Hierarchical Bayesian estimation of metacognitive efficiency from confidence ratings

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ABSTRACT

Metacognition refers to the ability to reflect on and monitor one's cognitive processes, such as perception, memory and decision-making. Metacognition is often assessed in the lab by whether an observer's confidence ratings are predictive of objective success, but simple correlations between performance and confidence are susceptible to undesirable influences such as response biases. Recently an alternative approach to measuring metacognition has been developed (Maniscalco & Lau, 2012) that characterises metacognitive sensitivity (meta-*d'*) by assuming a generative model of confidence within the framework of signal detection theory. However, current estimation routines require an abundance of confidence rating data to recover robust parameters, and only provide point estimates of meta-*d'*. In contrast, hierarchical Bayesian estimation methods provide opportunities to enhance statistical power, incorporate uncertainty in group-level parameter estimates and avoid edge-correction confounds. Here I introduce such a method for estimating meta-*d'* from confidence ratings and demonstrate its application for assessing group differences. A tutorial is provided on both the meta-*d'* model and the preparation of behavioural data for model fitting. A hierarchical approach may be particularly useful in situations where limited data are available, such as when quantifying metacognition in patient populations. In addition, the model may be flexibly expanded to estimate parameters encoding other influences on metacognitive efficiency. MATLAB software and documentation for implementing hierarchical meta-*d'* estimation (HMeta-d) can be downloaded at https://github.com/smfleming/HMM.

Keywords: Metacognition; Confidence; Signal Detection Theory; Bayes; MCMC

INTRODUCTION

Metacognition is defined as "knowledge of one's own cognitive processes" (Flavell, 1979). For example, humans can reflect on whether a particular memory is accurate or inaccurate, a capacity that is routinely tested in courts of law. Understanding the relationship between metacognition and cognitive performance is a key goal for diverse research areas including judgment and decision-making (Lichtenstein, Fischhoff, & Phillips, 1982), developmental psychology (Goupil, Romand-Monnier, & Kouider, 2016; L. G. Weil et al., 2013), social psychology (Heatherton, 2011), consciousness science (Lau & Rosenthal, 2011) and clinical disorders (David, Bedford, Wiffen, & Gilleen, 2012; Moeller & Goldstein, 2014). Metacognitive *sensitivity* can be assessed by the extent to which an observer's confidence ratings are predictive of their actual success. Consider a simple decision task such as whether a stimulus is familiar or novel, followed by a confidence rating in being correct. The task of discriminating one's correct and incorrect judgments using confidence ratings is often called the "type 2 task" (Clarke, Birdsall, & Tanner, 1959; Galvin, Podd, Drga, & Whitmore, 2003) to differentiate it from the "type 1 task" of discriminating between stimulus classes (such as old or new). If higher confidence ratings are given after correct judgments and lower confidence ratings after incorrect judgments we can ascribe high metacognitive sensitivity to the subject. Thus a simple and intuitive way of assessing metacognitive sensitivity is to correlate confidence with accuracy (Nelson, 1984).

However, confidence-accuracy correlations (e.g. gamma and phi correlations) are affected by the confounding factors of type 1 performance (*d'*) and type 2 response bias (overall level of confidence; Fleming & Lau, 2014; Masson & Rotello, 2009). Consider two subjects A and B performing the same task but with different baseline levels of performance. A and B may have the same underlying metacognitive ability, but their confidence-accuracy correlations may differ due to differing performance levels. In this situation we may erroneously conclude that A and B have different metacognition, despite their underlying metacognitive ability being equal. More generally, an important lesson from the signal detection theory (SDT) approach to modelling type 1 and type 2 tasks is that type 1 sensitivity (*d'*) and type 1 criterion (*c*) influence measures of type 2 sensitivity (Galvin et al., 2003).

Recently an alternative approach to measuring metacognitive sensitivity has been developed by Maniscalco & Lau (2012). This approach posits a generative model of the confidence data within the framework of signal detection theory (Figure 1A). Fitting the model to data returns a parameter, meta-*d'*, that reflects an individual's metacognitive sensitivity. Specifically, meta-*d'* is the value of type 1 performance (*d'*) that would have been predicted to give rise to the observed confidence rating data assuming an ideal observer with type 1 *d'* = meta-*d'*. Meta-*d'* can then be compared with actual *d'* and a relative measure of metacognitive sensitivity can then be calculated as a ratio (*meta-d'/d'*) or subtraction (meta-*d'*-*d'*). Meta-*d'/d'* is a measure of *metacognitive efficiency* – given a particular level of task performance, how efficient is the individual's metacognition? If meta-*d'* = *d'*, then the observer is metacognitively "ideal", using all the information available for the type 1 task when reporting type 2 confidence. However we might find that meta-*d'* < *d'*, due to some degree of noise or imprecision introduced when rating one's confidence. Conversely we may find that meta-*d'* > *d'* if subjects are able to draw on additional information such as hunches (Scott, Dienes, Barrett, Bor, & Seth, 2014) or further processing of stimulus information (Charles, Van Opstal, Marti, & Dehaene, 2013; Rabbitt & Vyas, 1981) when making their metacognitive judgments (Fleming & Daw, submitted).

The properties of the meta-*d'* model have been thoroughly explored in previous articles (Barrett, Dienes, & Seth, 2013; Fleming & Lau, 2014; Maniscalco & Lau, 2014; 2012). The goal of the present paper is twofold. First, I introduce a new method for estimating meta-*d'* from confidence ratings using hierarchical Bayes, and provide a tutorial on its usage. Second, I demonstrate the application of this method to derive group-level estimates of metacognitive efficiency in situations where data are limited, such as in the study of patient populations.

Previously *meta-d'* has been fitted using gradient descent on the likelihood (maximum likelihood estimation; MLE), sum-of-squared error (SSE) or using analytic approximation (Barrett et al., 2013; Maniscalco & Lau, 2012). However several factors make a Bayesian approach attractive for typical metacognition studies:

1) Point estimates of meta-*d'* are inevitably noisy. Several parameters must be estimated in the signal detection model, including multiple type 2 criteria

(specifically, (k*2)-2, where k = number of confidence ratings available). One common issue in cognitive neuroscience is that trial numbers per condition are also low (e.g. in patient studies, or tasks conducted in conjunction with neuroimaging), and frequentist estimates of hit and false-alarm rates fail to account for uncertainty about these rates that is a consequence of finite data. A Bayesian analysis incorporates such uncertainty into parameter estimates.

2) A hierarchical Bayesian approach is the correct way to combine information about within- and between-subject uncertainty. In a typical study, the metacognitive sensitivities of two groups (e.g. patients and controls) are compared. Single-subject maximum likelihood fits are carried out, and the fitted meta-*d'* parameters are entered into an independent samples t-test. Any information about the uncertainty in each subject's parameter fits is discarded in this procedure. In contrast, using hierarchical Bayes, information about uncertainty is retained, such that group-level parameters are less influenced by single-subject fits that have a high degree of uncertainty. In turn, hierarchical model fits are able to capitalize on the statistical strength offered by the degree to which subjects are similar with respect to one or more model parameters, mutually constraining the subject-level model fits.
3) In fitting SDT models to data, padding (edge correction) is often applied to avoid zero counts of confidence ratings in particular cells (e.g. high confidence error trials; Hautus, 1995; Macmillan & Creelman, 2005). This padding may bias subject-specific parameter estimates particularly when the overall trial number is low. A Bayesian approach avoids the need for edge correction as the generative multinomial model naturally handles zero cell counts (Lee, 2008).
4) The very nature of a hierarchical model makes testing group-level hypotheses natural and straightforward. For example, say we are interested in testing whether a particular patient group has lower metacognitive sensitivity compared to controls. Hierarchical Bayes allows us to directly estimate a parameter that characterises the differences between groups, and provides a principled framework for hypothesis testing. Finally, a Bayesian framework for cognitive modelling enjoys other advantages that have been outlined in detailed elsewhere (Kruschke, 2014; Lee & Wagenmakers, 2014). Briefly, they include the ability

to gain evidence in favour of the null hypothesis as well as against it; the ability to combine prior information (for example, a prior on the distribution of metacognitive sensitivity in a healthy population) with new data; and the flexible extension of the model to estimate subject- and trial-level influences on metacognition within a general linear model (Kruschke, 2014).

The basics of Bayesian cognitive models are intuitive and straightforward to understand. First, prior information is specified in the form of probability distributions over model parameters, and observed data are used to update beliefs to construct a posterior distribution or belief in a particular parameter. The "hierarchical" component of hierarchical Bayes simply indicates that multiple instances of a particular parameter (for example, across different subjects) are estimated in the same model. The development of efficient sampling routines for arbitrary models such as Markov chain Monte Carlo (MCMC), their inclusion in freely available software packages such as JAGS (http://mcmc-jags.sourceforge.net) and STAN (http://mc-stan.org) and advances in computing power means that arbitrary Bayesian cognitive models are now straightforward to implement in practice (Kruschke, 2014).

In this paper I briefly introduce the meta-*d'* (HMeta-d) model and its hierarchical Bayesian variant (further details of the model can be found in the Supplementary Material and in Maniscalco & Lau, 2014). I then provide a step-by-step MATLAB tutorial for fitting meta-*d'* to single-subject and group data. Finally I carry out parameter-recovery simulations to validate the model and demonstrate its flexibility in addressing population-level experimental questions. Model code and examples are freely available online at https://github.com/smfleming/HMM.

METHODS

Outline of the meta-d' model

The meta-*d'* model is summarized in graphical form in Figure 1A. The raw data for the model fit is the observed distribution of confidence ratings conditional on whether a decision is correct or incorrect. Intuitively, if a subject has greater metacognitive sensitivity, they are able to monitor their decision performance by providing higher

confidence ratings when they are correct, and lower ratings when incorrect, and these distributions will only weakly overlap (solid lines). Conversely a subject with poorer metacognitive sensitivity will show greater overlap between these distributions (dotted lines). The overlap between distributions can be calculated through type 2 receiver operating characteristic (ROC) analysis. The conditional probability P(confidence = y | accuracy) is calculated for each confidence level; cumulating these conditional probabilities and plotting them against each other produces the type 2 ROC function. A type 2 ROC that bows sharply upwards indicates a high degree of sensitivity to correct/incorrect decisions; a type 2 ROC closer to the major diagonal indicates weaker metacognitive sensitivity.

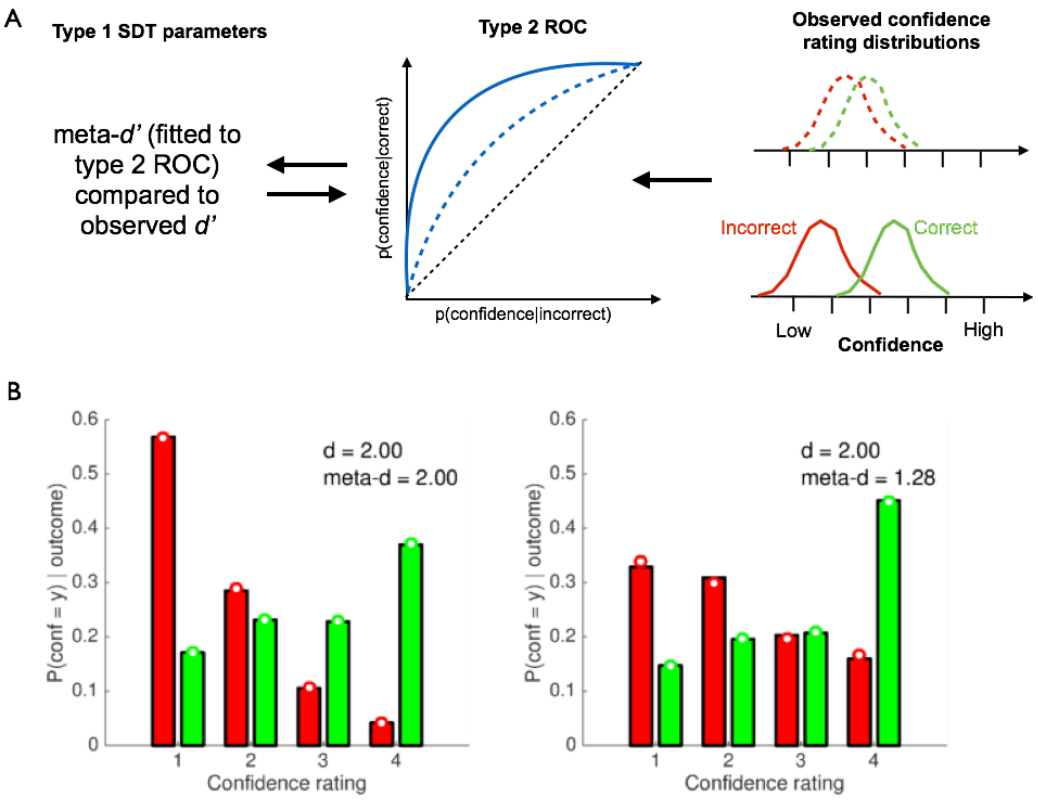

Figure 1. The meta-*d' *model. *A) The right-hand panel shows schematic confidence-rating distributions conditional on correct and incorrect decisions. A subject with good metacognitive sensitivity will provide higher confidence ratings when they are correct, and lower ratings when incorrect, and these distributions will only weakly overlap (solid lines). Conversely a subject with poorer metacognitive sensitivity will show greater overlap between these distributions (dotted lines). The overlap between*

distributions can be calculated through type 2 receiver operating characteristic (ROC) analysis (middle panel). The theoretical type 2 ROC is completely determined by an equal-variance Gaussian signal detection theory model; we can therefore invert the model to determine the type 1 d' that best fits the observed confidence rating data, which is labelled meta-d'. Meta-d' can be directly compared to the type 1 d' calculated from the subject's decisions – if meta-d' is equal to d', then the subject approximates the ideal SDT prediction of metacognitive sensitivity. B) Simulated data from a SDT model with d' = 2. The y-axis plots the conditional probability of a particular rating given the first-order response is correct (green) or incorrect (red). In the right-hand panel, Gaussian noise has been added to the internal state underpinning the confidence rating (but not the decision) leading to a blurring of the correct/incorrect distributions. Open circles show fits of the meta-d' model to each simulated dataset.

The area under the type 2 ROC (AUROC2) is itself a useful non-parametric measure of metacognitive sensitivity, indicating how well an observer's ratings discriminate between correct and incorrect decisions. However, as outlined in the introduction, AUROC2 is affected by type 1 performance. In other words, a change in task performance (*d'* or criterion) is expected, *a priori*, to lead to changes in AUROC2 despite endogenous metacognitive efficiency remaining unchanged. By explicitly modelling the connection between performance and metacognition we can appropriately handle this confound. The core idea behind the meta-*d'* approach is that a single theoretical type 2 ROC is completely determined by an equal-variance Gaussian signal detection theory model with parameters *d'*, criterion *c* and confidence criteria c_2 (the arrow going from left to right in Fig. 1A). The converse is therefore also true: an observed type 2 ROC implies a particular type 1 *d'* (the arrow going from right to left in Fig. 1A). We can therefore invert the model to determine the type 1 *d'* that best fits the observed confidence rating data. As this pseudo-*d'* is fit only to confidence rating data, and not the subject's decisions, we label it meta-*d'*. Meta-*d'* can be directly compared to the type 1 *d'* calculated from the subject's decisions – if meta-*d'* is equal to *d'*, then the subject approximates the ideal SDT prediction of metacognitive sensitivity. The relative values of *d'* and meta-*d'* thus quantify the relative sensitivity of decisions and confidence ratings respectively. A ratio of these quantities (meta-*d'/d'*) provides a summary measure of "metacognitive efficiency".

Figure 1B provides a concrete example. The data in both panels are simulated from a SDT model with *d'* = 2 and symmetric flanking confidence criteria positioned such that stronger internal signals lead to higher confidence ratings on a 1-4 scale. The y-axis plots the conditional probability of a particular rating given the first-order response is correct (green) or incorrect (red). In both panels, the simulations return higher confidence ratings more often on correct trials and lower confidence more often on incorrect trials. However, in the right-hand panel, Gaussian noise has been added to the internal state underpinning the confidence rating (but not the decision). This leads to a blurring of the correct/incorrect distributions, such that higher confidence ratings are used even when the decision is incorrect. The open circles show fits of the meta-*d'* model to each simulated dataset. While both fits return type 1 *d'* values of 2.0, the meta-*d'* value in the right-hand panel is much lower than on the left, leading to a meta-*d'*/*d'* ratio of ~64% of optimal. Notably meta-*d'* in the left panel is equal to *d'*, as expected if confidence ratings are generated from an ideal observer model without any additional noise. This example illustrates how meta-*d'* can appropriately recover changes in the fidelity of confidence ratings independently of changes in performance.

Single-subject optimisation of meta-d'

I first briefly review the standard meta-*d'* model and methods for obtaining single-subject parameter estimates. The model contains free parameters for meta-*d'* and the positions of the (k-1)*2 confidence criteria, where k = number of confidence ratings available. These criteria are response-conditional, with k-1 criteria following an S1 response and k-1 criteria following an S2 response (c_{S1} and c_{S2}). The raw data comprise counts of confidence ratings conditional on both the stimulus category (S1 or S2) and response (S1 or S2). Type 1 criterion c and sensitivity *d'* are estimated from the data using standard formulae (Macmillan & Creelman, 2005)[1].

The fitting of meta-*d'* rests on calculating the likelihood of the confidence rating data given a particular type 2 ROC generated by systematic variation of type 1 SDT parameters *d'* and c, and type 2 criteria c_2. Describing the observed type 2 ROC in terms

[1] In HMeta-d there is also a user option for jointly estimating both *d'* and meta-*d'* in a hierarchical framework.

of these type 1 SDT parameters underpins the meta-*d'* model. By convention, the prefix "meta-" is added to each type 1 SDT parameter in order to indicate that the parameter is being used to fit type 2 ROC curves. Thus, the type 1 SDT parameters *d' c*, and c_2, when used to characterize type 2 ROC curves, are named meta-*d'* meta-*c*, and meta-c_2.

The Appendix contains equations for deriving type 2 probabilities from the type 1 SDT model for both S1 and S2 responses. Given a particular setting of meta-*d'*, meta-*c* and meta-c_2 these equations specify a multinomial probability distribution $P(conf = y \mid stim = i\,, resp = j)$ over observed confidence counts. The likelihood of the type 2 confidence data can be characterized using the multinomial model as:

$$L(\theta|data) \propto \prod_{y,i,j} P_\theta(conf = y \mid stim = i\,, resp = j)^{n_{data}(conf=y|stim=i\,,resp=j)}$$

Best-fitting parameters are then obtained by maximising the likelihood of the data:

$$\theta^* = \arg\max_{\theta} L(\theta|data), \qquad \text{subject to:} \quad \text{meta}c' = c', \quad \gamma\left(\text{meta}\underline{\boldsymbol{c}}_{ascending}\right)$$

where $\gamma\left(\text{meta}\underline{\boldsymbol{c}}_{ascending}\right)$ is a Boolean function which returns a value of "true" only if the type 1 and type 2 criteria stand in appropriate ordinal relationships, i.e. each element in $\underline{\boldsymbol{c}}_{ascending}$ is at least as large as the previous element, and c' is a measure of type 1 response bias.

Hierarchical Bayesian estimation of meta-d'

In hierarchical Bayesian estimation of meta-*d'* (HMeta-d), the model is similar except group-level prior densities are specified over each of the subject-level parameters referred to in the previous section. A further difference between HMeta-d and single-subject estimation is that the group-level parameter of interest is the ratio meta-*d'*/*d'* rather than meta-*d'* itself. The rationale for this modelling choice is that while each subject or group may differ in type 1 *d'*, our parameter of interest is metacognitive efficiency at the group level, not meta-*d'* (which itself will be influenced by subject- or

group-level variability in *d'*). Thus *d'* is treated as a subject-level nuisance parameter[2]. An advantage of this scheme is that group-level inference is carried out directly on metacognitive efficiency rather than a transformed parameter. I specified weakly informative Gaussian priors on log(meta-*d'*/*d'*) that encompassed 167 MLE parameter estimates aggregated from behavioural studies of metacognition of perceptual decision-making in our laboratory (Fleming, Huijgen, & Dolan, 2012; Fleming, Weil, Nagy, Dolan, & Rees, 2010; Palmer, David, & Fleming, 2014; L. G. Weil et al., 2013; Figure 2B). A log-normal prior is appropriate for a ratio parameter, ensuring that increases and decreases relative to the expected value of 1 are given equal weight (Howell, 2009; Keene, 1995).

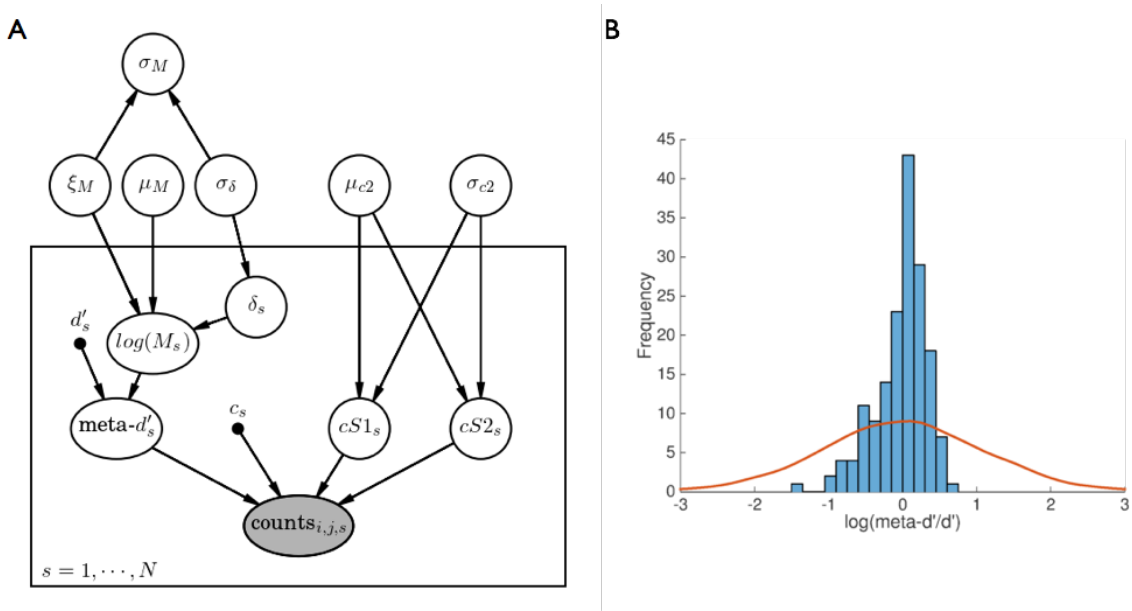

Figure 2. The hierarchical meta-*d' *model. *A) Probabilistic graphical model for estimating metacognitive efficiency using hierarchical Bayes* (HMeta-d). *The nodes represent all the relevant variables for parameter estimation, and the graph structure is used to indicate dependencies between the variables as indicated by directed arrows. As is convention, unobserved variables are represented without shading and observed variables (in this case, confidence rating counts) are represented with shading. Discrete*

[2] Alternative estimation schemes are possible; for instance, calculating the ratio of hierarchical nodes independently encoding meta-*d'* and *d'*. I chose the "nuisance parameter" scheme as it stays closest to the standard MLE approach while directly estimating a group-level node for meta-*d'*/*d'*.

variables for type 1 d' and criterion are represented as individual points, and the box encloses participant-level parameters subscripted with s. *The main text contains a description of each node and its prior distribution. Figure created using the Daft package in Python (http://daft-pgm.org). B) Prior over the group-level estimate of log(meta-*d'/d'*) (μ_M). The solid line shows a kernel density estimate of samples from the prior; the histogram represents empirical meta-*d'/d' *estimates obtained from 167 subjects (see main text for details).*

Dependencies between nodes in the HMeta-d model are illustrated as a probabilistic graphical model in Figure 2A. The box encloses participant-level parameters subscripted with *s*. Each node is specified as follows (where *M* denotes meta-*d'/d'*):

$$\mu_{c2} \sim \mathcal{N}(0, 10)$$

$$\sigma_{c2} \sim \mathcal{HN}(10)$$

$$\mu_M \sim N(0, 1)$$

$$\sigma_\delta \sim \mathcal{HN}(1)$$

$$\xi_M \sim Beta(1,1)$$

$$cS1_s[1: k-1] \sim N(-\mu_{c2}, \sigma_{c2})$$

$$cS2_s[1: k-1] \sim N(\mu_{c2}, \sigma_{c2})$$

$$\delta_s \sim N(0, \sigma_\delta)$$

$$\log(M_s) = \mu_M + \xi_M \times \delta_s$$

$\mathcal{N}$ represents a normal distribution parmeterised by mean and standard deviation; $\mathcal{HN}$ represents a positive-only, half-normal parameterised by standard deviation. To avoid the hierarchical variance parameter on log(*M*) becoming trapped near zero parameter expansion is employed (Gelman & Hill, 2007; Lee & Wagenmakers, 2014) such that the mean and variance of $\log(M_s)$ are scaled by a redundant multiplicative parameter ξ_M.

The HMeta-d toolbox uses Markov chain Monte Carlo (MCMC) implemented in JAGS (Plummer, 2003) to estimate the joint posterior distribution of all model parameters.

Preparing confidence rating data

Fitting of group-level data in the HMeta-d toolbox requires identical data preparation to that required when obtaining single-subject fits using MLE or SSE using Maniscalco & Lau's MATLAB code (available at http://www.columbia.edu/~bsm2105/type2sdt/). I therefore start with a short tutorial on preparing data for estimating single-subject meta-*d'*, before explaining how to input data from a group of subjects into the hierarchical model.

Data from each subject need to be coerced into two vectors, `nR_S1` and `nR_S2`, which contain confidence-rating counts for when the *stimulus* was S1 and S2, respectively. Each vector has length $k \times 2$, where k is the number of ratings available. Confidence counts are entered such that the first entry refers to counts of maximum confidence in an S1 response, and the last entry to maximum confidence in an S2 response. For example, if three levels of confidence rating were available and `nR_S1 = [100 50 20 10 5 1]`, this corresponds to the following rating counts following S1 presentation:

responded S1, rating=3 : 100 times

responded S1, rating=2 : 50 times

responded S1, rating=1 : 20 times

responded S2, rating=1 : 10 times

responded S2, rating=2 : 5 times

responded S2, rating=3 : 1 time

This pattern of responses corresponds to responding "high confidence, S1" most often following S1 presentations, and least often with "high confidence, S2". A mirror image of this vector would be expected for `nR_S2`. For example, `nR_S2 = [3 7 8 12 27 89]` corresponds to the following rating counts following S2 presentation:

responded S1, rating=3 : 3 times

responded S1, rating=2 : 7 times

responded S1, rating=1 : 8 times

responded S2, rating=1 : 12 times

responded S2, rating=2 : 27 times

responded S2, rating=3 : 89 times

Together these vectors specify the confidence × stimulus × response matrix that is the basis of the meta-*d'* fit, and can be passed directly into Maniscalco & Lau's `fit_meta_d_MLE` function to estimate meta-*d'* on a subject by subject basis.

Fitting a hierarchical model

Estimating a group-level model using HMeta-d requires very little extra work. In HMeta-d, the `nR_S1` and `nR_S2` variables are cell arrays of vectors, with each entry in the cell containing confidence counts for a single subject. For example, to specify the confidence counts following S1 presentation listed above for subject 1, one would enter in MATLAB:

```
nR_S1{1} = [100 50 20 10 5 1]
```

and so on for each subject in the dataset. These cell arrays then contain confidence counts for all subjects, and are passed in one step to the main HMeta-d function:

```
fit = fit_meta_d_mcmc_group(nR_S1, nR_S2)
```

An optional third argument to this function is `mcmc_params` which is a structure containing flags for choosing different model variants, and for specifying the details of the MCMC routine. If omitted reasonable default settings are chosen.

The call to `fit_meta_d_mcmc_group` returns a "fit" structure with several subfields. The key parameter of interest is `fit.mu_logMratio`, which is the mean of the posterior distribution of the group-level log(meta-*d'/d'*). `fit.mcmc` contains the samples of each

parameter, which can be plotted with the helper function `plotSamples`. For instance to plot the MCMC samples of μ_M, one would enter:

```
plotSamples(exp(fit.mcmc.samples.mu_logMratio))
```

Note the "exp" to allow plotting of meta-*d'/d'* rather than log(meta-*d'/d'*). The `exampleFit_` scripts in the toolbox provide other examples, such as how to set up response-conditional models and to visualise subject-level fits.

An important step in model fitting is checking that the MCMC chains have converged. While there is no way to guarantee convergence for a given number of MCMC samples, some heuristics can help identify problems. By using `plotSamples`, we can visualise the traces to check that there are no drifts or jumps and that each chain occupies a similar position in parameter space. Another useful statistic is Gelman & Rubin's scale-reduction statistic $\hat{R}$, which is stored in the field `fit.mcmc.Rhat` for each parameter (Gelman & Rubin, 1992). This provides a formal test of convergence that compares within-chain and between-chain variance of different runs of the same model, and will be close to 1 if the samples of the different chains are similar. Large values of $\hat{R}$ indicate convergence problems and values < 1.1 suggest convergence.

As well as obtaining an estimate for group-level meta-*d'/d'*, we are often interested in our certainty in this parameter value. This can be estimated by computing the 95% high-density interval (HDI) from the posterior samples (Kruschke, 2014). The helper function `calc_HDI` takes as input a vector of samples and returns the 95% HDI:

```
calc_HDI(exp(fit.mcmc.samples.mu_logMratio(:))
```

The colon in the brackets selects all samples in the array regardless of its chain of origin. As HMeta-d uses Bayesian estimation it is straightforward to use the group-level posterior density for hypothesis testing. For instance, if the question is whether one group of subjects has greater metacognitive efficiency than a second group, we can ask whether the HDI of the difference overlaps with zero (see "Empirical examples" below for an example of this). However, note that it is incorrect to use the subject-level parameters estimated as part of the hierarchical model in a frequentist test (e.g. a *t*-test); this violates the independence assumption.

In addition to enabling inference on individual parameter distributions, there may be circumstances in which we wish to compare models of different complexity (see Discussion). To enable this, JAGS returns the deviance information criteria for each model (Spiegelhalter, Best, Carlin, & Van Der Linde, 2002; DIC; lower is better). While the DIC is known to be somewhat biased towards models with greater complexity, it is a common method for assessing model fit in hierarchical models. In HMeta-d the DIC for each model can be obtained in `fit.mcmc.dic`.

RESULTS

Example fit

Figure 3A shows the output of a typical call to HMeta-d and the resultant posterior samples of the population-level estimate of metacognitive efficiency, $\mu_{meta-d'/d'}$, plotted with `plotSamples`. The data were generated as 20 simulated subjects, each with 400 trials and four possible confidence levels (confidence criteria $c_2 = \pm[0.5\ 1\ 1.5]$; type 1 criterion $c = 0$). For each subject, type 1 *d'* was sampled from a normal distribution $N(2, 0.2)$, and meta-*d'/d'* was fixed at 0.8. The chains show excellent mixing with a modest number of samples ($\hat{R}$ = 1.000) and the posterior is centred around the ground truth simulated value.

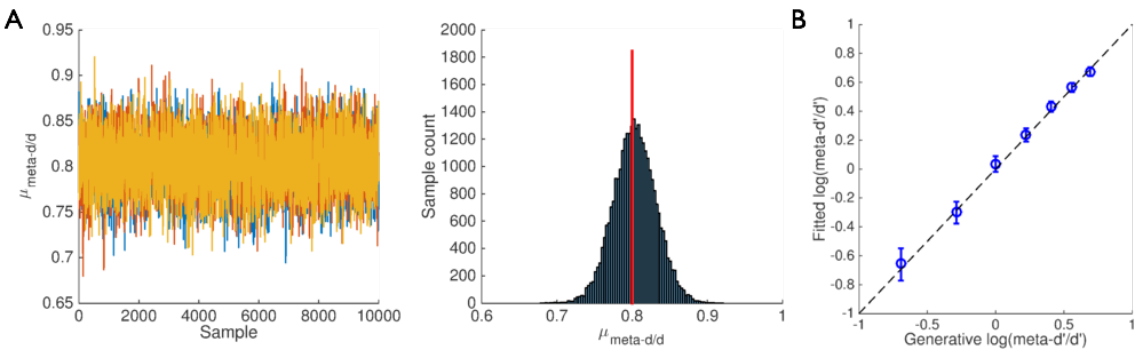

Figure 3. HMeta-d output. *A) Example output from HMeta-d fit to simulated data with ground truth* meta-*d'/d' fixed at 0.8 for 20 subjects. The left panel shows the three MCMC chains for parameter* $\mu_{meta-d'/d'}$*; the right panel shows the same samples aggregated in a histogram. B) Parameter recovery exercise using HMeta-d to fit data*

simulated from 7 groups of 20 subjects with different levels of meta-d'/d' = [0.5 0.75 1.0 1.25 1.5 1.75 2]. Error bars denote 95% HDI.

Parameter recovery

To further validate the model a parameter recovery exercise was carried out in which data was simulated from 7 groups of 20 subjects with different levels of meta-*d'/d'* = [0.5 0.75 1.0 1.25 1.5 1.75 2]. All other settings were as described in the previous section. Figure 3B plots the fitted group-level $\mu_{meta-d'/d'}$ and its associated 95% HDI for each of the simulated datasets against the empirical ground truth, demonstrating robust parameter recovery.

Empirical examples

To illustrate the practical application of HMeta-d I fit data from a recent experiment that examined metacognitive sensitivity in perceptual and mnemonic tasks in patients with post-surgical lesions and controls (Fleming, Ryu, Golfinos, & Blackmon, 2014). This study found that metacognitive efficiency in patients with lesions to anterior prefrontal cortex (aPFC) was selectively compromised on a perceptual task but unaffected on a memory task, suggesting that the neural architecture supporting metacognition may comprise domain-specific components differentially affected by neurological insult.

For didactic purposes here I restrict comparison of metacognition in the aPFC patients (N=7) and healthy controls (HC; N=19) on the perceptual task. The task required a two-choice discrimination as to which of two briefly presented patches contained a greater number of small white dots, followed by a continuous confidence rating on a sliding scale from 1 (low confidence) to 6 (high confidence). For analysis these confidence ratings were binned into four quantiles. For each subject confidence rating data (levels 1-4) were sorted according to the position of the target stimulus (L/R) and the subject's response (L/R), thereby specifying the two nR_S1 and nR_S2 arrays required for estimating meta-*d'*.

For each group I constructed cell arrays of confidence counts and estimated $\mu_{meta-d'/d'}$ with the default settings in HMeta-d. The resultant posterior distributions are plotted in Figure 4A, and the posterior distribution of the difference is shown in Figure 4B.

Several features are evident from these outputs. First, there is a reduced metacognitive efficiency in the aPFC group compared to controls, as revealed by the 95% HDI of the difference being greater than zero (right-hand panel). Second, the posterior distribution of metacognitive efficiency in the healthy controls is overlapping with the optimal estimate of 1. Finally, for the aPFC group, which compromises fewer subjects, there is a higher degree of uncertainty about the true metacognitive efficiency – the width of the posterior distribution is greater. This is a consequence of the parameter estimate being constrained by fewer data points and is a natural consequence of the Bayesian approach.

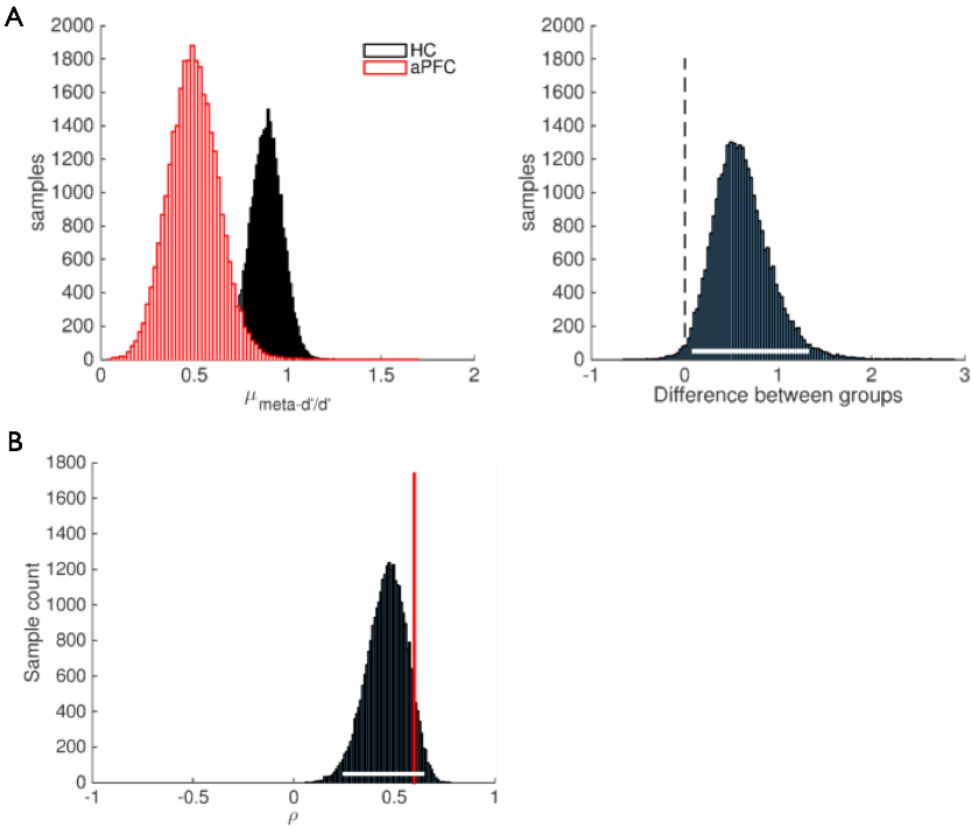

Figure 4. Empirical applications of HMeta-d. *A) HMeta-d fits to data from the perceptual metacognition task reported in Fleming et al. (2014). Each histogram represents posterior densities of* $\mu_{meta-d'/d'}$ *for two groups of subjects: HC = healthy controls; aPFC = anterior prefrontal cortex lesion patients. The right panel shows the difference (in log units) between the group posteriors. The white bar indicates the 95% HDI which excludes zero. B) Example of extending HMeta-d to estimate the correlation coefficient ρ between metacognitive efficiencies in two domains. The red line shows the ground-truth correlation between pairs of meta-d'/d' values for 100 simulated subjects.*

Flexible extensions of the basic model

An advantage of working with Bayesian graphical models is that they are easily extendable to estimate other influences on metacognitive efficiency in the context of the same model (Lee & Wagenmakers, 2014). For instance, one question of interest is whether metacognitive ability in one domain, such as perception, is predictive of metacognitive ability in another domain, such as memory. Evidence pertaining to this question is mixed: some studies have found evidence for a modest correlation in metacognitive efficiency across domains (Ais, Zylberberg, Barttfeld, & Sigman, 2016; McCurdy et al., 2013) whereas others have reported a lack of correlation (Baird, Smallwood, Gorgolewski, & Margulies, 2013; Kelemen, Frost, & Weaver, 2000). One critical issue in testing this hypothesis is that the uncertainty in the model's estimate of meta-*d'* should be incorporated into an assessment of any correlation between the two domains. This is naturally accommodated by embedding an estimate of the correlation coefficient in a hierarchical estimation of metacognitive efficiency.

To expand the model, each subject's metacognitive efficiencies in the two domains (*M1, M2*) are specified as draws from a bivariate Gaussian[3]:

$$[\log(M1_s)\ \log(M2_s)] \sim N\left(\begin{bmatrix}\mu_{M1}\\ \mu_{M2}\end{bmatrix}, \begin{bmatrix}\sigma_{M1}^2 & \rho\sigma_{M1}\sigma_{M2}\\ \rho\sigma_{M1}\sigma_{M2} & \sigma_{M2}^2\end{bmatrix}\right)$$

Priors were specified as follows:

$$\mu_M^1, \mu_M^2 \sim N(0,1)$$

$$\sigma_{M1}, \sigma_{M2} \sim \text{InvSqrtGamma}(0.001, 0.001)$$

$$\rho \sim \text{Uniform}(-1,1)$$

To demonstrate the application of this expanded model I simulated 100 subjects' confidence data from the type 2 SDT model in two "tasks". Each task's generative meta-*d'/d'* was drawn from a bivariate Gaussian with mean = $\mu_{M1} = \mu_{M2} = 0.8$ and standard deviations $\sigma_{M1} = \sigma_{M2} = 0.5$. Type 1 *d'* was generated separately for each task

[3] Note parameter expansion is omitted here for clarity

from a $N(2, 0.2)$ distribution. The generative correlation coefficient ρ was set to 0.6. Data from both domains are then passed into the model simultaneously, and a group-level posterior distribution on the correlation coefficient ρ is returned. Figure 4B shows this posterior together with the 95% HDI, which encompasses the generative correlation coefficient.

DISCUSSION

The quantification of metacognition from confidence ratings is a question with application in several subfields of psychology and neuroscience, including perception, decision-making, memory, education, aging, and psychiatric disorders. There are now several tools in the psychologist's armoury for estimating how closely a subject's subjective reports track task performance (Fleming & Lau, 2014). An important advance is the recognition that simple correlation coefficients are affected by fluctuations in performance and confidence bias, and the meta-*d'* model was developed to allow correction of metacognitive sensitivity for these potential confounds (Maniscalco & Lau, 2012).

The hierarchical Bayesian approach to estimating metacognitive efficiency introduced here enjoys several advantages. It naturally incorporates variable uncertainty about finite hit and false-alarm rates; it is the correct way to incorporate information about within- and between-subject uncertainty; it avoids the need for edge correction or data modification, and provides a flexible framework for hypothesis testing and model expansion. The toolbox introduced here provides a simple MATLAB implementation that harnesses the MCMC sampler JAGS to return posterior distributions over group-level model parameters. The tutorial outlined how data preparation is identical to that required for the existing maximum-likelihood routines, allowing the user to easily apply both approaches once their data are in the correct format.

Practical recommendations for quantifying metacognition

If group-level estimates of meta-*d'/d'* are of primary interest, HMeta-d allows direct inference at this upper level of the hierarchy while appropriately handling participant-level uncertainty. The HMeta-d toolbox also allows Bayesian estimation of single-subject meta-*d'*, but if single-subject estimates are of primary interest, the MLE approach may be simpler and computationally less expensive (http://www.columbia.edu/~bsm2105/type2sdt/).

More generally, whether one should use metacognitive sensitivity (e.g. meta-*d'* or AUROC2) or metacognitive efficiency (meta-*d'/d'*) as the measure of metacognition

depends on the goal of an analysis. For example, if we are interested in establishing the presence or absence of metacognition in a particular condition, such as when performance is particularly low (Scott et al., 2014) or in particular subject groups such as human infants (Goupil et al., 2016), computing metacognitive sensitivity may be sufficient. However, when comparing experimental conditions or groups which may differ systematically in performance, estimating metacognitive efficiency appropriately controls for the undesirable confounds introduced by type 1 performance and response biases. Note however there are also limitations in the applicability of the meta-*d'* model. First and foremost, the task should be amenable to analysis in a 2-choice SDT framework, as fitting meta-*d'* requires specification of a 2 (stimulus) x 2 (response) x N (confidence rating) matrix. If a task does not conform to these specifications (such as one with N alternative responses) then employing an alternative bias-free measure of metacognitive sensitivity such as the area under the type 2 ROC (AUROC2) may be preferable (Fleming & Lau, 2014). In addition, like all analysis approaches, meta-*d'* assumes a particular generative model of the confidence data that is at best incomplete, and untenable in certain circumstances. For instance, equal variance is specified for S1 and S2 distributions[4] and stable confidence criteria are assumed which may be at odds with findings of serial adjustments in criteria (Rahnev, Koizumi, McCurdy, D'Esposito, & Lau, 2015; Treisman, 1984). More broadly, meta-*d'* is primarily a tool for estimating metacognitive sensitivity, and additional considerations are needed when developing a complete model of confidence (Pouget, Drugowitsch, & Kepecs, 2016; Fleming & Daw, submitted).

Future directions

The HMeta-d model code can be flexibly extended to allow estimation of other influences on metacognitive sensitivity. Here one simple example is explored, the

[4] Maniscalco & Lau's `fit_meta_d_MLE` code allows setting the ratio of S1 and S2 variances as a free parameter; it would be possible to incorporate a similar parameter in future versions of HMeta-d. However as described by Maniscalco & Lau (2014) there is ambiguity between changes in response-specific metacognitive efficiency and the variance ratio, and therefore we recommend users employ the equal-variance model unless they have access to independent estimates of the variance inequality.

specification of a population-level correlation coefficient relating metacognitive efficiencies across domains. More broadly, it is possible to specify flexible general linear models linking trial- or subject-level variables to meta-*d'* (Kruschke, 2014). Currently this requires bespoke model specification, but in future work we hope to provide a flexible user interface for the specification of arbitrary models (cf. Wiecki, Sofer, & Frank, 2013). Estimation of single-trial influences on metacognitive efficiency, such as attentional state or brain activity, is a particularly intriguing proposition. Currently, estimation of meta-*d'* requires many trials, restricting studies of the neural basis of metacognitive efficiency to between-condition or between-subject analyses. Extending the HMeta-d framework to estimate trial-level effects on meta-*d'* may therefore accelerate our understanding of the neural basis of metacognitive efficiency.

Also naturally accommodated in a hierarchical framework is the comparison of different model structures for metacognition within and across tasks. A currently open question is whether metacognition relies on common or distinct processes across different domains, such as perception or memory (Ais et al., 2016; Baird et al., 2013; Fleming et al., 2014; McCurdy et al., 2013). One approach to addressing this question is to specify variants of the HMeta-d model in which different parameters are shared across domains, such as meta-*d'* and/or the confidence criteria. Through model comparison, one could then obtain the model that best accounted for the relationship between metacognitive performance across different domains, and shed light on the common and distinct components.

Conclusions

This paper introduces a hierarchical Bayesian approach to estimating metacognitive efficiency. This approach has several methodological advantages in comparison to current approaches that focus on single-subject point estimates, and may prove particularly beneficial for studies of metacognition in patient populations and cognitive neuroscience experiments where often only limited data are available. More broadly, this framework can be flexibly extended to specify and compare different models of meta-*d'* within a common scheme, thereby advancing our understanding of the neural and computational basis of self-evaluation.

APPENDIX

Type 2 SDT model equations

For a discrete confidence scale ranging from 1 to k, $k - 1$ type 2 criteria are required to rate confidence for each response type. We define type 2 confidence criteria for S1 and S2 responses as:

$$\underline{\dot{\boldsymbol{c}}}_{2,"S1"} = \left(c, c_{2,"S1"}^{conf=2}, c_{2,"S1"}^{conf=3}, \dots, c_{2,"S1"}^{conf=k}, -\infty\right)$$
$$\underline{\dot{\boldsymbol{c}}}_{2,"S2"} = \left(c, c_{2,"S2"}^{conf=2}, c_{2,"S2"}^{conf=3}, \dots, c_{2,"S2"}^{conf=k}, \infty\right)$$

And

$$\underline{\boldsymbol{c}}_{ascending} = \left(c_{2,"S1"}^{conf=k}, c_{2,"S1"}^{conf=k-1}, \dots, c_{2,"S1"}^{conf=1}, c, c_{2,"S2"}^{conf=1}, c_{2,"S2"}^{conf=2}, \dots, c_{2,"S2"}^{conf=k}\right)$$

Then the probabilities of each confidence rating conditional on a given stimulus and response are:

$$Prob(conf = y \mid stim = S1, resp = "S1") = \frac{\Phi\left(\underline{\dot{\boldsymbol{c}}}_{2,"S1"}(y), -\frac{d\prime}{2}\right) - \Phi\left(\underline{\dot{\boldsymbol{c}}}_{2,"S1"}(y+1), -\frac{d\prime}{2}\right)}{\Phi\left(c, -\frac{d\prime}{2}\right)}$$

$$Prob(conf = y \mid stim = S2, resp = "S1") = \frac{\Phi\left(\underline{\dot{\boldsymbol{c}}}_{2,"S1"}(y), \frac{d\prime}{2}\right) - \Phi\left(\underline{\dot{\boldsymbol{c}}}_{2,"S1"}(y+1), \frac{d\prime}{2}\right)}{\Phi\left(c, \frac{d\prime}{2}\right)}$$

$$Prob(conf = y \mid stim = S1, resp = "S2") = \frac{\Phi\left(\underline{\dot{\boldsymbol{c}}}_{2,"S2"}(y+1), -\frac{d\prime}{2}\right) - \Phi\left(\underline{\dot{\boldsymbol{c}}}_{2,"S2"}(y), -\frac{d\prime}{2}\right)}{1 - \Phi\left(c, -\frac{d\prime}{2}\right)}$$

$$Prob(conf = y \mid stim = S2\,, resp = "S2")$$

$$= \frac{\Phi\left(\underline{\dot{\boldsymbol{c}}}_{2,"S2"}(y+1), \frac{d\prime}{2}\right) - \Phi\left(\underline{\dot{\boldsymbol{c}}}_{2,"S2"}(y), \frac{d\prime}{2}\right)}{1 - \Phi\left(c, \frac{d\prime}{2}\right)}$$

where $\Phi()$ is the cumulative distribution function of the standard normal distribution.

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