CopyMix: Mixture Model Based Single-Cell Clustering and Copy Number Profiling using Variational Inference

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Abstract

Single-cell sequencing technologies are becoming increasingly more established, in particular, in the study of tumor heterogeneity. Investigating tumor heterogeneity is imperative to better understand how tumors evolve since each cell subpopulation harbors a unique set of genomic features that yields a unique phenotype, an issue that is bound to have clinical relevance. Clustering of cells based on copy number data, obtained from single-cell DNA sequencing, provides an opportunity to assess different tumor cell subpopulations. Accordingly, computational methods have emerged for single-cell copy number profiling as well as clustering; however, these two tasks have up to now been handled sequentially, and, included various ad-hoc preprocessing steps; hence, prone to introduce clustering artifacts. We propose CopyMix, a mixture model-based method using Variational Inference, to simultaneously cluster cells and infer a single copy number profile for each cluster, revealing the copy number variation pattern behind each cluster. CopyMix is evaluated using simulated data as well as published biological data from metastatic colorectal cancer and ovarian cancer. The results reveal high clustering performance and low errors in copy number profiling. These favorable results indicate a considerable potential to obtain clinical impact by using CopyMix in studies of cancer tumor heterogeneity.
Keywords: Mixture Models; Variational Inference; Copy number profiling; Single-cell

1 Introduction

A tumor typically consists of a collection of heterogeneous cell populations, each having distinct genetic and phenotypic properties, in particular, concerning the capacity to promote cancer progression, metastasis, and therapy resistance Eirew et al. (2015); Nowell (1976). Single-cell sequencing technologies Gawad et al. (2016); Navin et al. (2011); Shapiro et al. (2013); Zahn et al. (2017) provide a new opportunity to investigate the genomic profile of individual cells both regarding single nucleotide variation (SNV) and copy number variation (CNV). CNVs and SNVs have been shown to be important contributors to phenotypic variation relating to health and disease Baslan et al. (2012); Lawson et al. (2018). Moreover, although single-cell SNV profiling is hampered by experimental imperfections such as drop-outs, copy number profiling (detecting single-cell copy number variations) is feasible at least at coarser resolutions.

Clustering of cells based on their individual copy number profiles provides the opportunity for SNV profiling of the clusters and, in general, opportunities to better understand tumor subpopulations and tumor heterogeneity, issues that are bound to have clinical relevance.

Current single-cell datasets pose a wealth of computational challenges. As answers to some of those, methods have emerged for single-cell copy number profiling as well as clustering of cells based on their already inferred copy number profiles, e.g., Garvin et al. (2015); Zahn et al. (2017); Leung et al. (2017); Vitak et al. (2017); however, these two tasks have up to now been handled sequentially with various ad-hoc preprocessing steps. The sequential approach is vulnerable to clustering artifacts since preprocessing decisions constrain all later analyses and are typically irreversible. Even if each task has maximized its performance, the final result may still fall short of the best possible in the traditional single-phase inference framework Blocker and Meng (2013). As a solution to this issue, We propose a novel framework, referred to as CopyMix, performing the two tasks simultaneously. As to the best of our knowledge, no earlier work has simultaneously performed the two tasks, and, the need is listed in Zhang and Campbell (2020) as a future direction. Similar to CopyMix, simultaneous inference have been considered but for single-cell DNA methylation Kapourani and Sanguinetti (2019); de Souza et al. (2020), single-cell SNV data Roth et al. (2016), and bulk chIP-seq data from several replicates Zuo et al. (2016).

CopyMix is a model-based approach enjoying the advantages of a fully probabilistic framework such as transparency, uncertainty measurements, and modeling flexibility. CopyMix uses a mixture model with components, corresponding to clusters, each having a specific copy number profile modeled by a sequence of latent variables which follows a Markov structure similar to the state-of-the-art methods using Hidden Markov Models (HMMs) Shah et al. (2006); Vitak et al. (2017); Zahn et al. (2017), revealing the copy number variation pattern behind each cluster. CopyMix combines statistical strength from data across all cells and neighboring sites. We deploy a Bayesian treatment and infer all quantities of interest using Variational Inference (VI) Jordan et al. (1999), which typically yields faster than inference methods like Markov chain Monte Carlo sampling Blei et al. (2017). Compared to Expectation-Maximization (EM), VI has multiple advantages, e.g., it estimates the posterior distributions rather than point estimates, protects against over-fitting, and allows for performing principled determination of the optimal number of mixture components Bishop (2006).

We perform experiments on biologically inspired simulated data. Moreover, we conduct experiments on the published biological data from Leung et al. (2017) and Laks and McPherson (2019), and evaluate CopyMix results. Note that it is not equitable to compare CopyMix to the state-of-the-art methods in copy number profiling and single-cell clustering since various ad-hoc processes (e.g., outlier removal, curve smoothing, etc.) are performed within these approaches, which add vital contributions to these sequential approaches. A sequential version of CopyMix, resembling Shah et al. (2006); Vitak et al. (2017) but excluding their ad-hoc processes, is compared to CopyMix, confirming the hypothesis that single-phase inference framework is superior over a sequential version of the framework.

2 CopyMix

CopyMix is a probabilistic clustering method based on a mixture of Markov chains. CopyMic is a mixture model with components corresponding to clusters, each having a specific copy number profile modeled by a sequence of latent variables. Similarly, as in Shah et al. (2006); Vitak et al. (2017); Zahn et al. (2017), we assume that each latent copy number sequence is governed by a Markov chain (describing a sequence

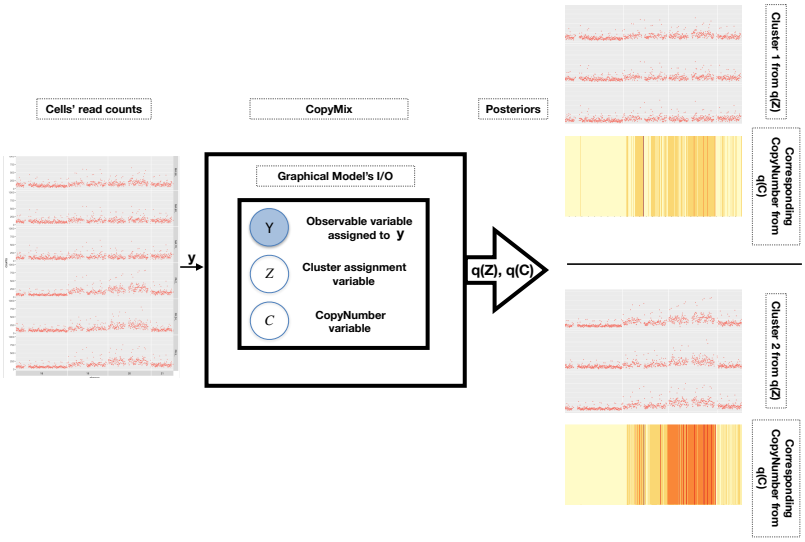

Figure 1: The overview of CopyMix with: input y, cells' reads; outputs $q(Z)$ and $q(C)$, posterior distributions resulting in clusters within y with their corresponding copy number profiles illustrated by heatmaps (this is an example of binary clustering).

of possible copy numbers in which the probability of each copy number value, defined as a state, depends only on the previous state in the sequence). We provide two observational model for CopyMix. First one is based on Poisson emission and the second one is based on Gaussian emission. In the following sections we describe these two choices of emission modelling.

2.1 Poisson emission

We assume that the data (read counts) follow a Poisson distribution as in Witten (2011); we consider read counts per fixed equal size genomic bin as in Leung et al. (2017). Moreover, we assume the read counts are emitted from a latent sequence of copy number states. Fig. 1 illustrates an overview of CopyMix; it analyzes the input cell sequences, containing read counts per predefined genomic bin, and produces clusters of cells with their corresponding copy number sequences (profiles) explaining the clusters.
Fig. 2 illustrates CopyMix graphical model where Y denotes the observable variables, read counts per predefined bins, C the latent copy number states forming a Markov chain, and Z the latent cell-specific cluster assignment variables. We notice in Fig. 2 that each Y has two levels of dependencies, which are reflections of the assumption that the Poisson distribution over the read counts depends on a latent cell-specific cluster assignment, Z, and a corresponding latent copy number state, C. Intuitively, a higher copy number should correspond to a higher read count. We incorporate this assumption in our model by defining the rate of Poisson distribution as the product of copy number state and cell-specific rate, θ_n, see Eq. (1); this also implies that θ_n corresponds to the average sequencing coverage for a haploid genome. Due to this multiplicative structure for the Poisson rate, a copy number state of zero results in

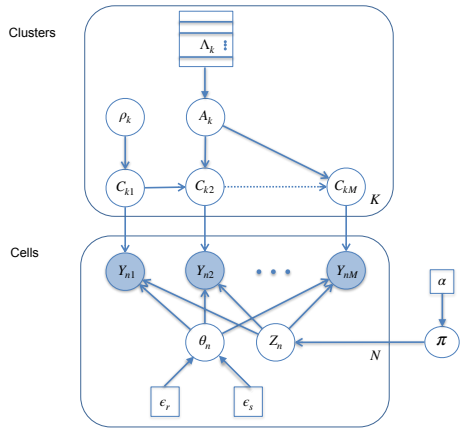

Figure 2: Probabilistic graphical model representing our proposed model. Shaded nodes represent observed values, the unshaded ones are the latent variables and the squares are the hyperparameters of the model; a posterior distribution over the values of the unshaded nodes is approximated using Variational Inference. Y_{nm}, observed read counts from cell n and bin m; C_{km}, corresponding latent copy number state forming a Markov chain; θ_n, cell-specific rate; Z_n, latent cell-specific cluster assignment variable. π and A_k, the priors over Z_n and C_{km} respectively; ρ_k, prior over the starting copy number state.

the lowest rate implying copy number deletion events; however, since the rate of Poisson cannot be zero, our implementation handles this by assigning a tiny number instead of zero. Our model is described in more detail in what follows, using conjugacy regarding the priors.

The genome considered is partitioned into M equally sized segments of consecutive positions called bins. Let $\mathbf{Y}_n = (Y_{n1}, \ldots, Y_{nM})$, where Y_{nm} is the random but observable number of reads aligned to bin m for cell n taking values in $\{0, 1, 2, \ldots\}$ for $m \in [M] = \{1, \ldots, M\}$ and $n \in [N] = \{1, \ldots, N\}$. We assume that $\mathbf{Y}_1, \ldots, \mathbf{Y}_N$ are independent and given a vector of true copy number states, $Y_{n1}, \ldots, Y_{nM}$ are also independent with the distribution of Y_{nm} depending on the true copy number state at bin m. The cluster membership of cell n is indicated by the hidden variable Z_n that takes values in $[K] = \{1, \ldots, K\}$. We assume there are $K << N$ vectors of true hidden copy number states, i.e., one for each cluster. The variables $Z_1, \ldots, Z_N$ are independent following a categorical distribution with $P(Z_n = k) = \pi_k$ and $\sum_{k=1}^{K} \pi_k = 1$. If $Z_n = k$ then the distribution of $\mathbf{Y}_n$ depends on the k-th vector of true hidden copy number states, defined as $\mathbf{C}_k = (C_{k1}, \ldots, C_{kM})$, with each C_{km} taking values in $[J] = \{1, \ldots, J\}$. We assume that $C_{k1}, \ldots, C_{kM}$ follow a discrete-time homogeneous Markov chain with initial probabilities $\rho_{kj} = P(C_{k1} = j)$, $j \in [J]$ and transition probabilities $a_{ij}^k = P(C_{km} = j | C_{km-1} = i)$, $i, j \in [J]$. Consequently, given the cluster assignment and the corresponding true hidden vector of copy number states, $Y_{n1}, \ldots, Y_{nM}$ are independent with Y_{nm} following a distribution with parameters depending on the hidden true state at bin m for cluster k, that is, $Y_{nm} | Z_n = k, C_{km} = j \sim F_{j\theta_n}$. As mentioned earlier, we assume $F_{j\theta_n}$ to be a Poisson distribution with rate $j \times \theta_n$, that is,

$$F_{j\theta_n}(y) \equiv \frac{(j \times \theta_n)^y e^{-(j \times \theta_n)}}{y!} \text{ with } j\theta_n > 0. \tag{1}$$

We, w.l.o.g., consider the initial probabilities, ρ_{kj}'s, to be fixed and known. We let $\mathbf{\Psi}$ be the set containing

all the unknown model parameters, i.e., $\mathbf{\Psi} = \{\mathbf{A}, \boldsymbol{\theta}, \boldsymbol{\pi}\}$, where $\mathbf{A} = \{\mathbf{A}_k : k \in [K]\}$ with $\mathbf{A}_k = \{a^k_{ij} : i, j \in [J]\}$; $\boldsymbol{\theta} = \{\theta_n : n \in [N]\}$; and $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_K)$. In order to infer $\mathbf{\Psi}$, the hidden states $\mathbf{C} = \{\mathbf{C}_1, \ldots, \mathbf{C}_K\}$, and $\mathbf{Z} = (Z_1, \ldots, Z_n)$ we apply VI; that is, we derive an algorithm that, for given data, approximates the posterior distribution of the parameters by finding the Variational Distribution (VD), $q(\mathbf{Z}, \mathbf{C}, \mathbf{\Psi})$, with smallest Kullback-Leibler divergence to the posterior distribution $P(\mathbf{Z}, \mathbf{C}, \mathbf{\Psi}|\mathbf{Y})$, which is equivalent to maximizing the evidence lower bound (ELBO) Blei et al. (2017) given by

$$\text{ELBO}(q) = \text{E}[\log P(\mathbf{Y}, \mathbf{Z}, C, \mathbf{\Psi})] - \text{E}[\log q(\mathbf{Z}, \mathbf{C}, \mathbf{\Psi})]. \tag{2}$$

We consider the following prior distributions for the parameters in $\mathbf{\Psi}$.

- $\mathbf{a}^k_i = (a^k_{i1}, \ldots, a^k_{iJ}) \sim \text{Dirichlet}(\mathbf{\Lambda}^{k0}_i)$.
- $\theta_n \sim \text{Gamma}(\epsilon^0_{sn}, \epsilon^0_{rn})$, with ϵ^0_{sn}, shape, and ϵ^0_{rn}, rate, hyperparameters.
- $\boldsymbol{\pi} \sim \text{Dirichlet}(\alpha^0_1, \ldots, \alpha^0_K)$

The main steps of our VI approach, for inferring $\mathbf{Z}, \mathbf{C}$, and $\mathbf{\Psi}$, are described below.

Step 1. VD factorization We assume the following factorization of the VD:

$$q(\mathbf{Z}, \mathbf{C}, \mathbf{\Psi}) = q(\boldsymbol{\pi}) \prod_{n=1}^{N} q(Z_n) \prod_{k=1}^{K} q(\mathbf{C}_k) \prod_{k=1}^{K} \prod_{i=1}^{J} q(\mathbf{a}^k_i) \prod_{n=1}^{N} q(\theta_n) \ . \tag{3}$$

Step 2. Joint distribution

The logarithm of the joint distribution satisfies: $\log P(\mathbf{Y}, \mathbf{Z}, \mathbf{C}, \mathbf{\Psi}) = \log P(\mathbf{Y}|\mathbf{Z}, \mathbf{C}, \mathbf{\Psi}) + \log P(\mathbf{C}|\mathbf{\Psi}) + \log P(\mathbf{Z}|\mathbf{\Psi}) + \log P(\mathbf{\Psi})$. For details of the calculations see Appendix A.1.

Step 3. VD computation

We now derive a coordinate ascent algorithm for the VD. That is, we derive an update equation for each term in the factorization, Eq. (3), by calculating the expectation of $\log P(\mathbf{Y}, \mathbf{Z}, \mathbf{C}, \mathbf{\Psi})$ over the VD of all random variables except the one currently being updated Bishop (2006). For example, we obtain the update equation for $\boldsymbol{\pi}$, $q(\boldsymbol{\pi})$, by calculating $E_{-\boldsymbol{\pi}}(\log P(\mathbf{Y}, \mathbf{Z}, \mathbf{C}, \mathbf{\Psi}))$, where $-\boldsymbol{\pi}$ indicates that the expectation is taken with respect to the VD of all other random variables than $\boldsymbol{\pi}$, i.e., $\mathbf{Z}, \mathbf{C}$ and $\{\mathbf{A}, \boldsymbol{\theta}\}$ except $\boldsymbol{\pi}$. See A.2 and A.3 for the update equation of each term in Eq. (3).

Step 4. Summary of updates

We update the parameters of each posterior distribution presented in Step 3; for the summary of the update equations, see Alg. 2, in Appendix A.2. A shorter version of Alg. 2, providing an overview of CopyMix, is shown in Alg. 1. As shown in lines 8 and 9 of Alg. 1, we aim to find the posterior distributions of cluster assignments and copy number states. Note that the update equations in line 5 of Alg. 1 can be run in a parallel fashion, keeping the algorithm's computational complexity low. The implementation of CopyMix is available at **CopyMix**.

Algorithm 1 CopyMix: a Variational Inference Algorithm

procedure ESTIMATE-POSTERIORS(Y): ▷ Y input is N sequences of length M
Assign priors hyperparameters
Initialise variational distributions parameters
  repeat
    Update variational parameters (using **Step 4. Summary of updates**)
  until convergence of the ELBO in Eq. 2
Output all posterior distributions including:
$q(Z_n = k) \quad \forall n, k;$ ▷ cluster assignment probabilities
$q(C_{km} = j) \quad \forall k, m, j;$ ▷ copy number probabilities
return posterior distribution

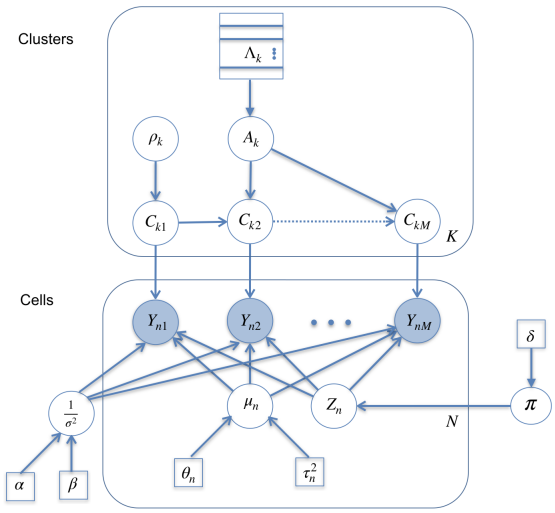

Figure 3: In this version Y_{nm} are observed corrected read ratios from cell n and bin m; C_{km}, corresponding latent copy number state forming a Markov chain; μ_n, cell-specific rate; Z_n, latent cell-specific cluster assignment variable. π and A_k, the priors over Z_n and C_{km} respectively; ρ_k, prior over the starting copy number state.

2.2 Gaussian emission

In this version, we assume that the read counts are GC-corrected (a bias correction performed in advance), which means that the data values are real numbers. Thus, it is meaningful to assume that the data follow a Gaussian distribution; again, we assume fixed equal size genomic bin as in the Poisson version. The rest of the model is similar to the previous section. The new probabilistic graphical model is illustrated in Fig. 3, where we model the average sequencing coverage as Gaussian with independent conjugate priors; this means that the mean of the Gaussian follows itself a Gaussian distribution and the precision of the Gaussian follows a Gamma distribution.

Let Y_{nm} be the observed gc-corrected reads ratio to bin m for cell n for $n = 1, \ldots, N$ and $m = 1, \ldots, M$. Let $\mathbf{Y}_n = (Y_{n1}, \ldots, Y_{nM})$ be vector with all observed data for cell n. Given the cluster assignment and the corresponding true hidden vector of copy number states, $Y_{n1}, \ldots, Y_{nM}$ are independent with Y_{nm} following

a distribution with parameters depending on the hidden true state at bin m for cluster population k, that is,

$$Y_{nm}|Z_n = k, C_{km} = j \sim F_{j\mu_n,\sigma^2}.$$

We assume $F_{j\mu_n,\sigma^2}$ to be a Gaussian distribution with $j\mu_n = j \times \mu_n$ (we add noise .001 to j=0 for computational purposes), i.e.,

$$F_{j\mu_n,\sigma^2}(r) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(r-j\times\mu_n)^2}{2\sigma^2}} \tag{4}$$

Let $\mathbf{\Psi}$ be the set containing all the unknown model parameters, i.e., $\mathbf{\Psi} = \{\mathbf{A}, \boldsymbol{\mu}, \sigma^2, \boldsymbol{\pi}\}$, where

- $\mathbf{A} = \{\mathbf{A}_k \text{ for } k = 1, \ldots, K\}$ with $\mathbf{A}_k = \{a_{ij}^k \text{ for } i = 1, \ldots, J \text{ and } j = 1, \ldots, J\}$;
- $\boldsymbol{\mu} = \{\mu_n \text{ for } n = 1, \ldots, N\}$;
- σ^2;
- $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_K)$.

We consider the following prior distributions for the parameters in $\mathbf{\Psi}$.

- $\mathbf{a}_i^k = (a_{i1}^k, \ldots, a_{iJ}^k) \sim \text{Dirichlet}(\mathbf{\Lambda})$
- $\mu_n \sim \text{Normal}(\theta_n, \tau_n^2)$. The conjugate prior concerning the mean of Normal distribution is Normal distribution.
- $\frac{1}{\sigma^2} \sim \text{Gamma}(\alpha, \beta)$. The conjugate prior concerning the precision of Normal distribution is Gamma distribution.
- $\boldsymbol{\pi} \sim \text{Dirichlet}(\boldsymbol{\delta})$

The main steps of our VI approach, for inferring $\mathbf{Z}, \mathbf{C}$, and $\mathbf{\Psi}$, are described below.

Step 1. VD factorization We assume the following factorization of the VD:

$$q(\mathbf{Z}, \mathbf{C}, \mathbf{\Psi}) = \Big[\prod_{n=1}^{N} q(Z_n)\Big]\Big[\prod_{k=1}^{K} q(\mathbf{C}_k)\Big]\Big[\prod_{k=1}^{K}\prod_{i=1}^{J} q(\mathbf{a}_i^k)\Big]\Big[\prod_{n=1}^{N} q(\mu_n)\Big] q(\frac{1}{\sigma^2}) q(\boldsymbol{\pi}). \tag{5}$$

Step 2. Joint distribution

The logarithm of the joint distribution satisfies: $\log P(\mathbf{Y}, \mathbf{Z}, \mathbf{C}, \mathbf{\Psi}) = \log P(\mathbf{Y}|\mathbf{Z}, \mathbf{C}, \mathbf{\Psi}) + \log P(\mathbf{C}|\mathbf{\Psi}) + \log P(\mathbf{Z}|\mathbf{\Psi}) + \log P(\mathbf{\Psi})$. For details of the calculations see Appendix A.4.

Step 3. VD computation

We now derive a coordinate ascent algorithm for the VD; as we described this for the Poisson version, we derive an update equation for each term in the factorization, Eq. (5), by calculating the expectation of $\log P(\mathbf{Y}, \mathbf{Z}, \mathbf{C}, \mathbf{\Psi})$ over the VD of all random variables except the one currently being updated Bishop (2006). See A.5 and A.6 for the update equation of each term in Eq. (5).

Step 4. Summary of updates

We update the parameters of each posterior distribution presented in Step 3; for the details of the update equations, see Appendix A.5. We summarize the updates as shown in Alg. 1.

3 Empirical investigation

3.1 Experimental setup

3.1.1 Dataset

We perform experiments on both simulated data and two published biological datasets: (1) Count data from Leung et al. (2017) (available from the NCBI Sequence Read Archive (SRA) under accession number SRP074289); (2) Real value data from Laks and McPherson (2019) accessed via DLP .

The first biological data stem from single-cell whole-genome sequencing from 18 primary colon cancer tumor cells and 18 metastatic cells from matched liver samples for one patient referred to as CRC2 in Leung et al. (2017). In this work, we mainly analyze CRC2 patient data (more complex than CRC1 according to Leung et al. (2017)) considering chromosomes 18, 19, 20, and 21 comprising a total of 904 genomic bins of an approximate size of 200Kb each. Each data point corresponds to the number of reads aligned per bin per cell after GC correction Leung et al. (2017). These counts stemming from the tumor cell's genome reflect the hidden copy numbers—copy number integers are conventionally ranged from 0 to 6, as in Leung et al. (2017). The higher the counts are, the higher the copy numbers are expected to be. Regarding clustering in Leung et al. (2017), it is reasonable that the primary tumor cells from the colon and the metastatic cells from the liver should cluster separately, as inter-tumor copy number differences are expected to dominate over minor intra-tumor copy number variations; therefore, we use the terms primary and metastatic clusters.

The second biological data stem from single-cell whole-genome sequencing using DLP technology. The data is from three cell lines from ovarian cancer with total number of 891 cells Laks and McPherson (2019). It is meaningful to infer that each cell line is a separate cluster, however, the goal is to find sub-clusters as well.

For the simulated data, we examine both Poisson and Gaussian models. For the Poisson version, we randomly generate 30 datasets for each test case, following our graphical model and inspired by biological datasets from Leung et al. (2017); for data generation details, see README.txt in the **CopyMix**. We examine different test cases, orthogonal to each other, to show our approach's robustness; each scenario, for both clustering and copy number state inference, is reproducible. Regarding the Gaussian model, we similarly show the robustness of CopyMix by varying the number of clusters and clustering scenarios.

3.1.2 Experimental Protocol

For the Poisson version, we run VI for different numbers of clusters considering various random and informative initialization methods, e.g., k-means and Baum-Welch, as recommended by Blei et al. (2017). For each given number of clusters, we choose the best VI run based on the highest log-likelihood. Next, we select the number of clusters by an elbow selection method Johnson and Wichern (2007) based on the Deviance Information Criterion Spiegelhalter et al. (2002) over different numbers of clusters. We assess the clustering performance of CopyMix on 30 simulated datasets for each test case via V-measure

Rosenberg and Hirschberg (2007). The V-measure, a value between 0 and 1, quantifies to which extent a cluster only contains data points that are members of a single class (the higher V-measure, the better clustering). Regarding inference of copy number profiles, we evaluate the performance of CopyMix on simulated data by calculating the proportion of discordant (unequal) position values between true and inferred vectors, as in de Souza et al. (2020), of copy number states. For biological data evaluation, V-measure is used to assess the clustering performance where the true clusters are assumed to correspond to the primary and metastatic cells; note that these may not be ground truth, but we believe there is certain concordance with the unknown ground truth. Regarding state inference, we compare the copy number profiles between the results from CopyMix and the original paper Leung et al. (2017).

For the Gaussian version, we assume we know the number of clusters. We run for different numbers of clusters and choose the result with the highest ELBO. Note that this is a simplified version of the above. In addition to merely comparing to the clustering result in the original paper Laks and McPherson (2019), we examine if the original paper has inferred meaningful clusters; we obtain this by using metrics CH Index Calinski and Harabasz (1974) and Silhouette Rousseeuw (1987). To compare to the original paper Laks and McPherson (2019), we use likelihood ration test. Finally, we provide ELBO in the reports.

CH Index is the ratio of the sum of between-clusters dispersion and of inter-cluster dispersion for all clusters, the higher the score, the better the performances. Silhouette is calculated using two scores $a, \quad b$, as shown below. Note that a is the mean distance between a sample and all other points in the same class. This score measure the closeness of points in the same cluster. And, b is the mean distance between a sample and all other points in the next nearest cluster. This score measure the distance of points of different clusters.

$$s = \frac{b - a}{max(a, b)} \tag{6}$$

The s score is bounded between -1 for incorrect clustering and +1 for highly dense clustering. Scores around zero indicate overlapping clusters. The score is higher when clusters are dense and well separated, which relates to a standard concept of a cluster.

Finally, the likelihood ration test assesses the goodness of fit of two competing statistical models based on the ratio of their likelihoods. In our experiments, we examine how statically significant the winning model is; by the winning model we mean the one with the higher likelihood. The significance is measured by p-value; a high p-value indicates that the null hypothesis is preferred over the winning hypothesis.

3.2 Experimental Results for Poisson Version

We perform experiments on simulated data and biological data, and we evaluate a sequential version of CopyMix to verify the superiority of single-phase inference over the sequential framework.

Simulated data Here, we evaluate CopyMix, and compare it to the initialization methods used by VI on 30 simulated datasets for each test scenario. The comparison is to illustrate, firstly, that VI results are not already achieved by the initialization methods, and secondly, that VI can recover the poor initialization performance and achieve optimal performance. We select the number of cells, sequence lengths, rate parameters, transition probabilities based on Leung et al. (2017). We assume equal amplification on the read counts across bins since the bin size is high enough (200kb as in Leung et al. (2017)), meaning that the effect of unequal amplification is negligible. Figures 4, 5 and 6 provide summaries of V-measures and proportions of discordant positions; note that the median values in the boxplots, highlighted in red for

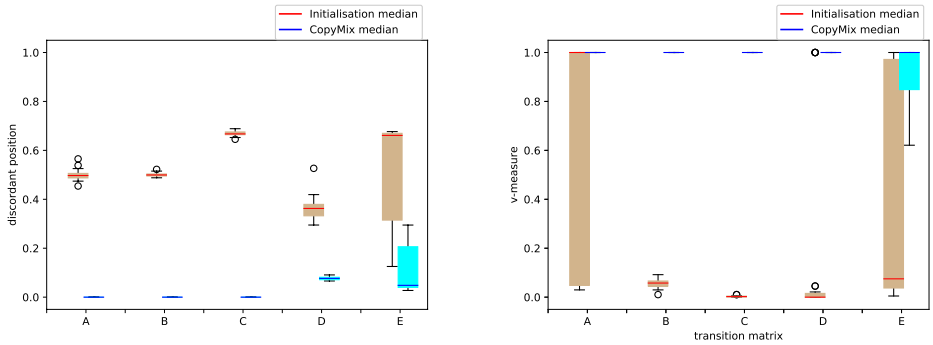

Figure 4: Performance of initialization methods (brown bars with red median values) and CopyMix (cyan bars with blue median values) across different sets of transition matrices. LHS: discordant positions between true and inferred vectors of copy number states. RHS: V-measure between true and inferred cell clustering assignments.

initialisation method and in blue for CopyMix, aim to facilitate the comparisons. The figures illustrate the performance comparison of CopyMix and the chosen initialization method, by CopyMix, for different settings: copy number transition matrices and number of clusters (Fig. 3); sequence length (Fig.4); and number of cells (Fig. 5). For all the tests, CopyMix results in V-measure of 1 except for one case (Fig. 4) where V-measure is in an interval of [.85, 1] with the median being 1; proportions of discordant positions are mostly 0 and otherwise are in the interval of [0, .2]. These results are optimal as the copy number profiling errors are near 0 and clustering near 1 (or 100%).

The results in Fig. 4 are obtained considering 45 cells—each with a sequence length of 800 bins and the mean value around 100—clustered into two and three clusters, wherein each clustering scenario, each cluster bears a different transition pattern than other clusters. The copy number transition patterns (forming the transition matrix of a Markov chain) vary across configurations A to E, which are orthogonal to each other; Configurations A to E are formed by combining the following state patterns.

- **single state**, a copy number state is inclined to stay at one certain state;
- **self-transition state**, a copy number state is inclined to stay at its current state for awhile;
- **oscillating state**, a copy number state changes shortly after being at a state;
- **spike state**, a sudden change of copy number state arises for a specific short range of positions in the copy number sequence;
- **scaled state**, a copy number changes by a multiplicative or additive constant.

The configurations A to E are: **A**, similar to chromosomes 11 in CRC1 and 4 in CRC2, (one cluster follows (**scaled state**); **B**, similar to chromosome 11 in CRC1 (one cluster follows **self-transition state** and the other one **single state**); **C**, similar to chromosome 3 in CRC1 (two clusters have identical transition matrices but one arrive at **spike state**); **D**, similar to chromosomes 18 to 21 in CRC2 (clusters combine **self-transition state**, **oscillating state**, and **scaled state**); **E**, two of the clusters have the transitions mentioned in case **C**, and, the third cluster has a combination of **oscillating state** and **single state**.

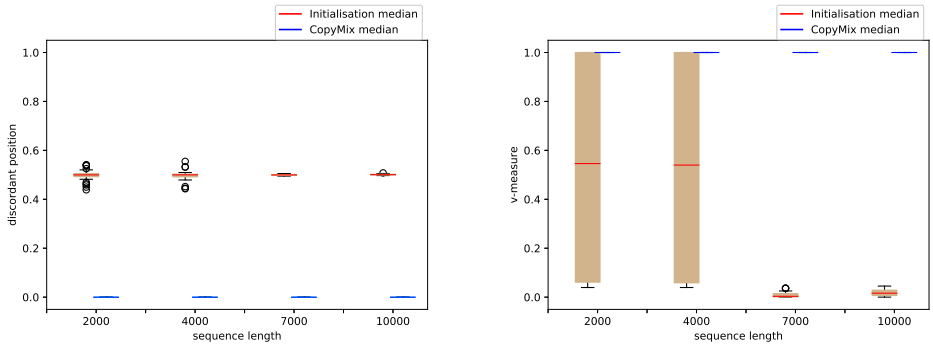

Figure 5: Performance of initialization methods and CopyMix, varying the sequence length. LHS and RHS figures, as well as color codes, are similar to Fig. 4.

We can observe that, in Fig. 4, CopyMix has a perfect performance in terms of clustering and copy number inference for easier cases and also a high performance concerning the harder cases (**D** and **E**). Next, however not shown here, we investigate decreasing the rate parameter, θ_n, for configuration **A** to see how our model handles shallow data as this can be a challenge in copy number profiling Zahn et al. (2017). Our model is robust towards lower rates up to $\theta_n = 3$, without reducing the quality of performance concerning V-measure and discordant positions. As the third set of tests, we rerun the test using configuration **A**, but we increase the size of the sequences up to 10000 (inspired by the total length of the genomic bins from biological data in Leung et al. (2017)), see Fig. 5. We can observe that our model handles these sequences without any decrease in performance. Finally, Fig. 6 illustrates the perfect performance of CopyMix when varying the number of cells from 15 to 500 when, again, using configuration **A**.

Biological data We evaluate CopyMix on the biological data from Leung et al. (2017). As aforementioned, we do not compare CopyMix to the earlier state-of-the-art methods due to their inclusion of ad-hoc processes in their framework; however, to evaluate the CopyMix results on the biological data, we use biologically verifiable results from the method in Leung et al. (2017). Using chromosomes 18 to 21, CopyMix clusters the cells into primary and metastatic groups, perfectly matching the corresponding primary and metastatic organs—V-measure is 1 w.r.t. the cells found in the two organs Leung et al. (2017). Adding chromosomes 3 and 8 to our test, metastatic tumor cells are clustered into two major subclusters as in Leung et al. (2017). As mentioned in Leung et al. (2017), amplifications on chromosomes 3 and 8, where some known genes are present, contribute to the detection of the metastatic subclusters. Note that, Leung et al. (2017) does not present a method to determine the threshold in hierarchical clustering, which results in the reported subclusters; moreover, there is no ground truth to verify these results.

Regarding the copy number profiling, Fig. 7 illustrates the copy number profiles inferred by CopyMix and the original work by Leung et al. (2017), here referred to as Navin. The heatmaps indicate a higher copy number state by a darker color. Note that in Navin, they perform ploidy scaling, which contributes to scaling differences compared to the results of CopyMix. Moreover, the Navin heatmap, here, illustrates an average of the heatmap in the original work across the cells since there is no single copy number profile per cluster. That is, Navin, unlike CopyMix, lacks a single copy number profile per cluster, revealing the copy number variation pattern behind each cluster. CopyMix detects the breakpoints (copy number

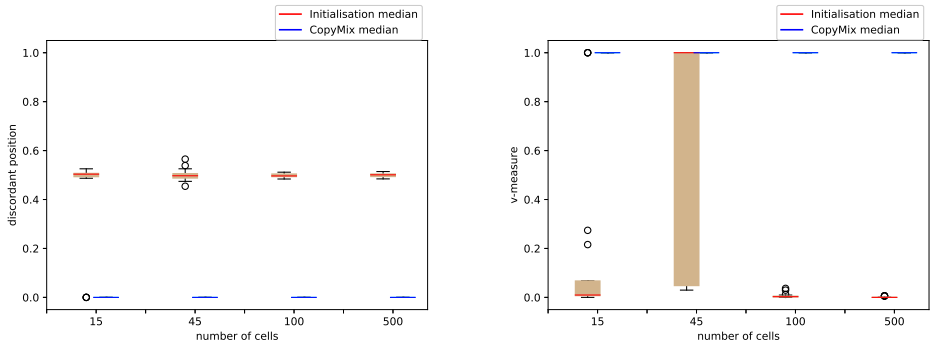

Figure 6: Performance of initialization methods and CopyMix varying the number of cells. LHS and RHS figures, as well as color codes, are similar to Fig. 4.

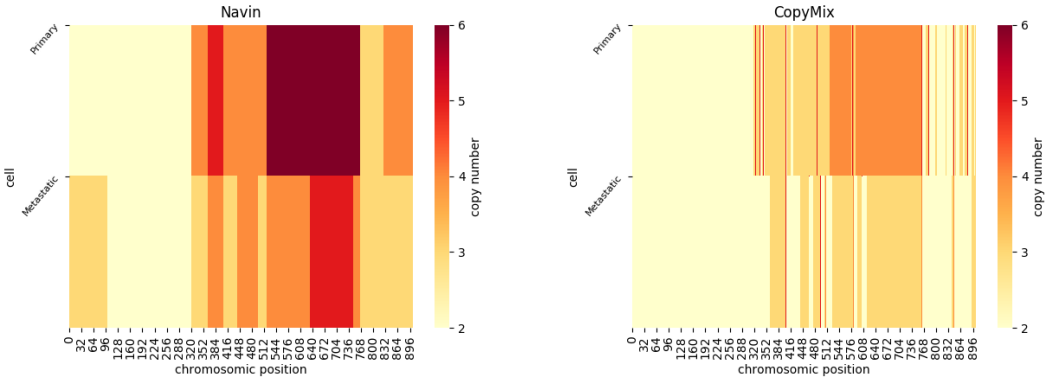

Figure 7: Copy number profiling comparison between the results from Navin Leung et al. (2017) (*LHS*) and CopyMix (*RHS*) for both primary and metastatic clusters.

changes) similar to Navin, with an accuracy of 76%, using the breakpoint margin (of a size corresponding to 5% of the sequence length). Again, there is no ground truth to verify the Navin results from Leung et al. (2017).

A sequential CopyMix Applying a sequential version of CopyMix (1. performing copy number profiling, using EM for HMMs; 2. performing clustering over the inferred copy numbers, using EM for a mixture model), resembling Shah et al. (2006); Vitak et al. (2017) but excluding their ad-hoc processes, on the biological data, results in poor clustering results (V-measure of .24) while, as mentioned, CopyMix gains V-measure of 1. This result confirms that simultaneous detection of copy number profiles and clustering is superior to the sequential approach. In particular, we conclude that methods that use EM for HMMs, using HMMcopy Shah et al. (2006) framework, benefit from the multiple ad-hoc transformations within the HMMcopy framework, much more than the pure computational model, e.g., HMM. Adding multiple ad-hoc rules within such computational models lacks elegance, integrity, and potential model-based approach benefits.

3.3 Experimental Results for Gaussian Version

We perform experiments on simulated data and biological data. For the biological data we evaluate CopyMix against the original paper, referring to as DLP, using CH Index, Silhouette, and likelihood ratio test.

Simulated data The experiments are similar to those shown for the Poisson version. Here, we, moreover, compare the ELBO. For the sake of investigating the model behavior w.r.t. various quality of initializations, we report the results for three different initializations. Table 1 illustrates the performance of CopyMix and the chosen initialization method, by CopyMix, for 19 different settings. The plots in Fig. 8 to 9 shows these configurations of clustering scenarios. In Table 1, the highlighted ELBO values (the best ones per configuration) guide us which λ for each configuration is desirable. The three λ are basically in the hyperparameter value of the Dirichlet for the edge weight (λ_1: 0.1, λ_2: 0.5, λ_3: 10). The overall V-measure concerning the Init method based on the three different hyperparameters is also reported.

We can observe that CopyMix, in general, performs well w.r.t. clustering. There are transition scenarios which are challenging for CopyMix, but they are still clustered with good V-measures. Finally, there are scenarios in which CopyMix cannot handle as good. For example, CONF 5 is hard for CopyMix, where the requirement of the successful clustering is to distinguish two shifted monotone sequence patterns. Another challenge for CopyMix, like CONF 17, is when the data has more variance in the transition patterns across the sequences and the clusters are overlapping.

Biological data We evaluate CopyMix on the biological data from Laks and McPherson (2019). We execute CopyMix on the whole genome with length of 6206 bins. We run 50 initializations for 50 iterations, and pick the ones corresponding to the 10 highest ELBO; subsequently, we let the 10 ones run until convergence. We pick the winning initialization method which has the highest final ELBO. We detect 4 clusters in total; the three original cell lines are fully recovered and one of the cell lines is clustered into two sub-clones. The original DLP co-clustering results (between the cell lines and the 9 clones detected by DLP), and the CopyMix co-clustering results (between 4 CopyMix clones and 9 DLP clones) are shown in Fig. 10. Note that the CopyMix clones 0 and 1 corresponds to DLP clones E and F, respectively. That is, 145 cells from cell line OV2295(R2) belong to DLP clone E and 75 cells from the same cell line belong to DLP clone F. The rest of the cell line are also matched correctly, as it can be seen by comparing the plots. The resulting ELBO for CopyMix is -4957712 and the V-measure when comparing

Table 1: Clustering result of CopyMix vs initialization method, for Conf 1 to Conf 17.

	conf 1			conf 2			conf 3		
Model	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
CopyMix	**100**%	**100**%	0%	**100**%	**100**%	**100**%	**100**%	**100**%	**100**%
Init method	2%	57%	**87**%	2%	57%	87%	2%	57%	87%
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
ELBO	**−58937**	**−58937**	−59372	−62199.04	**−62196**	−62199	**−63737**	**−63737**	**−63737**

	conf 4			conf 5			conf 6		
Model	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
CopyMix	**100**%	**100**%	**100**%	0%	0%	0%	**100**%	**100**%	**100**%
Init method	2%	57%	87%	**2**%	**57**%	**87**%	2%	57%	87%
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
ELBO	−82485	−82560	**−82421**	−82200	**−82198**	−82199	−90638	**−90638**	−90638

	conf 7			conf 8			conf 9		
Model	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
CopyMix	**67**%	**67**%	**67**%	**72**%	0%	0%	0	**72**%	**72**%
Init method	1%	2%	0%	1%	1%	1%	0%	1%	0%
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
ELBO	−384978	**−378882**	−382544	**−119482**	−193878	−193353	−189912	**−132598**	−133213

	conf 10			conf 11			conf 12		
Model	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
CopyMix	**100**%	0%	**100**%	**86**%	**86**%	**86**%	56%	**86**%	59%
Init method	1%	2%	1%	1%	1%	0%	1%	1%	**0**%
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
ELBO	**−50418**	−65537	−50472	**−71023**	−75495	−78568	−79648	**−70837**	−83825

	conf 13			conf 14			conf 15		
Model	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
CopyMix	**67**%	**67**%%	50%	85%	**100**%	84%	**77**%	0%	72%
Init method	2%	4%	2%	1%	2%	1%	1%	1%	1%
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
ELBO	**−152026**	−155452	−153478	−123102	**−106973**	−111778	−139992	−165247	**−137978**

	conf 16			conf 17		
Model	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
λ_1	λ_2	λ_3				
CopyMix	78%	45%	**81**%	**48**%	**48**%	**48**%
Init method	2%	2%	2%	2%	1%	1%
	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
ELBO	**−191852**	−204358	−202328	**−2250183**	−2261522	−2252863

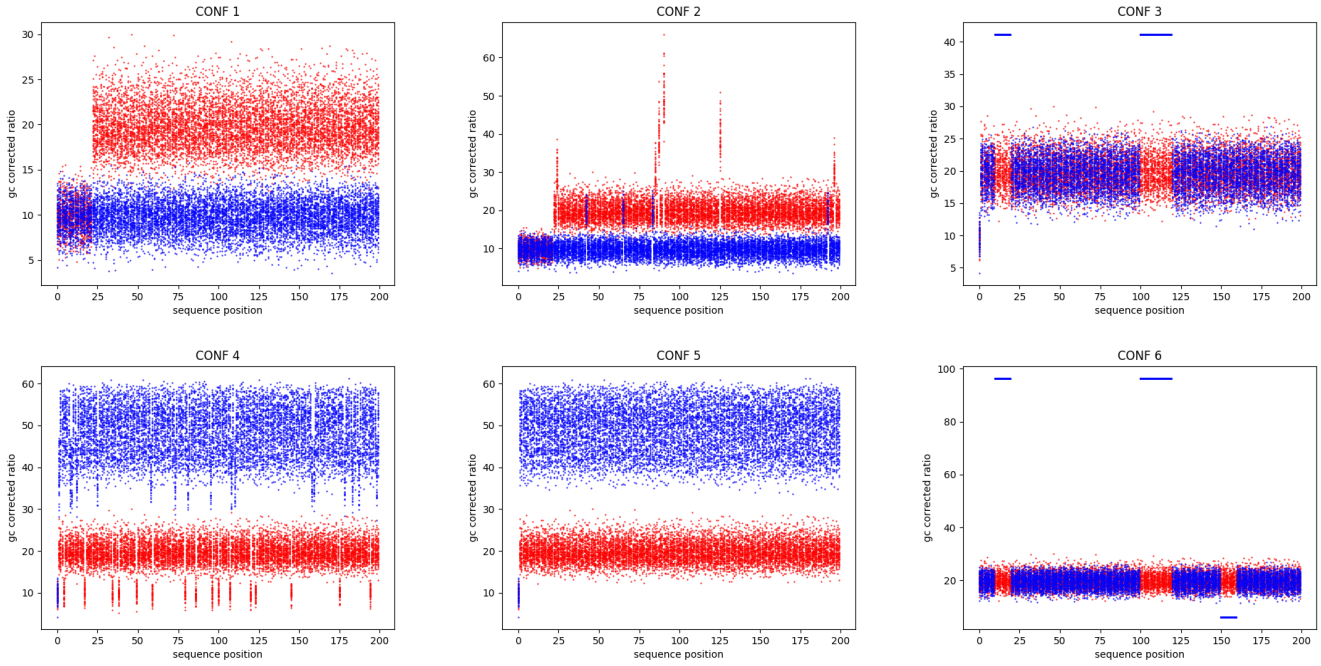

Figure 8: Binary clustering scenarios, using different CNV (transition) patterns, which form CONF 1 to CONF 6.

CopyMix to DLP is 67%. The initialization method provides only V-measure of 1%. Fig 11 shows how some clones are too overlapping, which explains why CopyMix did not account for them as separate clusters. We now show the results of three metrics which confirm this observation, statistically speaking. Fig. 12 shows that CopyMix outperforms DLP method w.r.t. both CH Index and Silhouette. However, regarding Silhouette, the score .49 is not the perfect score, i.e., 1. These metrics both show that DLP method has detected overlapping clusters which is not in consensus with the definition of clustering in machine learning literature; however, biologically speaking, DLP results can make sense. Finally, we show that performing a likelihood ration test confirms that CopyMix as a null hypothesis is favored over DLP. Assuming each clone follows an empirical Gaussian distribution (the mean and variance can be calculated by MLE), the DLP log likelihood is -8980284.42 and the CopyMix log likelihood is -9023905.94. The test results in p-value 1 which means that DLP likelihood is not significantly higher that CopyMix likelihood.

Regarding the copy number profiling, Fig. 13 illustrates the copy number profiles inferred by CopyMix.

4 Conclusion

The tasks of single-cell copy number profiling and clustering have so far been sequential, requiring multiple ad-hoc processing of data; hence prone to introduce clustering artifacts. We introduce a mixture model-based framework, CopyMix, to allow for a fully probabilistic and simultaneous single-cell copy number profiling and clustering using VI. Our approach is evaluated on both simulated and biological data resulting in good performance w.r.t. both of the tasks.

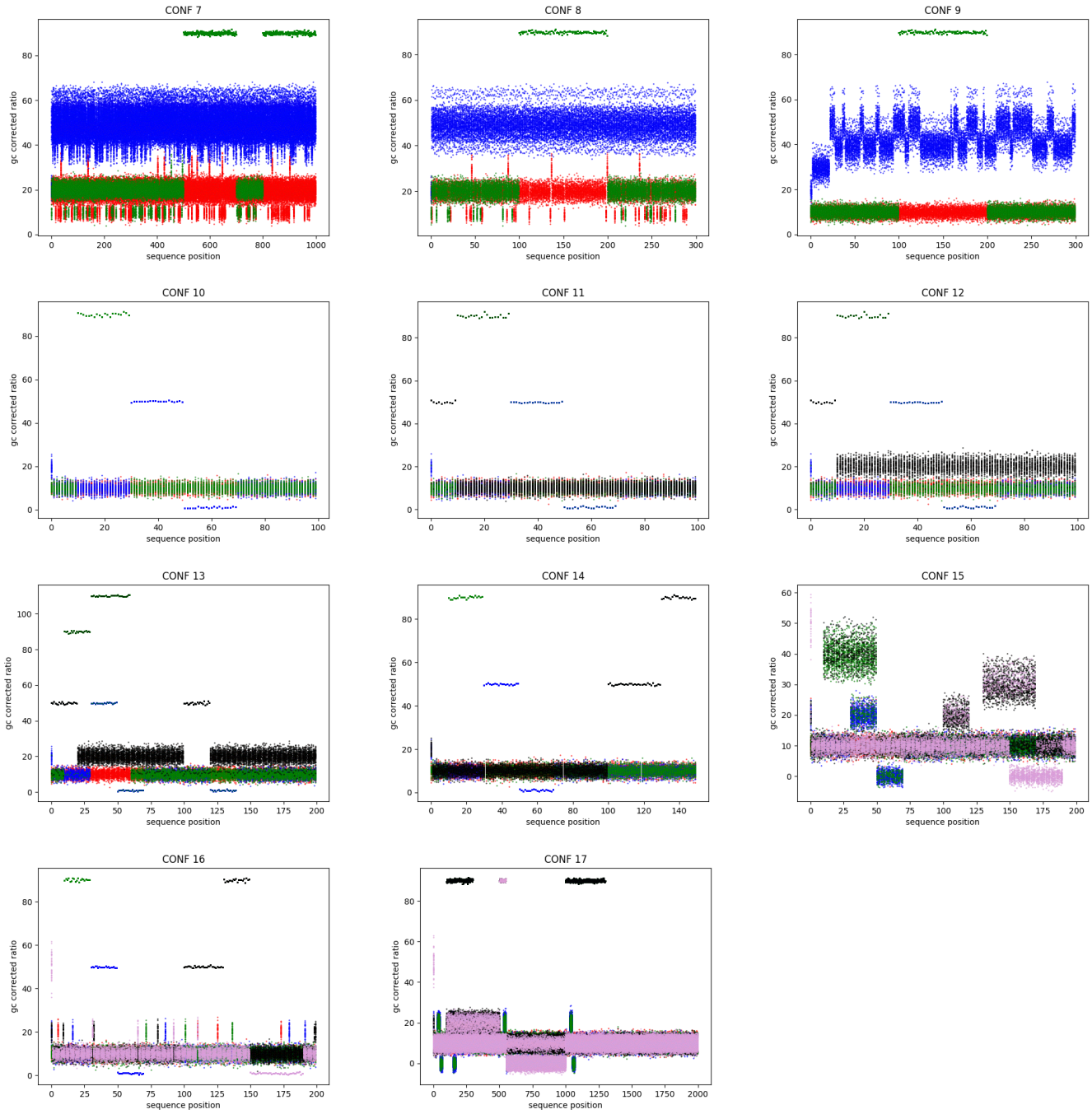

Figure 9: Clustering scenarios with 3 components (CONF 7 to 10), with 4 components (CONF 11 to 14), and with 5 components (CONF 15 to 17).

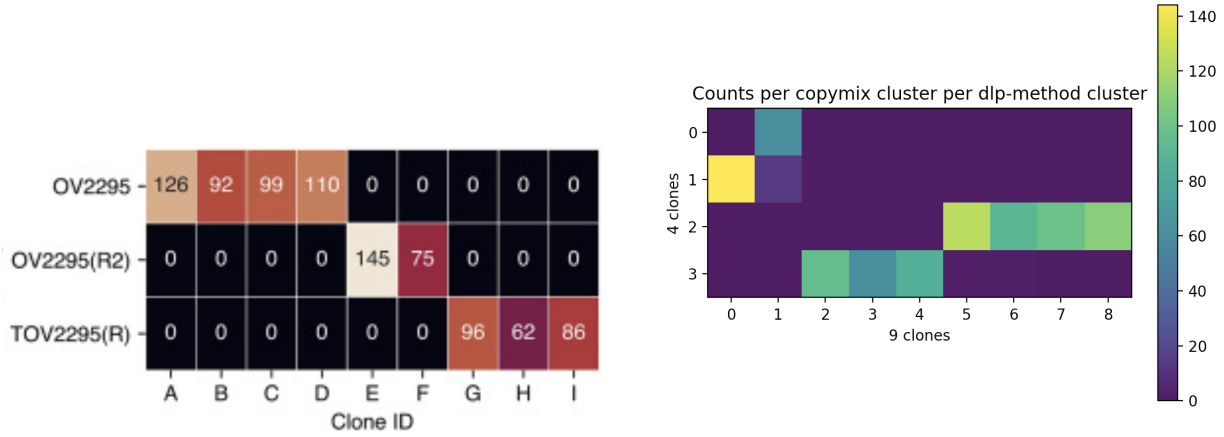

Figure 10: The original three cell lines and the co-clustering of CopyMix and DLP.

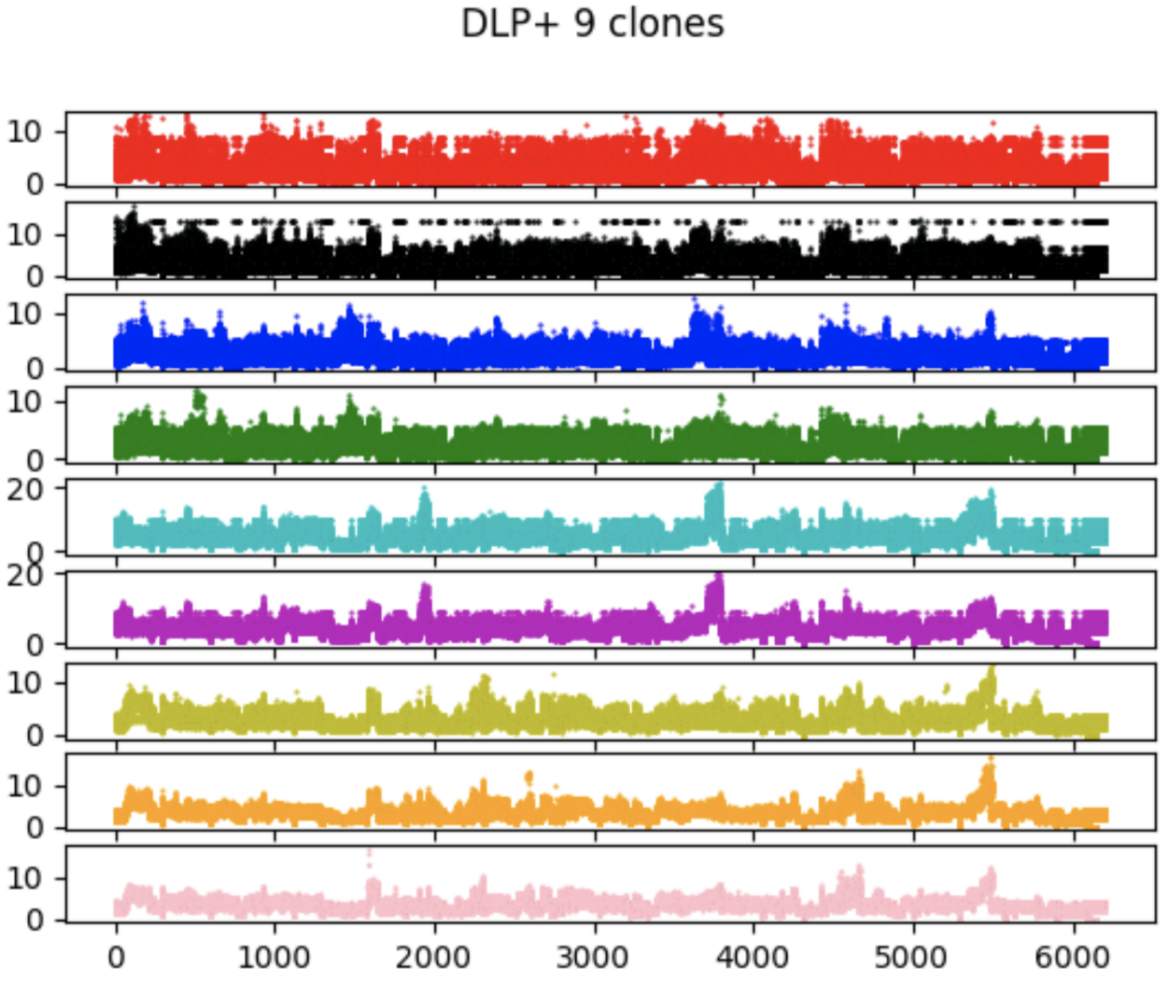

Figure 11: The GC-corrected data presented in 9 clones detected by DLP.

CH index (the higher the better)

Dlp+ paper with 9 clusters	462.6
The three cell lines (3 clusters)	623.1
CopyMix with 4 clusters	**1082.7**

Silhouette (1 is the best)

Dlp+ paper with 9 clusters	0.06	Overlapping clusters
Dlp+ paper with 7 clones (based on their phylogenetic tree)	0.21	
Dlp+ paper with 5 clones (based on their phylogenetic tree)	0.25	
CopyMix with 4 clusters	**.49**	Nicely separated clusters

Figure 12: The result of CH Index and Silhouette w.r.t CopyMix, DLP, and DLP with some major sub-clones.

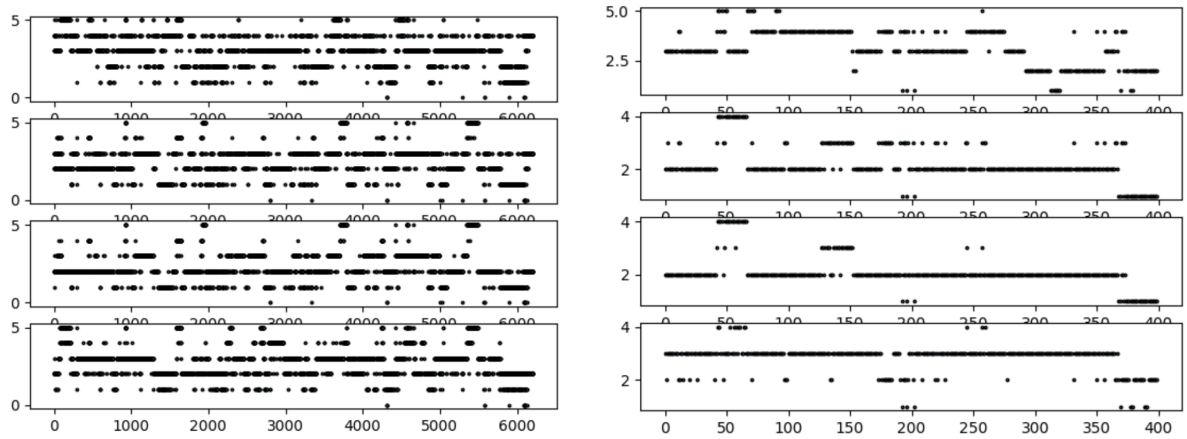

Figure 13: The copy number profiles detected by CopyMix corresponding to 4 clones. The bottom figure shows a higher resolution version by showing bins 400 to 800.

A future direction is to apply a biological indicator to detect clustering artifacts, where the contribution of non-sequential methods can be additionally illustrated. Overall, CopyMix provides various research opportunities concerning clustering and copy number profiling, such as improving clustering, computational efficiency, augmenting and refining the model, and modeling biallelic deletions and aneuploidy inspired by a recently proposed CopyKat Gao et al. (2021).

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A Appendix: Derivation details

A.1 Step 2 of Section 2.1

In this section, we provide the calculation of each decomposed term of $\log P(\mathbf{Y}, \mathbf{Z}, \mathbf{C}, \mathbf{\Psi})$. Each term of $\log P(\mathbf{Y}, \mathbf{Z}, \mathbf{C}, \mathbf{\Psi})$ presented in Step 2 of Section 2 is calculated as follows: $\log P(\mathbf{Y}|\mathbf{Z}, \mathbf{C}, \mathbf{\Psi}) =$
$\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{k=1}^{K}\sum_{j=1}^{J} \mathrm{I}(Z_n = k)\mathrm{I}(C_{km} = j) \log F_{j\theta_n}(y_{nm}) =$
$\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{k=1}^{K}\sum_{j=1}^{J} \mathrm{I}(Z_n = k)\mathrm{I}(C_{km} = j)\Big[y_{nm}\log(j\theta_n) - j\theta_n - \log(y_{nm}!))\Big]$;
$\log P(\mathbf{C}|\mathbf{\Psi}) = \sum_{k=1}^{K}\big[\sum_{j=1}^{J}\mathrm{I}(C_{k1} = j)\log\rho_{kj} + \sum_{m=2}^{M}\sum_{i=1}^{J}\sum_{j=1}^{J}\mathrm{I}(C_{km-1} = i, C_{km} = j)\log a_{ij}^{k}\big]$;

$$\log P(\mathbf{Z}|\mathbf{\Psi}) = \sum_{n=1}^{N}\sum_{k=1}^{K}\mathrm{I}(Z_n = k)\log\pi_k;$$

$$\log P(\mathbf{\Psi}) = \log P(\mathbf{A}) + \log P(\boldsymbol{\theta}) + \log P(\boldsymbol{\pi}); \tag{7}$$

where,

$$\log P(\boldsymbol{\theta}) \quad = \quad \sum_{n=1}^{N} \big[(\epsilon_{sn}^0 - 1)\log\theta_n - \epsilon_{rn}^0\theta_n\big] + C,$$

$$\log P(\mathbf{A}) \quad = \quad \sum_{k=1}^{K}\sum_{i=1}^{J}\big[\sum_{j=1}^{J}(\Lambda_{ij}^{k0} - 1)\log a_{ij}^k\big] + C,$$

and

$$\log P(\boldsymbol{\pi}) \quad = \quad \sum_{k=1}^{K}(\alpha_k^0 - 1)\log\pi_k + C.$$

A.2 Step 3 and 4 of Section 2.1

In this section we first present the details of the step 3 of section 2, i.e., the derivation of each posterior update equation (VD computation). We then summarise them in the form of a pseudo code, covering the details of step 4 in Section 2. For convenience, we use $_+\!\approx$ to denote equality up to a constant additive factor.

Update equation for $\boldsymbol{\pi}$: $\log q(\boldsymbol{\pi}) {}_+\!\approx \mathrm{E}_{-\boldsymbol{\pi}}\big(\log P(\mathbf{Y},\mathbf{Z},\mathbf{C},\boldsymbol{\Psi})\big)$
${}_+\!\approx \mathrm{E}_{-\boldsymbol{\pi}}\big(\log P(\mathbf{Z}|\boldsymbol{\Psi})\big) \; + \; \mathrm{E}_{-\boldsymbol{\pi}}\big(\log P(\boldsymbol{\pi})\big)$
$= \sum_{n=1}^{N}\sum_{k=1}^{K}\mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big)\log\pi_k + \log P(\boldsymbol{\pi})$
$= \sum_{k=1}^{K}\log\pi_k\Big[\sum_{n=1}^{N}\mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big)\Big] + \sum_{k=1}^{K}\log\pi_k(\alpha_k^0 - 1)$
$= \sum_{k=1}^{K}\log\pi_k\Big[\Big(\sum_{n=1}^{N}\mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) + \alpha_k^0\Big) - 1\Big]$. Let $\mathrm{I}(\cdot)$ be the indicator function. $q(\boldsymbol{\pi})$ is a Dirichlet distribution with parameters $\alpha = (\alpha_1, \ldots, \alpha_K)$, where

$$\alpha_k = \alpha_k^0 + \sum_{n=1}^{N}\mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big). \tag{8}$$

Update equation for Z_n*:*

$$\begin{aligned}\log q(Z_n) \quad {}_+\!\approx \quad & \mathrm{E}_{-Z_n}\big(\log P(\mathbf{Y},\mathbf{Z},\mathbf{C},\boldsymbol{\Psi})\big) \\ {}_+\!\approx \quad & \mathrm{E}_{-Z_n}\big(\log P(\mathbf{Y}|\mathbf{Z},\mathbf{C},\boldsymbol{\Psi})\big) + \mathrm{E}_{-Z_n}\big(\log P(\mathbf{Z}|\boldsymbol{\Psi})\end{aligned}$$

Let y_{nm} be the observed number of reads aligned to bin m for cell n corresponding to the random variable Y_{nm} and let $D_{n,m,j} \equiv y_{nm}\log\theta_n + y_{nm}\log j - j\theta_n - \log(y_{nm}!)$. Note that $\log P(\mathbf{Y}|\mathbf{Z},\mathbf{C},\boldsymbol{\Psi})$ and $\log P(\mathbf{Z}|\boldsymbol{\Psi})$ can be written as the sum of two terms, one that depends on Z_n and one that does not, i.e., $\log P(\mathbf{Y}|\mathbf{Z},\mathbf{C},\boldsymbol{\Psi}) = \sum_{m=1}^{M}\sum_{k=1}^{K}\sum_{j=1}^{J}\mathrm{I}(Z_n = k)\mathrm{I}(C_{km} = j)D_{n,m,j} \; +$
$\sum_{l\neq n}\sum_{m=1}^{M}\sum_{k=1}^{K}\sum_{j=1}^{J}\mathrm{I}(Z_l = k)\mathrm{I}(C_{km} = j)D_{n,m,l}$
and

$$\log P(\mathbf{Z}|\boldsymbol{\Psi}) = \sum_{k=1}^{K}\mathrm{I}(Z_n = k)\log\pi_k + \sum_{l\neq n}\sum_{k=1}^{K}\mathrm{I}(Z_l = k)\log\pi_k.$$

Consequently, $\log q(Z_n) {}_+\!\approx$
$\sum_{k=1}^{K}\mathrm{I}(Z_n = k)\big\{\mathrm{E}_{q(\boldsymbol{\pi})}(\log\pi_k) + \sum_{m=1}^{M}\sum_{j=1}^{J}\mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km} = j)\big) \times \mathrm{E}_{q(\theta_n)}(D_{n,m,j})\big\}$.

The distribution $q(Z_n)$ is a Categorical distribution with parameters $\boldsymbol{\pi}_n = (\pi_{n1}, \ldots, \pi_{nK})$, where

$$\pi_{nk} = \frac{\exp(\tilde{\pi}_{nk})}{\sum_{k'=1}^{K} \exp(\tilde{\pi}_{nk'})} \tag{9}$$

with

$$\tilde{\pi}_{nk} = \mathrm{E}_{q(\boldsymbol{\pi})}(\log \pi_k) + \sum_{m=1}^{M} \sum_{j=1}^{J} \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km} = j)\big) \mathrm{E}_{q(\theta_n)}\big(D_{n,m,j}\big).$$

Update equation for θ_n:

$$\begin{aligned} \log q(\theta_n) \quad &{}_{+}\!\approx \quad \mathrm{E}_{-\theta_n}\big(\log P(\mathbf{Y}, \mathbf{Z}, \mathbf{C}, \boldsymbol{\Psi}) \\ &{}_{+}\!\approx \quad \mathrm{E}_{-\theta_n}\big(\log P(\mathbf{Y}|\mathbf{Z}, \mathbf{C}, \boldsymbol{\Psi})\big) + \mathrm{E}_{-\theta_n}\big(\log P(\boldsymbol{\theta})\big) \end{aligned}$$

We can write $\log P(\mathbf{Y}|\mathbf{Z}, \mathbf{C}, \boldsymbol{\Psi})$ as in Eq. 22 and $\log P(\boldsymbol{\theta})$ which depends on θ_n as,

$$\log P(\boldsymbol{\theta}) = (\epsilon_{sn}^0 - 1) \log \theta_n - \epsilon_{rn}^0 \theta_n + \sum_{l \neq n} \big[(\epsilon_{sn}^0 - 1) \log \theta_l - \epsilon_{rn}^0 \theta_l\big] + C.$$

Therefore, $\log q(\theta_n) {}_{+}\!\approx$
$\log \theta_n \Big[\sum_{m=1}^{M} y_{nm} \sum_{k=1}^{K} \sum_{j=1}^{J} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km} = j)\big) \Big]$
$- \theta_n \Big[\sum_{m=1}^{M} \sum_{k=1}^{K} \sum_{j=1}^{J} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km} = j)\big) j \Big]$
$+ \log \theta_n (\epsilon_{sn}^0 - 1) - \theta_n \epsilon_{rn}^0$.
Note that $\sum_{k=1}^{K} \sum_{j=1}^{J} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km} = j)\big) = 1$.
The distribution $q(\theta_n)$ is a Gamma distribution with shape and rate parameters:

$$\epsilon_{sn} = \epsilon_{sn}^0 + \sum_{m=1}^{M} y_{nm} \tag{10}$$

$$\epsilon_{rn} = \epsilon_{rn}^0 + \Big\{ \sum_{m=1}^{M} \sum_{k=1}^{K} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \sum_{j=1}^{J} \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km} = j)\big) j \Big\}.$$

Update equation for $\mathbf{a}_i^k$:

$$\begin{aligned} \log q(\mathbf{a}_i^k) \quad &{}_{+}\!\approx \quad \mathrm{E}_{-\mathbf{a}_i^k}\big(\log P(\mathbf{Y}, \mathbf{Z}, \mathbf{C}, \boldsymbol{\Psi})\big) \\ &{}_{+}\!\approx \quad \mathrm{E}_{-\mathbf{a}_i^k}\big(\log P(\mathbf{C}|\boldsymbol{\Psi})\big) + \mathrm{E}_{-\mathbf{a}_i^k}\big(\log P(\mathbf{A})\big) \end{aligned}$$

Disregarding the terms that do not depend on $\mathbf{a}_i^k$ in $\log P(\mathbf{C}|\boldsymbol{\Psi})$ and $\log P(\mathbf{A})$, we obtain:
$\log q(\mathbf{a}_i^k) {}_{+}\!\approx$
$\sum_{j=1}^{J} \sum_{m=2}^{M} \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km-1} = i, C_{km} = j)\big) \log a_{ij}^k + \sum_{j=1}^{J} (\Lambda_{ij}^{k0} - 1) \log a_{ij}^k =$
$\sum_{j=1}^{J} \log a_{ij} \Big\{ \Big[\Lambda_{ij}^{k0} + \sum_{m=2}^{M} \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km-1} = i, C_{km} = j)\big) \Big] - 1 \Big\}$. The distribution $q(\mathbf{a}_i^k)$ is Dirichlet with parameters $\boldsymbol{\Lambda}_i^k = (\Lambda_{i1}^k, \ldots, \Lambda_{iJ}^k)$, where

$$\Lambda_{ij}^k = \Lambda_{ij}^{k0} + \sum_{m=2}^{M} \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km-1} = i, C_{km} = j)\big). \tag{11}$$

Update equation for $\mathbf{C}_k$: In the calculation of $q(\mathbf{C}_k)$, the resulting distribution resembles the posterior probability of

the hidden variables in an HMM and only differs in the normalization constant. Similarly to the approach adopted in MacKay (1997), we use a slightly different implementation of forward-backward algorithm. More specifically, we define a graph (one per cluster k) $G = \{V, E\}$ with vertices $V = \{C_{mj} : m \in [M], j \in [J]\}$, having weights denoted as $w_k(C_{mj})$, and edges $E = \{C_{m-1i}C_{mj} : m \in \{2, \ldots, M\}, j \in [J]\}$, having weights denoted as $w_k(C_{m-1i}C_{mj})$. Note that for the simplicity of notation we use C_{mj} instead of $C_{km} = j$ for edges and vertices. We define forward (ϕ^k_{mj}) and backward (β^k_{mj}) quantities similar to HMMs Bishop (2006). Instead of using the terminologies of transition and emission probabilities, we formulate a corresponding algorithm in terms of the weights of the graph, $w_k(C_{m-1i}C_{mj})$ and $w_k(C_{mj})$. The initial transition probability can be defined as $w_k(C_0C_{1j})$. Notice that k indicates the cluster which the graph belongs to. We calculate forward and backward quantities using the weights, and there is no need to normalize the graph weights since a normalization is performed later when calculating the VD. Having the weights, we compute the two posterior probabilities $q(C_{km} = j)$ and $q(C_{km-1} = i, C_{km} = j)$; note that they are normalized by summing over all $j \in [J]$. McGrory and Titterington (2009) derive the exact normalization of the graph weights, for a similar but less complex graphical model; however, this is unnecessary since the normalization can be performed later, as we do.

$$\begin{aligned}
\log q(\mathbf{C}_k) \quad &{}_{+}\!\approx \quad \mathrm{E}_{-\mathbf{C}_k}\big(\log P(\mathbf{Y}, \mathbf{Z}, \mathbf{C}, \mathbf{\Psi})\big) \\
&{}_{+}\!\approx \quad \mathrm{E}_{-\mathbf{C}_k}\big(\log P(\mathbf{Y}|\mathbf{Z}, \mathbf{C}, \mathbf{\Psi})\big) + \mathrm{E}_{-\mathbf{C}_k}\big(\log P(\mathbf{C}|\mathbf{\Psi})\big).
\end{aligned}$$

We can write $\log P(\mathbf{Y}|\mathbf{Z}, \mathbf{C}, \mathbf{\Psi})$ as the sum of two terms, one that depends on $\mathbf{C}_k$ and another that does not, that is, $\log P(\mathbf{Y}|\mathbf{Z}, \mathbf{C}, \mathbf{\Psi}) = \sum_{n=1}^N \sum_{m=1}^M \sum_{j=1}^J \mathrm{I}(Z_n = k)\mathrm{I}(C_{km} = j)D_{n,m,j}$
$+ \sum_{n=1}^N \sum_{m=1}^M \sum_{k' \neq k} \sum_{j=1}^J \mathrm{I}(Z_n = k)\mathrm{I}(C_{k'm} = j)D_{n,m,j}$.

Let $f(k)$ be a function of k as $\sum_{j=1}^J \mathrm{I}(C_{k1} = j) \log \rho_{kj} + \sum_{m=2}^M \sum_{i=1}^J \sum_{j=1}^J \mathrm{I}(C_{km-1} = i, C_{km} = j) \log a^k_{ij}$, We can write $\log P(\mathbf{C}|\mathbf{\Psi}) = f(k) + \sum_{k' \neq k} f(k')$.

Thus, disregarding the terms that do not depend on $\mathbf{C}_k$ we obtain: $\log q(\mathbf{C}_k) {}_{+}\!\approx$
$\sum_{m=1}^M \sum_{j=1}^J \mathrm{I}(C_{km} = j)\Big\{ \sum_{n=1}^N \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big)\mathrm{E}_{q(\theta_n)}\big(D_{n,m,j}\big)\Big\} +$
$\sum_{j=1}^J \mathrm{I}\big(C_{k1} = j\big) \log \rho_{kj} + \sum_{m=2}^M \sum_{j=1}^J \sum_{i=1}^J \mathrm{I}\big(C_{km-1} = i, C_{km} = j\big) \times \mathrm{E}_{q(\mathbf{a}^k_i)}\big(\log a^k_{ij}\big)$. Calculations regarding directed graph are as follows: $\mathrm{u}_k(\mathbf{C}_{1:m-1}, C_m = j) = \prod_{i=1}^{m-1} \prod_{s=1}^J \exp\Big\{ \sum_{n=1}^N \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big)\mathrm{E}_{q(\theta_n)}\big(D_{n,i,s}\big)\Big\}^{\mathrm{I}(C_{ki}=s)}$
$\exp\Big\{ \sum_{n=1}^N \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big)\mathrm{E}_{q(\theta_n)}\big(D_{n,m,j}\big)\Big\}$
$\prod_{j=1}^J \exp\Big\{ \log \rho_{kj}\Big\}^{\mathrm{I}\big(C_{k1}=j\big)}$
$\prod_{t=2}^{m-1} \prod_{s=1}^J \prod_{i=1}^J \exp\Big\{\mathrm{E}_{q(\mathbf{a}^k_i)}\big(\log a^k_{is}\big)\Big\}^{\mathrm{I}\big(C_{kt-1}=i, C_{kt}=s\big)}$
$\prod_{i=1}^J \exp\Big\{\mathrm{E}_{q(\mathbf{a}^k_i)}\big(\log a^k_{ij}\big)\Big\}^{\mathrm{I}\big(C_{km-1}=i, C_{km}=j\big)}$.
and
$\mathrm{v}_k(\mathbf{C}_{m+1:M}, C_m = j) = \prod_{i=m+1}^M \prod_{s=1}^J \exp\Big\{ \sum_{n=1}^N \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big)\mathrm{E}_{q(\theta_n)}\big(D_{n,i,s}\big)\Big\}^{\mathrm{I}(C_{ki}=s)}$
$\prod_{r=m+1}^M \prod_{s=1}^J \prod_{i=1}^J \exp\Big\{\mathrm{E}_{q(\mathbf{a}^k_i)}\big(\log a^k_{is}\big)\Big\}^{\mathrm{I}\big(C_{kr}=i, C_{kr+1}=s\big)}$
$\prod_{i=1}^J \exp\Big\{\mathrm{E}_{q(\mathbf{a}^k_i)}\big(\log a^k_{ji}\big)\Big\}^{\mathrm{I}\big(C_{km}=j, C_{km+1}=i\big)}$.

We define $\phi^k_{mj} = \sum_{C_1=1}^J .. \sum_{C_{m-1}=1}^J u_k(\mathbf{C}_{1:m-1}, C_m = j)$ and
$\beta^k_{mj} = \sum_{C_{m+1}=1}^J .. \sum_{C_M=1}^J v_k(\mathbf{C}_{m+1:M}, C_m = j)$.

The graph weights are (we assume $w_k(C_0C_{1j})$ to be fixed):
$w_k(C_{m-1i}C_{mj}) = \exp\big\{\mathrm{E}_{q(\mathbf{a}_i^k)}(\log a_{ij}^k)\big\}$
and
$w_k(C_{mj}) = \exp\big\{\sum_{n=1}^{N} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big)\mathrm{E}_{q(\theta_n)}\big(D_{n,m,j}\big)\big\}$.

Note that we skip writing i, j in the calculations to make them short and more readable:

$\phi_m^k = \sum_{C_1=1}^{J} .. \sum_{C_{m-1}=1}^{J} u_k(\mathbf{C}_{1:m-1}, C_m) = \sum_{C_1=1}^{J} .. \sum_{C_{m-1}=1}^{J} u_k(\mathbf{C}_{1:m-2}, C_{m-1}, C_m) =$
$\sum_{C_1=1}^{J} .. \sum_{C_{m-1}=1}^{J} u_k(\mathbf{C}_{1:m-2}, C_{m-1})w_k(C_m)w_k(C_{m-1}C_m) = w_k(C_m)\sum_{C_{m-1}=1}^{J} \phi_{m-1}^k w_k(C_{m-1}C_m)$.

$\beta_m^k = \sum_{C_{m+1}=1}^{J} .. \sum_{C_M=1}^{J} v_k(\mathbf{C}_{m+1:M}, C_m) = \sum_{C_{m+1}=1}^{J} .. \sum_{C_M=1}^{J} v_k(\mathbf{C}_{m+2:M}, C_{m+1}, C_m) =$
$\sum_{C_{m+1}=1}^{J} .. \sum_{C_M=1}^{J} v_k(\mathbf{C}_{m+2:M})w_k(C_mC_{m+1})w_k(C_{m+1}) = \sum_{C_{m+1}=1}^{J} \beta_{m+1}^k w_k(C_mC_{m+1})w_k(C_{m+1})$.

We now calculate the posteriors: $\mathrm{q}(\mathrm{C}_{km} = j) = \sum_{C_{k,1:m-1}}^{J} \sum_{C_{k,m+1:M}}^{J} q(C_{k,1:m-1}, C_{km} = j, C_{k,m+1:M}) =$
$\sum_{C_{1:m-1}}^{J} u_k(\mathbf{C}_{1:m-1}, C_m = j) \sum_{C_{m+1:M}}^{J} v_k(\mathbf{C}_{m+1:M}, C_m = j) = \phi_{mj}^k \beta_{mj}^k$.
$q(C_{km-1} = i, C_{km} = j) =$
$\sum_{C_{k,1:m-2}}^{J} \sum_{C_{k,m+1:M}}^{J} q(C_{k,1:m-2}, C_{k,m-1} = i, C_{km} = j, C_{k,m+1:M}) =$
$\sum_{C_{1:m-2}}^{J} u_k(\mathbf{C}_{1:m-2}, C_{m-1} = i)w_k(C_{m-1i}C_{mj})w_k(C_{mj})$
$\sum_{C_{m+1:M}}^{J} v_k(\mathbf{C}_{m+1:M}, C_m = j) = w_k(C_{m-1i}C_{mj})w_k(C_{mj})\phi_{m-1i}^k \beta_{mj}^k$.

Finally, the following are the resulted update equations. Subsequently, the detailed algorithm used in CopyMix is shown in Algorithm 2.

$$\mathrm{w}_k(C_{m-1i}C_{mj}) = \exp\big\{\mathrm{E}_{q(\mathbf{a}_i^k)}(\log a_{ij}^k)\big\}$$
$$w_k(C_{mj}) = \exp\big\{\sum_{n=1}^{N} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big)\mathrm{E}_{q(\theta_n)}\big(D_{n,m,j}\big)\big\}$$
$$q(C_{km} = j) = \frac{\phi_{mj}^k \beta_{mj}^k}{\sum_{j=1}^{J} \phi_{mj}^k \beta_{mj}^k}$$
$$q(C_{km-1} = i, C_{km} = j) = \frac{w_k(C_{m-1i}C_{mj})w_k(C_{mj})\phi_{m-1i}^k \beta_{mj}^k}{\sum_{j=1}^{J} w_k(C_{m-1i}C_{mj})w_k(C_{mj})\phi_{m-1i}^k \beta_{mj}^k}$$

A.3 Calculating expectations for section 2.1

In this section we list the calculations of the expected values that are used in the update equations in the previous Appendix. Let $\mathbf{\Phi}$ be the digamma function defined as $\mathbf{\Phi}(x) = \frac{d}{dx}\log\Gamma(x)$ which can be easily calculated via numerical approximation. The values of the expectations taken with respect to the approximated distributions are given as follows where $\mathbf{\Phi}$ is used for some of them.

- $\mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) = \pi_{nk}$
- Due to computational issues in calculating $\mathrm{E}_{q(\theta_n)}\big(D_{n,m,j}\big) = y_{nm}E_{q(\theta_n)}(\log\theta_n) + y_{nm}\log j - jE_{q(\theta_n)}(\theta_n) - \log(y_{nm}!)$, we can instead compute it using the fact that
  $\mathrm{E}_{q(\theta_n)}(\theta_n) = \epsilon_{sn}/\epsilon_{rn}$ and
  $\mathrm{E}_{q(\theta_n)}\big(\log\theta_n\big) = \mathbf{\Phi}(\epsilon_{sn}) + \log E_{q(\theta_n)}(\theta_n) - \log\epsilon_{sn}$.

Algorithm 2 CopyMix: a Variational Inference Algorithm

procedure ESTIMATE-POSTERIORS(Y): ▷ Y input is N sequences of length M
Initialization and assigning priors ϵ^0_{sn}, ϵ^0_{rn}, $\mathbf{\Lambda}^{k0}_i$, α^0_k, π_{nk}, $w_k(C_0C_{1j})$
Update ϵ_{sn} using y and ϵ^0_{sn} in Eq. 10
Update $w_k(C_{m-1i}C_{mj})$ using $\mathbf{\Lambda}^{k0}_i$ and π_{nk} in Eq. A.2
Update $w_k(C_{mj})$ using $\mathbf{\Lambda}^{k0}_i$, π_{nk}, ϵ^0_{rn}, ϵ_{sn} and y in Eq. A.2
  repeat
    Update $\alpha^{(c)}_k$ using $\pi^{(c-1)}_{nk}$ and α^0_k in Eq. 8
    Update $q(C_{km}=j)^{(c)}$ and $q(C_{km-1}=i, C_{km}=j)^{(c)}$ using
      $w_k(C_0C_{1j})^{(c-1)}$, $w_k(C_{m-1i}C_{mj})^{(c-1)}$ and
      $w_k(C_{mj})^{(c-1)}$ (Eq. A.2 to A.2)
    Update $\mathbf{\Lambda}^{k(c)}_i$ using $\mathbf{\Lambda}^{k0}_i$ and $q(C_{km-1}=i, C_{km}=j)^{(c)}$ in Eq. 11
    Update $\epsilon^{(c)}_{rn}$ using ϵ^0_{rn}, $\pi^{(c-1)}_{nk}$ and $q(C_{km}=j)^{(c)}$ in Eq. A.2
    Update $\pi^{(c)}_{nk}$ using $\alpha^{(c)}_k$, ϵ_{sn}, $\epsilon^{(c)}_{rn}$, $q(C_{km}=j)^{(c)}$ and y in Eq. 9
    Update $w_k(C_{mj})^{(c)}$ using $\mathbf{\Lambda}^{k(c)}_i$, $\pi^{(c)}_{kn}$, $\epsilon^{(c)}_{rn}$, ϵ_{sn} and y in Eq. A.2
    Update $w_k(C_{m-1i}C_{mj})^{(c)}$ using $\mathbf{\Lambda}^{k(c)}_i$ and $\pi^{(c)}_{nk}$ in Eq. A.2
    $c \leftarrow c+1$
  until convergence of the ELBO in Eq. 2
Output all posterior distributions including:
$q(Z_n=k) \quad \forall n,k;$ ▷ cluster assignment probabilities
$q(C_{km}=j) \quad \forall k,m,j;$ ▷ copy number probabilities
return posterior distribution

Therefore, we obtain $\mathrm{E}_{q(\theta_n)}\big(D_{n,m,j}\big) = \log F_{\tilde{\theta}}(y_{nm}) + y_{nm} \times \big[\mathbf{\Phi}(\epsilon_{sn}) - \log \epsilon_{sn}\big]$, with $F_{\tilde{\theta}}$ being a Poisson p.m.f. with rate parameter $\tilde{\theta} = j \times (\epsilon_{sn}/\epsilon_{rn})$.

- $\mathrm{E}_{q(\boldsymbol{\pi})}(\log \pi_k) = \mathbf{\Phi}(\alpha_k) - \mathbf{\Phi}\Big(\sum_{k=1}^K \alpha_k\Big)$
- $\mathrm{E}_{q(\mathbf{a}^k_i)}(\log a^k_{ij}) = \mathbf{\Phi}(\Lambda^k_{ij}) - \mathbf{\Phi}\Big(\sum_{j=1}^J \Lambda^k_{ij}\Big)$
- $\mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km}=j)\big) = q(C_{km}=j)$ and $\mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km-1}=i, C_{km}=j)\big) = q(C_{km-1}=i, C_{km}=j)$, which are calculated in Eq. A.2 and A.2.

A.4 Step 2 of Section 2.2

The logarithm of the joint distribution of $\mathbf{Y}$, $\mathbf{Z}$, $\mathbf{C}$ and $\mathbf{\Psi}$ satisfies

$$\log P(\mathbf{Y},\mathbf{Z},\mathbf{C},\mathbf{\Psi}) = \log P(\mathbf{Y}|\mathbf{Z},\mathbf{C},\mathbf{\Psi}) + \log P(\mathbf{C}|\mathbf{\Psi}) + \log P(\mathbf{Z}|\mathbf{\Psi}) + \log P(\mathbf{\Psi}), \tag{12}$$

where

$$\begin{aligned}\log P(\mathbf{Y}|\mathbf{Z},\mathbf{C},\mathbf{\Psi}) &= \sum_{n=1}^N\sum_{m=1}^M\sum_{k=1}^K\sum_{j=1}^J \mathrm{I}(Z_n=k)\mathrm{I}(C_{km}=j)\log F_{\mu_{nj},\sigma^2}(r_{nm}) \\ &= \sum_{n=1}^N\sum_{m=1}^M\sum_{k=1}^K\sum_{j=1}^J \mathrm{I}(Z_n=k)\mathrm{I}(C_{km}=j)(-\frac{1}{2})\Big[\log(2\pi\sigma^2) + \frac{(r_{nm}-j\mu_n)^2}{\sigma^2}\Big], \end{aligned}\tag{13}$$

$$\log P(\mathbf{C}|\mathbf{\Psi}) = \sum_{k=1}^K\left[\sum_{j=1}^J \mathrm{I}(C_{k1}=j)\log\rho_{kj} + \sum_{m=2}^M\sum_{i=1}^J\sum_{j=1}^J \mathrm{I}(C_{km-1}=i, C_{km}=j)\log a^k_{ij}\right], \tag{14}$$

and

$$\log P(\mathbf{Z}|\mathbf{\Psi}) \quad = \quad \sum_{n=1}^{N}\sum_{k=1}^{K} \mathrm{I}(Z_n = k)\log \pi_k \tag{15}$$

Moreover,

$$\log P(\mathbf{\Psi}) = \log P(\mathbf{A}) + \log P(\boldsymbol{\mu}) + \log P(\frac{1}{\sigma^2}) + \log P(\boldsymbol{\pi}), \tag{16}$$

where, if B is the multivariate Beta function, i.e.,

$$B(\boldsymbol{x}) = \frac{\prod_{k=1}^{K}\Gamma(x_k)}{\Gamma(\sum_{k=1}^{K} x_k)},$$

$$\log P(\boldsymbol{\mu}) \quad = \quad \sum_{n=1}^{N}\big[(-\frac{1}{2})\Big[\log(2\pi\tau^2) + \frac{(\mu_n-\theta)^2}{\tau^2}\Big]\big], \tag{17}$$

$$\log P(\frac{1}{\sigma^2}) \quad = \quad \big[-\frac{\beta}{\sigma^2} + (\alpha-1)\log(\frac{1}{\sigma^2})\big] + C, \tag{18}$$

$$\log P(\mathbf{A}) \quad = \quad \sum_{k=1}^{K}\sum_{i=1}^{J}\big[\sum_{j=1}^{J}(\Lambda_j - 1)\log a_{ij}^k - \log B(\mathbf{\Lambda})\big], \tag{19}$$

and

$$\log P(\boldsymbol{\pi}) \quad = \quad \sum_{k=1}^{K}(\delta_k - 1)\log \pi_k - \log B(\boldsymbol{\delta}). \tag{20}$$

A.5 Step 3 and 4 of Section 2.2

Update equation for π: Since (15) and (20) are the only terms in (12) that depend on $\boldsymbol{\pi}$, the update equation $q^*(\boldsymbol{\pi})$ can be derived as follows.

$$\begin{aligned}
&\log q^*(\boldsymbol{\pi}) \\
&= \mathrm{E}_{-\boldsymbol{\pi}}\big(\log P(\mathbf{Y},\mathbf{Z},\mathbf{C},\mathbf{\Psi})\big) \\
&+\approx \mathrm{E}_{-\boldsymbol{\pi}}\big(\log P(\mathbf{Z}|\mathbf{\Psi})\big) \;+\; \mathrm{E}_{-\boldsymbol{\pi}}\big(\log P(\boldsymbol{\pi})\big) \\
&= \sum_{n=1}^{N}\sum_{k=1}^{K}\mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n=k)\big)\log\pi_k + \log P(\boldsymbol{\pi}) \\
&= \sum_{k=1}^{K}\log\pi_k\Big[\sum_{n=1}^{N}\mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n=k)\big)\Big] + \sum_{k=1}^{K}\log\pi_k(\delta_k-1) \\
&= \sum_{k=1}^{K}\log\pi_k\Big[\Big(\sum_{n=1}^{N}\mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n=k)\big)+\delta_k\Big)-1\Big].
\end{aligned}$$

Therefore, $q^*(\boldsymbol{\pi})$ is a Dirichlet distribution with parameters $\delta^* = (\delta_1^*, \ldots, \delta_K^*)$, where

$$\delta_k^* = \delta_k + \sum_{n=1}^{N}\mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n=k)\big). \tag{21}$$

Update equation for Z_n: Since (13) and (15) are the only terms in (12) that depend on Z_n, $q^*(Z_n)$ can be obtained as follows.

$$\log q^*(Z_n)$$

$$
\begin{aligned}
&= \mathrm{E}_{-Z_n}\big(\log P(\mathbf{Y},\mathbf{Z},\mathbf{C},\mathbf{\Psi})\big)\\
&\overset{+}{\approx} \mathrm{E}_{-Z_n}\big(\log P(\mathbf{Y}|\mathbf{Z},\mathbf{C},\mathbf{\Psi})\big) + \mathrm{E}_{-Z_n}\big(\log P(\mathbf{Z}|\mathbf{\Psi})\big)
\end{aligned}
$$

Note that $\log P(\mathbf{Y}|\mathbf{Z},\mathbf{C},\mathbf{\Psi})$ and $\log P(\mathbf{Z}|\mathbf{\Psi})$ can be written as the sum of two terms, one that depends on Z_n and one that does not, i.e.,

$$
\begin{aligned}
\log P(\mathbf{Y}|\mathbf{Z},\mathbf{C},\mathbf{\Psi}) &= \sum_{m=1}^{M}\sum_{k=1}^{K}\sum_{j=1}^{J}\mathrm{I}(Z_n=k)\mathrm{I}(C_{km}=j)(-\frac{1}{2})\Big[\log(2\pi\sigma^2)+\frac{(r_{nm}-j\mu_n)^2}{\sigma^2}\Big] \qquad (22)\\
&+\sum_{l\neq n}\sum_{m=1}^{M}\sum_{k=1}^{K}\sum_{j=1}^{J}\mathrm{I}(Z_l=k)\mathrm{I}(C_{km}=j)(-\frac{1}{2})\Big[\log(2\pi\sigma^2)+\frac{(r_{lm}-j\mu_l)^2}{\sigma^2}\Big]
\end{aligned}
$$

and

$$
\log P(\mathbf{Z}|\mathbf{\Psi}) = \sum_{k=1}^{K}\mathrm{I}(Z_n=k)\log\pi_k + \sum_{l\neq n}\sum_{k=1}^{K}\mathrm{I}(Z_l=k)\log\pi_k.
$$

Consequently,

$$
\log q^*(Z_n) \overset{+}{\approx} \sum_{k=1}^{K}\mathrm{I}(Z_n=k)\Big\{\mathrm{E}_{q^*(\boldsymbol{\pi})}(\log\pi_k)+\sum_{m=1}^{M}\sum_{j=1}^{J}\mathrm{E}_{q^*(\mathbf{C}_k)}\big(\mathrm{I}(C_{km}=j)\big)\mathrm{E}_{q^*(\sigma)q^*(\mu)}[D_{nmj}]\Big\},
$$

Where:

$$
D_{nmj} = -\frac{1}{2}\Big[\log(2\pi\sigma^2)+\frac{(r_{nm}-j\mu_n)^2}{\sigma^2}\Big] \qquad (23)
$$

We conclude that $q^*(Z_n)\sim\text{Categorical}(\boldsymbol{p}_n^*)$ with parameters $\boldsymbol{p}_n^*=(p_{n1}^*,\ldots,p_{nK}^*)$ where

$$
\tilde{p}_{nk} = \mathrm{E}_{q^*(\boldsymbol{\pi})}(\log\pi_k)+\sum_{m=1}^{M}\sum_{j=1}^{J}\mathrm{E}_{q^*(\mathbf{C}_k)}\big(\mathrm{I}(C_{km}=j)\big)\mathrm{E}_{q^*(\frac{1}{\sigma^2})q^*(\mu)}[D_{nmj}].
$$

and

$$
p_{nk}^* = \frac{\exp(\tilde{p}_{nk})}{\sum_{k'=1}^{K}\exp(\tilde{p}_{nk'})}. \qquad (24)
$$

Update equation for μ: Since (13) and (17) are the only terms in (12) that depend on μ_n, $q^*(\mu_n)$ can be obtained as follows.

$$
\begin{aligned}
&\log q^*(\mu_n)\\
&= \mathrm{E}_{-\mu_n}\big(\log P(\mathbf{Y},\mathbf{Z},\mathbf{C},\mathbf{\Psi})\\
&\overset{+}{\approx} \mathrm{E}_{-\mu_n}\big(\log P(\mathbf{Y}|\mathbf{Z},\mathbf{C},\mathbf{\Psi})\big)+\mathrm{E}_{-\mu_n}\big(\log P(\boldsymbol{\mu})\big)
\end{aligned}
$$

We can write $\log P(\mathbf{Y}|\mathbf{Z},\mathbf{C},\mathbf{\Psi})$ as in Eq. 22, and $\mathrm{E}_{-\mu_n}\big(\log P(\boldsymbol{\mu})\big)=\log P(\boldsymbol{\mu})$, which is:

$$
\log P(\boldsymbol{\mu}) = (-\frac{1}{2})\Big[\log(2\pi\tau^2)+\frac{(\mu_n-\theta)^2}{\tau^2}\Big]+\sum_{l\neq n}(-\frac{1}{2})\Big[\log(2\pi\tau^2)+\frac{(\mu_l-\theta)^2}{\tau^2}\Big]
$$

Therefore, $\log q^*(\mu_n)$

$$
\overset{+}{\approx}(-\tfrac{1}{2})\Bigg(\big(\tfrac{\mu_n^2-2\theta\mu_n}{\tau^2}\big)+\textstyle\sum_{m=1}^{M}\sum_{k=1}^{K}\sum_{j=1}^{J}\mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n=k)\big)\mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km}=j)\big)\mathrm{E}_{q^*(\frac{1}{\sigma^2})}[\tfrac{j^2\mu_n^2-2j\mu_n r_{nm}}{\sigma^2}]\Bigg)
$$

$$
\overset{+}{\approx}(-\tfrac{1}{2})\Bigg(\big(\tfrac{\mu_n^2-2\theta\mu_n}{\tau^2}\big)+\textstyle\sum_{m=1}^{M}\sum_{k=1}^{K}\sum_{j=1}^{J}\mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n=k)\big)\mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km}=j)\big)\tfrac{\alpha^*}{\beta^*}(j^2\mu_n^2-2j\mu_n r_{nm})\Bigg)
$$

$$
\overset{+}{\approx}(-\tfrac{1}{2})\Bigg(\mu_n^2\Bigg[\tfrac{1}{\tau^2}+\tfrac{\alpha^*}{\beta^*}\textstyle\sum_{m=1}^{M}\sum_{k=1}^{K}\sum_{j=1}^{J}\mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n=k)\big)\mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km}=j)\big)j^2\Bigg]
$$

$$- 2\mu_n \left[\frac{\theta}{\tau^2} + \frac{\alpha^*}{\beta^*} \sum_{m=1}^{M} \sum_{k=1}^{K} \sum_{j=1}^{J} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km} = j)\big) j r_{nm} \right] \Bigg)$$

Thus, $q^*(\mu_n)$ is a Normal distribution with parameters:

$$\theta_n^* = \frac{\frac{\theta}{\tau^2} + \sum_{m=1}^{M} \sum_{k=1}^{K} \sum_{j=1}^{J} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km} = j)\big) j r_{nm} \frac{\alpha^*}{\beta^*}}{\frac{1}{\tau^2} + \sum_{m=1}^{M} \sum_{k=1}^{K} \sum_{j=1}^{J} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km} = j)\big) \frac{\alpha^*}{\beta^*} j^2} \tag{25}$$

$$\tau_n^{2*} = \frac{1}{\frac{1}{\tau^2} + \sum_{m=1}^{M} \sum_{k=1}^{K} \sum_{j=1}^{J} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km} = j)\big) \frac{\alpha^*}{\beta^*} j^2} \tag{26}$$

Update equation for $\frac{1}{\sigma^2}$: Similar to the previous calculations, we do as follows.

$$\begin{aligned} &\log q^*(\frac{1}{\sigma^2}) \\ &= \mathrm{E}_{-\frac{1}{\sigma^2}}\big(\log P(\mathbf{Y}, \mathbf{Z}, \mathbf{C}, \mathbf{\Psi}) \\ &\underset{+}{\approx} \mathrm{E}_{-\frac{1}{\sigma^2}}\big(\log P(\mathbf{Y}|\mathbf{Z}, \mathbf{C}, \mathbf{\Psi})\big) + \mathrm{E}_{-\frac{1}{\sigma^2}}\big(\log P(\frac{1}{\sigma^2})\big) \end{aligned}$$

We can write $\log P(\mathbf{Y}|\mathbf{Z}, \mathbf{C}, \mathbf{\Psi})$ and $\log P(\sigma)$ as they are in Eq. 13, 18. Putting them together, we achieve the following.

$$\begin{aligned} &\log q^*(\tfrac{1}{\sigma^2}) \\ &\underset{+}{\approx} \big[-\tfrac{\beta}{\sigma^2} + (\alpha - 1)\log(\tfrac{1}{\sigma^2})\big] + \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{k=1}^{K} \sum_{j=1}^{J} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km} = j)\big) \mathrm{E}_{q^*(\mu_n)}[D_{nmj}] \\ &\underset{+}{\approx} -\tfrac{\beta}{\sigma^2} + (\alpha - 1)\log(\tfrac{1}{\sigma^2}) \\ &- \tfrac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{k=1}^{K} \sum_{j=1}^{J} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km} = j)\big) \big[-\log(\tfrac{1}{\sigma^2}) + \tfrac{r_{nm}^2 - 2 j r_{nm} \theta_n^* + j^2(\tau_n^{*2} + \theta_n^{*2})}{\sigma^2} \big] \\ &= \log(\tfrac{1}{\sigma^2}) \Big\{ \alpha^* - 1 \Big\} - \tfrac{1}{\sigma^2} \Big\{ \beta^* \Big\} \end{aligned}$$

Therefore, $q^*(\frac{1}{\sigma^2})$ is a Gamma distribution with parameters:

$$\alpha^* = \alpha + \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{k=1}^{K} \sum_{j=1}^{J} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km} = j)\big) = \alpha + \frac{NM}{2} \tag{27}$$

$$\beta^* = \beta + \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{k=1}^{K} \sum_{j=1}^{J} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km} = j)\big) \Big(r_{nm}^2 - 2 j r_{nm} \theta_n^* + j^2(\tau_n^{*2} + \theta_n^{*2}) \Big) \tag{28}$$

Update equation for $\mathbf{a}_i^k$: Because (14) and (19) are the only terms in (12) that depend on $\mathbf{a}_i^k$, we calculate $q^*(\mathbf{a}_i^k)$ as follows.

$$\begin{aligned} &\log q^*(\mathbf{a}_i^k) \\ &= \mathrm{E}_{-\mathbf{a}_i^k}\big(\log P(\mathbf{Y}, \mathbf{Z}, \mathbf{C}, \mathbf{\Psi})\big) \\ &\underset{+}{\approx} \mathrm{E}_{-\mathbf{a}_i^k}\big(\log P(\mathbf{C}|\mathbf{\Psi})\big) + \mathrm{E}_{-\mathbf{a}_i^k}\big(\log P(\mathbf{A})\big) \end{aligned}$$

Disregarding the terms that do not depend on $\mathbf{a}_i^k$ in $\log P(\mathbf{C}|\mathbf{\Psi})$ and $\log P(\mathbf{A})$, we obtain:

$$\begin{aligned} \log q^*(\mathbf{a}_i^k) \quad &\underset{+}{\approx} \sum_{j=1}^{J} \sum_{m=2}^{M} \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km-1} = i, C_{km} = j)\big) \log a_{ij}^k + \sum_{j=1}^{J} (\Lambda_j - 1) \log a_{ij}^k \\ &= \sum_{j=1}^{J} \log a_{ij} \Big\{ \Big[\Lambda_j + \sum_{m=2}^{M} \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km-1} = i, C_{km} = j)\big) \Big] - 1 \Big\}. \end{aligned}$$

Therefore, $q^*(\mathbf{a}_i^k)$ is Dirichlet with parameters $\mathbf{\Lambda}_i^{k*} = (\Lambda_{i1}^{k*}, \ldots, \Lambda_{iJ}^{k*})$ where

$$\Lambda_{ij}^{k*} = \Lambda_j + \sum_{m=2}^{M} \mathrm{E}_{q(\mathbf{C}_k)}\big(\mathrm{I}(C_{km-1} = i, C_{km} = j)\big) \tag{29}$$

Update equation for $\mathbf{C}_k$: Since only (13) and (14) depend on $\mathbf{C}_k$, $\log q^*(\mathbf{C}_k)$ can be calculated as follows.

$$\begin{aligned} &\log q^*(\mathbf{C}_k) \\ &= \mathrm{E}_{-\mathbf{C}_k}\big(\log P(\mathbf{Y}, \mathbf{Z}, \mathbf{C}, \mathbf{\Psi})\big) \\ &\overset{+}{\approx} \mathrm{E}_{-\mathbf{C}_k}\big(\log P(\mathbf{Y}|\mathbf{Z}, \mathbf{C}, \mathbf{\Psi})\big) + \mathrm{E}_{-\mathbf{C}_k}\big(\log P(\mathbf{C}|\mathbf{\Psi})\big). \end{aligned}$$

Similarly we can write $\log P(\mathbf{C}|\mathbf{\Psi})$ as

$$\begin{aligned} \log P(\mathbf{C}|\mathbf{\Psi}) &= \sum_{j=1}^{J} \mathrm{I}(C_{k1} = j) \log \rho_{kj} + \sum_{m=2}^{M}\sum_{i=1}^{J}\sum_{j=1}^{J} \mathrm{I}(C_{km-1} = i, C_{km} = j) \log a_{ij}^k, \\ &+ \sum_{k' \neq k} \Big[\sum_{j=1}^{J} \mathrm{I}(C_{k'1} = j) \log \rho_{k'j} + \sum_{m=2}^{M}\sum_{i=1}^{J}\sum_{j=1}^{J} \mathrm{I}(C_{k'm-1} = i, C_{k'm} = j) \log a_{ij}^{k'} \Big]. \end{aligned}$$

Thus, disregarding the terms that do not depend on $\mathbf{C}_k$ we obtain:

$$\begin{aligned} &\log q^*(\mathbf{C}_k) \\ &\overset{+}{\approx} \sum_{m=1}^{M}\sum_{j=1}^{J} \mathrm{I}(C_{km} = j) \Big\{ \sum_{n=1}^{N} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \mathrm{E}_{q^*(\mu_n), q^*(\frac{1}{\sigma^2})}(D_{n,m,j}) \Big\} \\ &\quad + \sum_{j=1}^{J} \mathrm{I}\big(C_{k1} = j\big) \mathrm{E}_{q^*(\boldsymbol{\rho}_k)}\big(\log \rho_{kj}\big) + \sum_{m=2}^{M}\sum_{j=1}^{J}\sum_{i=1}^{J} \mathrm{I}\big(C_{km-1} = i, C_{km} = j\big) \mathrm{E}_{q^*(\mathbf{a}_i^k)}\big(\log a_{ij}^k\big), \end{aligned}$$

Calculations regarding directed graph: We define graph $G = \{V, E\}$ where we have each vertex as $V = \{C_{mj} : m \in M, j \in J\}$ with weight $w(C_{mj})$ and each edge as $E = \{C_{mj}C_{m+1j} : m \in M, j \in J\}$ with weight $w(C_{mj}C_{m+1j})$. We define quantities of forward, ϕ_{mj}, and backward, β_{mj}, similar to HMM as below (the calculations are based on $\log q^*(\mathbf{C}_k)$):

$$\begin{aligned} &u_k(\mathbf{C}_{1:m-1}, C_m = j) \\ &\overset{+}{\approx} \prod_{i=1}^{m-1}\prod_{s=1}^{J} \exp\Big\{ \sum_{n=1}^{N} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) E_{q^*(\mu_n), q^*(\frac{1}{\sigma^2})(D_{n,i,s})} \Big\}^{\mathrm{I}(C_{ki}=s)} \exp\Big\{ \sum_{n=1}^{N} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \mathrm{E}_{q^*(\mu_n), q^*(\frac{1}{\sigma^2})}(D_{n,m,j}) \Big\} \\ &\quad \prod_{j=1}^{J} \exp\Big\{ \mathrm{E}_{q^*(\boldsymbol{\rho}_k)}\big(\log \rho_{kj}\big) \Big\}^{\mathrm{I}\left(C_{k1}=j\right)} \\ &\quad \prod_{t=2}^{m-1}\prod_{s=1}^{J}\prod_{i=1}^{J} \exp\Big\{ \mathrm{E}_{q^*(\mathbf{a}_i^k)}\big(\log a_{is}^k\big) \Big\}^{\mathrm{I}\left(C_{kt-1}=i, C_{kt}=s\right)} \prod_{i=1}^{J} \exp\Big\{ \mathrm{E}_{q^*(\mathbf{a}_i^k)}\big(\log a_{ij}^k\big) \Big\}^{\mathrm{I}\left(C_{km-1}=i, C_{km}=j\right)} \end{aligned}$$

$\phi_{mj} = \sum_{C_1=1}^{J} .. \sum_{C_{m-1}=1}^{J} u_k(\mathbf{C}_{1:m-1}, C_m = j)$ and
$\beta_{mj} = \sum_{C_{m+1}=1}^{J} .. \sum_{C_M=1}^{J} v_k(\mathbf{C}_{m+1:M}, C_m = j)$.

$$\begin{aligned} &v_k(\mathbf{C}_{m+1:M}, C_m = j) \\ &\overset{+}{\approx} \prod_{i=m+1}^{M}\prod_{s=1}^{J} \exp\Big\{ \sum_{n=1}^{N} \mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n = k)\big) \mathrm{E}_{q^*(\mu_n), q^*(\frac{1}{\sigma^2})}(D_{n,i,s}) \Big\}^{\mathrm{I}(C_{ki}=s)} \end{aligned}$$

$$\prod_{r=m+1}^{M}\prod_{s=1}^{J}\prod_{i=1}^{J}\exp\left\{\mathrm{E}_{q^*(\mathbf{a}_i^k)}\big(\log a_{is}^k\big)\right\}^{\mathrm{I}\left(C_{kr}=i,C_{kr+1}=s\right)}\prod_{i=1}^{J}\exp\left\{\mathrm{E}_{q^*(\mathbf{a}_i^k)}\big(\log a_{ji}^k\big)\right\}^{\mathrm{I}\left(C_{km}=j,C_{km+1}=i\right)}$$

Instead of using terminologies of transition and emission probabilities we formulate those as weights of the graph; $w_k(C_{mj}C_{m+1j})$ and $w_k(C_{mj})$ respectively. The initial transition probability can be define as $w_k(C_0C_{1j})$ assuming the source starts at 1. Notice that k shows the cluster which the graph belongs to. We can calculate forward and backwards using dynamic programming; having those values we can compute the two posterior probabilities of $q_k^*(C_m = j)$ and $q_k^*(C_{m-1} = i, C_m = j)$ which are the expectations of the indicator functions; we then normalized them by summing over all $j \in J$. The graph weights are: $w_k(C_0C_{1j}) = \exp\left\{\mathrm{E}_{q^*(\boldsymbol{\rho}_k)}(\log\rho_{kj})\right\}$; $w_k(C_{m-1i}C_{mj}) = \exp\left\{\mathrm{E}_{q^*(\mathbf{a}_i^k)}(\log a_{ij}^k)\right\}$ and $w_k(C_{mj}) = \exp\left\{\sum_{n=1}^{N}\mathrm{E}_{q(Z_n)}\big(\mathrm{I}(Z_n=k)\big)\mathrm{E}_{q^*(\mu_n),q^*(\frac{1}{\sigma^2})}(D_{n,i,s})\right\}$.

Note that we skip writing i, j, k in the calculations to make them short and more readable: $\phi_m = \sum_{C_1=1}^{J}..\sum_{C_{m-1}=1}^{J}u_k(\mathbf{C}_{1:m-1}, C$
$\sum_{C_1=1}^{J}..\sum_{C_{m-1}=1}^{J}u_k(\mathbf{C}_{1:m-2}, C_{m-1}, C_m) =$
$\sum_{C_1=1}^{J}..\sum_{C_{m-1}=1}^{J}u_k(\mathbf{C}_{1:m-2}, C_{m-1})w(C_m)w(C_{m-1}C_m) = w(C_m)\sum_{C_{m-1}=1}^{J}\phi_{m-1}w(C_{m-1}C_m)$

$\beta_m = \sum_{C_{m+1}=1}^{J}..\sum_{C_M=1}^{J}v_k(\mathbf{C}_{m+1:M}, C_m) = \sum_{C_{m+1}=1}^{J}..\sum_{C_M=1}^{J}v_k(\mathbf{C}_{m+2:M}, C_{m+1}, C_m) =$
$\sum_{C_{m+1}=1}^{J}..\sum_{C_M=1}^{J}v_k(\mathbf{C}_{m+2:M})w(C_mC_{m+1})w(C_{m+1}) = \sum_{C_{m+1}=1}^{J}\beta_{m+1}w(C_mC_{m+1})w(C_{m+1})$ We now calculate the posteriors: $\mathrm{q}_k(C_m = j) = \sum_{C_{1:m-1}}^{J}\sum_{C_{m+1:M}}^{J}q_k(C_{1:m-1}, C_m = j, C_{m+1:M}) = \sum_{C_{1:m-1}}^{J}u_k(\mathbf{C}_{1:m-1}, C_m = j)\sum_{C_{m+1:M}}^{J}v_k(\mathbf{C}_{m+1:M}, C_m = j)$
$= \phi_{mj}\beta_{m,j}$

$\mathrm{q}_k(C_{m-1} = i, C_m = j) = \sum_{C_{1:m-2}}^{J}\sum_{C_{m+1:M}}^{J}q_k(C_{1:m-2}, C_{m-1} = i, C_m = j, C_{m+1:M}) =$
$\sum_{C_{1:m-2}}^{J}u_k(\mathbf{C}_{1:m-2}, C_{m-1} = i)w(C_{m-1i}C_{mj})w(C_{mj})\sum_{C_{m+1:M}}^{J}v_k(\mathbf{C}_{m+1:M}, C_m = j) =$
$w(C_{m-1i}C_{mj})w(C_{mj})\phi_{m-1j}\beta_{m,j}$

A.6 Calculating expectations and ELBO for Section 2.2

Let $\boldsymbol{\Psi}$ be the digamma function defined as

$$\boldsymbol{\Psi}(x) = \frac{d}{dx}\log\Gamma(x), \tag{30}$$

which can be easily calculated via numerical approximation. The values of the expectations above taken with respect to the approximated distributions are given as follows.

$$\mathrm{E}_{q(Z_n)}(\mathrm{I}(Z_n = k)) = \pi_{nk}^*$$

$$\mathrm{E}_{q(\mu_n)}(\mu_n) = \theta_n^*$$

$$E_{q(\frac{1}{\sigma^2})}(\log\frac{1}{\sigma^2}) = \boldsymbol{\Psi}(\alpha^*) - \log\beta^*$$

$$\mathrm{E}_{q(\mu_n)}(\mu_n^2) = \theta_n^{*2} + \tau_n^{*2}$$

$$\mathrm{E}_{q(\frac{1}{\sigma^2})}(\frac{1}{\sigma^2}) = \frac{\alpha^*}{\beta^*}$$

$$\mathrm{E}_{q(\frac{1}{\sigma^2})}(D_{nmj}) = -\tfrac{1}{2}\left[\mathrm{E}\left[\log(2\pi\sigma^2) + \tfrac{(r_{nm}-j\mu_n)^2}{\sigma^2}\right]\right] =$$
$$-\tfrac{1}{2}\left[\log(2\pi) - \mathrm{E}[\log(\tfrac{1}{\sigma^2})] + \mathrm{E}[\tfrac{r_{nm}^2}{\sigma^2}] - 2jr_{nm}\mathrm{E}[\tfrac{\mu_n}{\sigma^2}] + \mathrm{E}[\tfrac{j^2\mu_n^2}{\sigma^2}]\right] =$$

$$-\frac{1}{2}\Bigg[\log(2\pi) - \mathrm{E}_{q(\sigma)}(\log\tfrac{1}{\sigma^2}) + r_{nm}^2\mathrm{E}_{q(\sigma)}(\tfrac{1}{\sigma^2}) - 2jr_{nm}(\mu_n)\mathrm{E}_{q(\sigma)}(\tfrac{1}{\sigma^2}) + (j^2\mu_n^2)\mathrm{E}_{q(\sigma)}(\tfrac{1}{\sigma^2})\Bigg]$$

$$\mathrm{E}_{q(\mu_n)}(D_{nmj}) = -\frac{1}{2}\Bigg[\mathrm{E}\Big[\log(2\pi\sigma^2) + \tfrac{(r_{nm}-j\mu_n)^2}{\sigma^2}\Big]\Bigg] =$$

$$-\frac{1}{2}\Bigg[\log(2\pi) - \log(\tfrac{1}{\sigma^2}) + \tfrac{r_{nm}^2}{\sigma^2} - 2jr_{nm}\mathrm{E}[\tfrac{\mu_n}{\sigma^2}] + \mathrm{E}[\tfrac{j^2\mu_n^2}{\sigma^2}]\Bigg] =$$

$$-\frac{1}{2}\Bigg[\log(2\pi) - \log\tfrac{1}{\sigma^2} + \tfrac{r_{nm}^2}{\sigma^2} - \tfrac{2jr_{nm}}{\sigma^2}\mathrm{E}_{q(\mu_n)}(\mu_n) + \tfrac{j^2}{\sigma^2}\mathrm{E}_{q(\mu_n)}(\mu_n^2)\Bigg]$$

$$\mathrm{E}_{q(\frac{1}{\sigma^2}),q(\mu_n)}(D_{nmj}) = -\frac{1}{2}\Bigg[\mathrm{E}\Big[\log(2\pi\sigma^2) + \tfrac{(r_{nm}-j\mu_n)^2}{\sigma^2}\Big]\Bigg] =$$

$$-\frac{1}{2}\Bigg[\log(2\pi) - \mathrm{E}[\log(\tfrac{1}{\sigma^2})] + \mathrm{E}[\tfrac{r_{nm}^2}{\sigma^2}] - 2jr_{nm}\mathrm{E}[\tfrac{\mu_n}{\sigma^2}] + \mathrm{E}[\tfrac{j^2\mu_n^2}{\sigma^2}]\Bigg] =$$

$$-\frac{1}{2}\Bigg[\log(2\pi) - \mathrm{E}_{q(\frac{1}{\sigma^2})}(\log\tfrac{1}{\sigma^2}) + r_{nm}^2\mathrm{E}_{q(\frac{1}{\sigma^2})}(\tfrac{1}{\sigma^2}) - 2jr_{nm}\mathrm{E}_{q(\mu_n)}(\mu_n)\mathrm{E}_{q(\frac{1}{\sigma^2})}(\tfrac{1}{\sigma^2}) + j^2\mathrm{E}_{q(\mu_n)}(\mu_n^2)\mathrm{E}_{q(\frac{1}{\sigma^2})}(\tfrac{1}{\sigma^2})\Bigg]$$

$$\mathrm{E}_{q(\boldsymbol{\pi})}(\log\boldsymbol{\pi}) = \boldsymbol{\Psi}(\delta_k^*) - \boldsymbol{\Psi}\Big(\sum_{k=1}^{K}\delta_k^*\Big) \quad \forall k \in K \tag{31}$$

$$\mathrm{E}_{q(\mathbf{a}_i^k)}(\log\mathbf{a}_i^k) = \boldsymbol{\Psi}(\Lambda_{ij}^{k*}) - \boldsymbol{\Psi}\Big(\sum_{j=1}^{J}\Lambda_{ij}^{k*}\Big) \quad \forall j \in J \tag{32}$$

Using the results above regarding the expectations, we update the parameters of the approximated distributions iteratively.
We then conduct many iterations until the convergence of the ELBO in Eq. 2 which is calculated as below. An assumption in the ELBO calculation is that we ignore all the constants contributing in the ELBO value.

$$\begin{aligned}
\mathrm{ELBO(q)} = {} & \mathrm{E}\big[\log P(\mathbf{Y}|\mathbf{Z},\mathbf{C},\boldsymbol{\Psi}) + \log P(\mathbf{Z}|\boldsymbol{\Psi}) + \log P(\mathbf{C}|\boldsymbol{\Psi}) + \log P(\boldsymbol{A}) + \log P(\boldsymbol{\mu}) + \log P(\tfrac{1}{\sigma^2}) + \log P(\boldsymbol{\pi})\big] \\
& - \mathrm{E}\big[\log q(\mathbf{Z}) + \log q(\mathbf{C}) + \log q(\boldsymbol{A}) + \log q(\boldsymbol{\mu}) + \log q(\tfrac{1}{\sigma^2}) + \log q(\boldsymbol{\pi}))\big] = \\
& \mathrm{E}\Big[\textstyle\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{k=1}^{K}\sum_{j=1}^{J}\mathrm{I}(Z_n=k)\mathrm{I}(C_{km}=j)D_{nmj} + \sum_{n=1}^{N}\sum_{k=1}^{K}\mathrm{I}(Z_n=k)\log\pi_k \\
& + \textstyle\sum_{k=1}^{K}\sum_{j=1}^{J}\mathrm{I}(C_{k1}=j)\log\rho_{kj} + \sum_{k=1}^{K}\sum_{m=2}^{M}\sum_{i=1}^{J}\sum_{j=1}^{J}\mathrm{I}(C_{km-1}=i,C_{km}=j)\log a_{ij}^k \\
& + \textstyle\sum_{k=1}^{K}\sum_{i=1}^{J}\sum_{j=1}^{J}(\Lambda_j-1)\log a_{ij}^k + \sum_{n=1}^{N}(\frac{-1}{2})\Big(\log(2\pi\tau^2) + \frac{\mu_n^2-2\theta\mu_n+\theta^2}{\tau^2}\Big) \\
& + (\tfrac{-\beta}{\sigma^2}) + (\alpha-1)\log\tfrac{1}{\sigma^2} + \log P(\boldsymbol{\pi})\Big] \\
& - \mathrm{E}\Big[\textstyle\sum_{n=1}^{N}\log q(Z_n) + \sum_{k=1}^{K}\log q(\mathbf{C}_k) + \sum_{k=1}^{K}\sum_{i=1}^{J}\log q(\mathbf{a}_i^k) + \sum_{n=1}^{N}\log q(\mu_n) + \log q(\tfrac{1}{\sigma^2}) + \log q(\boldsymbol{\pi})\Big]
\end{aligned}$$

We calculate the $\mathrm{E}[\log q(\mathbf{C}_k)]$ by decomposing it into initial and transition components. Note that the initial probabilities, ρ_{kj}s, are fixed and, therefore, the term corresponding to that cancels out the corresponding term in $\mathrm{E}[\log P(\mathbf{C}|\boldsymbol{\Psi})]$. The remaining term, the transition component, is calculated by the following using $q(C_{km}=j|C_{km-1}=i) = \frac{q(C_{km}=j,C_{km-1}=i)}{q(C_{km-1}=i)}$, as below:

$$\begin{aligned}
& \mathrm{E}_{q(\mathbf{C}_k)}[\log q(\mathbf{C}_k)] = \mathrm{E}_{q(\mathbf{C}_k)}[\log\textstyle\prod_{m=2}^{M}\prod_{i=1}^{J}\prod_{j=1}^{J}q(C_{km}=j|C_{km-1}=i)^{\mathrm{I}(C_{km-1}=i,C_{km}=j)}] = \\
& \textstyle\sum_{m=2}^{M}\sum_{i=1}^{J}\sum_{j=1}^{J}\mathrm{E}_{q(C_{km-1},C_{km})}\Big[\mathrm{I}(C_{km-1}=i,C_{km}=j)\Big]\log q(C_{km}=j|C_{km-1}=i)
\end{aligned}$$

$\sum_{m=2}^{M}\sum_{i=1}^{J}\sum_{j=1}^{J}\mathrm{E}_{q(C_{km-1},C_{km})}\Big[\mathrm{I}(C_{km-1}=i,C_{km}=j)\Big]\log\frac{\mathrm{E}_{q(C_{km-1},C_{km})}\Big[\mathrm{I}(C_{km-1}=i,C_{km}=j)\Big]}{\mathrm{E}_{q(C_{km-1})}\Big[\mathrm{I}(C_{km-1}=i)\Big]}$

We calculate the $\mathrm{E}[\log p(Z_n|\Psi)]$ as below:

$$\mathrm{E}_{(q(Z_n),q(\boldsymbol{\pi}))}\Big[\log p(Z_n|\Psi)\Big]=\mathrm{E}_{(q(Z_n),q(\boldsymbol{\pi}))}\Big[\sum_{k=1}^{K}\mathrm{I}(Z_n=k)\log\pi_k\Big]=\sum_{k=1}^{K}\mathrm{E}_{q(Z_n)}\Big[\mathrm{I}(Z_n=k)\Big]\mathrm{E}_{q(\boldsymbol{\pi})}\Big[\log\pi_k\Big] \tag{33}$$

We calculate the $\mathrm{E}[\log q(Z_n)]$ as below. Note that the expectation of $q(Z_n)$ w.r.t. $q(Z_n)$ is equal to $q(Z_n)$. Also, we know that $\mathrm{E}_{q(Z_n)}\Big[\mathrm{I}(Z_n=k)\Big]=q(Z_n=k)$.

$$\mathrm{E}_{q(Z_n)}\Big[\log q(Z_n)\Big]=\mathrm{E}_{q(Z_n)}\Big[\sum_{k=1}^{K}\mathrm{I}(Z_n=k)\log q(Z_n=k)\Big]=\sum_{k=1}^{K}\mathrm{E}_{q(Z_n)}\Big[\mathrm{I}(Z_n=k)\Big]\log q(Z_n=k) \tag{34}$$

We calculate the $\mathrm{E}[\log P(\mu_n)]$ as below:

$$\mathrm{E}_{q(\mu_n)}[\log p(\mu_n)]=(\frac{-1}{2})\Bigg(\log(2\pi\tau^2)+\frac{\mathrm{E}_{q(\mu_n)}\Big[\mu_n^2\Big]-2\theta\mathrm{E}_{q(\mu_n)}\Big[\mu_n\Big]+\theta^2}{\tau^2}\Bigg) \tag{35}$$

We calculate the $\mathrm{E}[\log q(\mu_n)]$ as below:

$$\mathrm{E}_{q(\mu_n)}[\log q(\mu_n)]=(\frac{-1}{2})\Bigg(\log(2\pi\tau_n^{*2})+\frac{\mathrm{E}_{q(\mu_n)}\Big[\mu_n^2\Big]-2\theta_n^*\mathrm{E}_{q(\mu_n)}\Big[\mu_n\Big]+\theta_n^{*2}}{\tau_n^{*2}}\Bigg)=(\frac{-1}{2})\Bigg(\log(2\pi\tau_n^{*2})+1\Bigg) \tag{36}$$

We calculate the $\mathrm{E}[\log P(\boldsymbol{\pi})]$ as below:

$\mathrm{E}_{q(\boldsymbol{\pi})}(\log P(\boldsymbol{\pi}))=\sum_{k=1}^{K}(\delta_k-1)\mathrm{E}_{q(\boldsymbol{\pi})}(\log\boldsymbol{\pi})-\log B(\boldsymbol{\delta})=\sum_{k=1}^{K}(\delta_k-1)\Big[\boldsymbol{\Psi}(\delta_k^*)-\boldsymbol{\Psi}\big(\sum_{k=1}^{K}\delta_k^*\big)\Big]-\log B(\boldsymbol{\delta})$
$=\Big(\sum_{k=1}^{K}\delta_k-K\Big)\boldsymbol{\Psi}\Big(\sum_{k=1}^{K}\delta_k^*\Big)-\sum_{k=1}^{K}\boldsymbol{\Psi}(\delta_k^*)(\delta_k-1)-\log B(\boldsymbol{\delta})$

We calculate the $\mathrm{E}(\log q(\boldsymbol{\pi}))$ as below:

$$\mathrm{E}_{q(\boldsymbol{\pi})}(\log q(\boldsymbol{\pi}))=\Big(\sum_{k=1}^{K}\delta_k^*-K\Big)\boldsymbol{\Psi}\Big(\sum_{k=1}^{K}\delta_k^*\Big)-\sum_{k=1}^{K}\boldsymbol{\Psi}(\delta_k^*)(\delta_k^*-1)-\log B(\boldsymbol{\delta}^*) \tag{37}$$

$\mathrm{ELBO(q)}=\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{k=1}^{K}\sum_{j=1}^{J}\mathrm{E}_{q(Z_n)}\Big[\mathrm{I}(Z_n=k)\Big]\mathrm{E}_{q(C_{km})}\Big[\mathrm{I}(C_{km}=j)\Big]\mathrm{E}_{q(\mu_n)q(\frac{1}{\sigma^2})}\Big[D_{nmj}\Big]$
$+\sum_{n=1}^{N}\sum_{k=1}^{K}\mathrm{E}_{q(Z_n)}\Big[\mathrm{I}(Z_n=k)\Big]\mathrm{E}_{q(\boldsymbol{\pi})}\Big[\log\pi_k\Big]+\sum_{k=1}^{K}\sum_{j=1}^{J}\mathrm{E}_{q(C_{k1})}\Big[\mathrm{I}(C_{k1}=j)\Big]\log\rho_{kj}$
$+\sum_{k=1}^{K}\sum_{m=2}^{M}\sum_{i=1}^{J}\sum_{j=1}^{J}\mathrm{E}_{q(C_{km-1},C_{km})}\Big[\mathrm{I}(C_{km-1}=i,C_{km}=j)\Big]\mathrm{E}_{q(\mathbf{a}_i^k)}\Big[\log a_{ij}^k\Big]$
$+\sum_{k=1}^{K}\sum_{i=1}^{J}\sum_{j=1}^{J}(\Lambda_j-1)\mathrm{E}_{q(\mathbf{a}_i^k)}\Big[\log a_{ij}^k\Big]+\sum_{n=1}^{N}(\frac{-1}{2})\Bigg(\log(2\pi\tau^2)+\frac{\mathrm{E}_{q(\mu_n)}\Big[\mu_n^2\Big]-2\theta\mathrm{E}_{q(\mu_n)}\Big[\mu_n\Big]+\theta^2}{\tau^2}\Bigg)$
$+(-\beta)\mathrm{E}_{q(\frac{1}{\sigma^2})}\Big[\frac{1}{\sigma^2}\Big]+(\alpha-1)\mathrm{E}_{q(\frac{1}{\sigma^2})}\Big[\log\frac{1}{\sigma^2}\Big]+\mathrm{E}_{q(\boldsymbol{\pi})}\Big[\log P(\boldsymbol{\pi})\Big]$
$-\sum_{n=1}^{N}\Big[\sum_{k=1}^{K}p_{nk}^*\log p_{nk}^*\Big]-\sum_{k=1}^{K}\sum_{m=1}^{M}\sum_{j=1}^{J}\mathrm{E}_{q(C_{km})}\Big[\mathrm{I}(C_{km}=j)\Big]\mathrm{E}_{q(\mu_n)q(\frac{1}{\sigma^2})}\Big[D_{nmj}\Big]$
$-\sum_{k=1}^{K}\sum_{j=1}^{J}\mathrm{E}_{q(C_{k1})}\Big[\mathrm{I}(C_{k1}=j)\Big]\log\rho_{kj}$

$$- \sum_{k=1}^{K} \sum_{m=2}^{M} \sum_{j=1}^{J} \sum_{i=1}^{J} \mathrm{E}_{q(C_{km-1}, C_{km})} \Big[\mathrm{I}(C_{km-1} = i, C_{km} = j) \Big] \log \frac{\mathrm{E}_{q(C_{km-1}, C_{km})} \Big[\mathrm{I}(C_{km-1} = i, C_{km} = j) \Big]}{\mathrm{E}_{q(C_{km-1})} \Big[\mathrm{I}(C_{km-1} = i) \Big]}$$

$$- \sum_{k=1}^{K} \sum_{i=1}^{J} \Big[\sum_{j=1}^{J} \mathrm{E}_{q(\mathbf{a}_i^k)} \Big[\log a_{ij}^k \Big] \Big\{ \Lambda_j^* - 1 \Big\} \Big]$$

$$- \sum_{n=1}^{N} (\tfrac{-1}{2}) \Big(\log(2\pi\tau_n^{*2}) + 1 \Big)$$

$$- \Big(\alpha^* - \ln \beta^* + \ln \Gamma(\alpha^*) + (1 - \alpha^*) \boldsymbol{\Psi}(\alpha^*) \Big)$$

$$- \mathrm{E}_{q(\boldsymbol{\pi})} \Big[\log q(\boldsymbol{\pi}) \Big]$$

We now approximate the term, $\Big(\alpha^* - \ln \beta^* + \ln \Gamma(\alpha^*) + (1 - \alpha^*) \boldsymbol{\Psi}(\alpha^*) \Big)$ in the above, by using approximations:

1. Stirling's: $\ln \Gamma(\alpha^*) = (\alpha^* - 1) \ln(\alpha^* - 1) - \alpha^* + 1$

2. digamma-approximation: $\boldsymbol{\Psi}(\alpha^*) = \ln(\alpha^*) - \frac{1}{2\alpha^*}$

3. Also: $\ln \Gamma(\alpha^*) = \ln(\frac{\Gamma(\alpha^* + 1)}{\alpha^*}) = (\alpha^* \ln(\alpha^*) - \alpha^*) - \ln(\alpha^*)$

Putting 3 and 2 in the original formula we obtain:

$$\alpha^* - \ln \beta^* + \alpha^* \ln(\alpha^*) - \alpha^* - \ln(\alpha^*) + (\ln(\alpha^*) - \tfrac{1}{2\alpha^*}) - \alpha^* \ln(\alpha^*) + \tfrac{1}{2} = \tfrac{1}{2}(1 - \tfrac{1}{\alpha^*}) - \ln \beta^*$$