the typical fan shape for varying values of t_d (see Figure 1).

Computation of the expected value of signal-response RTs is straightforward:

$$\begin{aligned}
 E[T_{go} | T_{go} < T_{stop} + t_d] &= \int_0^{\infty} f_{sr}(t | t_d) dt \\
 &= \int_0^{\infty} t f_{go}(t) [1 - F_{stop}(t - t_d)] / p_r(t_d) \\
 &= \frac{1}{p_r(t_d)} \frac{\lambda_{go}}{(\lambda_{go} + \lambda_{stop})^2} \exp[\lambda_{stop} t_d] \\
 &= \frac{\lambda_{go} \exp[\lambda_{go} + \lambda_{stop}]}{(\lambda_{go} + \lambda_{stop})([\exp[\lambda_{go} t_d](\lambda_{go} + \lambda_{stop}) - \lambda_{stop})]}. \tag{29}
 \end{aligned}$$

In particular, for $t_d = 0$, we obtain $E[T_{go} | T_{go} < T_{stop} + t_d] = 1/(\lambda_{go} + \lambda_{stop})$, consistent with the density we mentioned above for this value of the stop signal delay.

Perfect negative dependence between exponentially distributed T_{go} and T_{stop}

For the bivariate distribution of (T_{go}, T_{stop}) , we have

$$\begin{aligned}
 H^-(s, t) &= \max\{F_{go}(s) + F_{stop}(t) - 1, 0\} \\
 &= \max\{1 - \exp[-\lambda_{go} s] + 1 - \exp[-\lambda_{stop} t] - 1, 0\} \\
 &= \max\{1 - \exp[-\lambda_{go} s] - \exp[-\lambda_{stop} t], 0\}. \tag{30}
 \end{aligned}$$

Then, from (16)

$$H_2^-(t, t - t_d) = \begin{cases} \lambda_{stop} \exp[-\lambda_{stop} (t - t_d)] & \text{if } \exp[-\lambda_{go} t] + \exp[-\lambda_{stop} (t - t_d)] < 1; \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

Defining set A as before in (17),

$$\begin{aligned}
 A &= \{t | F_{go}(t) + F_{stop}(t - t_d) - 1 > 0\} \\
 &= \{t | \exp[-\lambda_{go} t] + \exp[-\lambda_{stop} (t - t_d)] < 1\}, \tag{32}
 \end{aligned}$$

this can be rewritten as

$$H_2^-(t, t - t_d) = \lambda_{stop} \exp[-\lambda_{stop} t] \mathbb{1}_{\{A\}}(t). \tag{33}$$

Here, the smallest value t^* satisfying the condition in (32) is a function of the parameters:

$$t^*(t_d, \lambda_{go}, \lambda_{stop}) = \min\{t \mid \exp[-\lambda_{go} t] + \exp[-\lambda_{stop} (t - t_d)] < 1\}. \quad (34)$$

For $\lambda_{go} = \lambda_{stop} = \lambda$, solving for t^* yields

$$t^* = \lambda^{-1} \log(1 - \exp[-\lambda t_d]).$$

When $\lambda_{go} \neq \lambda_{stop}$, there is no closed-form solution but t^* is easily obtainable to arbitrary precision by numerical algorithms (e.g. Newton-Raphson).

For the signal-response RT distribution $F_{sr}(t \mid t_d)$ we obtain, from inserting exponential distributions into (21) and (22),

$$\begin{aligned} F_{sr}(t \mid t_d) &= \frac{F_{stop}(t - t_d) - F_{stop}(t^* - t_d)}{1 - F_{stop}(t^* - t_d)} \\ &= 1 - \exp[-\lambda_{stop} (t - t^*)]. \end{aligned} \quad (35)$$

Although not made explicit in (35), note that the signal-response distribution function F_{sr} does depend on t_d via the value of t^* , that is, there is a shift-dependency on t_d . Figure 2 depicts the no-stop signal distribution F_{go} together with signal-response distribution $F_{sr}(t \mid t_d)$ for different t_d values employing the same parameters as in the independence case above ($\lambda_{go} = .01, \lambda_{stop} = .02$). Computing the values of t^* according to (34) required finding the fixed point of recursion (Newton-Raphson)

$$t_{n+1} = t_n - \frac{1 - \exp[-\lambda_{go} t_n] - \exp[-\lambda_{stop} (t_n - t_d)]}{\exp[-\lambda_{go} t_n] + \exp[-\lambda_{stop} (t_n - t_d)]}, \quad (36)$$

which was determined using function `FixedPoint` of Mathematica[®]. From (25) the expected value for the signal-response distribution is very easy to determine:

$$\begin{aligned} \mathbb{E}[T_{go} \mid T_{go} < T_{stop} + t_d] &= \frac{1}{p_r(t_d)} \mathbb{E}[T_{stop}] \\ &= \frac{1}{p_r(t_d) \lambda_{stop}} \\ &= \frac{\exp[\lambda_{stop}(t^* - t_d)]}{\lambda_{stop}}. \end{aligned} \quad (37)$$

Final comments

- Just like the Independent Race Model, the *Dependent Race Model* provides a distribution-free measure of the latency of the STOP process;
- by assuming perfect negative dependence between GO and STOP process, the Dependent Race Model constitutes a behavioral framework that is consistent with the finding of mutually inhibitory gaze-holding and gaze-shifting neurons ([6]).

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Acknowledgments

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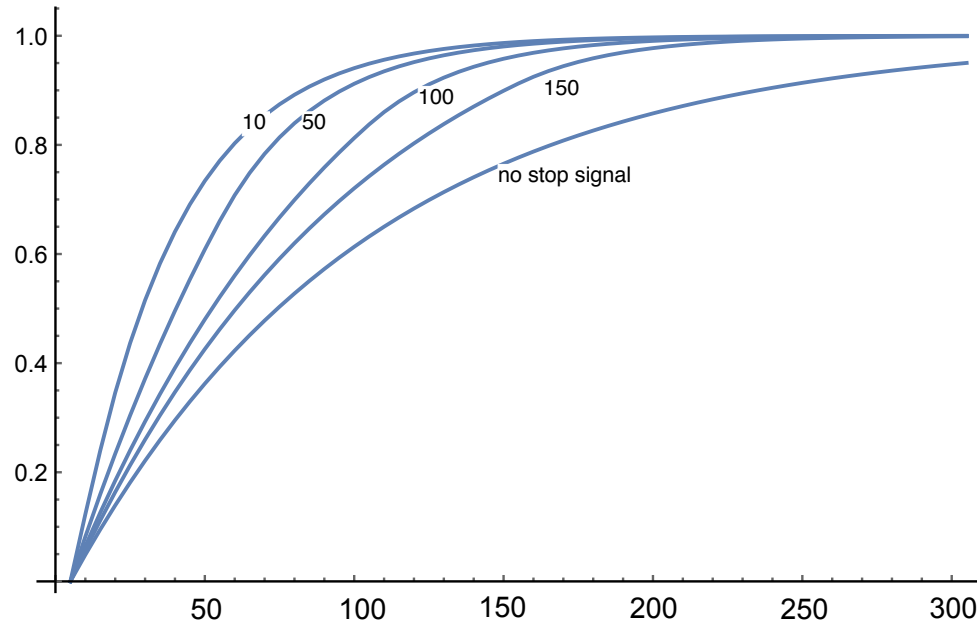


Figure 1. Independent (exponential) race model: $F_{go}(t)$ (no-stop signal distribution) compared to $F_{sr}(t | t_d)$ (signal-respond distribution) for $t_d = 10, 50, 100, 150$ [msec] with $\lambda_{go} = .01$ and $\lambda_{stop} = .02$.

DEPENDENT RACE MODEL

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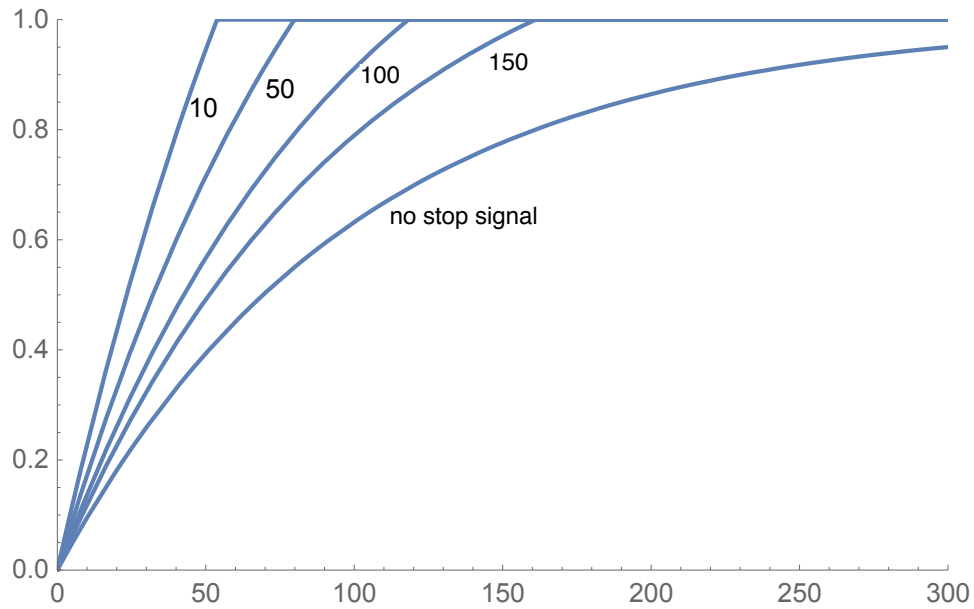


Figure 2. Negatively dependent (exponential) race model: $F_{go}(t)$ (no-stop signal distribution) compared to $F_{sr}(t|t_d)$ (signal-respond distribution) for $t_d = 10, 50, 100, 150$ [msec] with $\lambda_{go} = .01$ and $\lambda_{stop} = .02$.