Bayesian confidence for drift diffusion observers in dynamic stimuli tasks

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Abstract

Much work has explored the possibility that the drift diffusion model, a model of response times and choices, could be extended to account for confidence reports. Many methods for making predictions from such models exist, although these methods either assume that stimuli are static over the course of a trial, or are computationally expensive, making it difficult to capitalise on trial-by-trial variability in dynamic stimuli. Using the framework of the drift diffusion model with time-dependent thresholds, and the idea of a Bayesian confidence readout, we derive expressions for the probability distribution over confidence reports. In line with current models of confidence, the derivations allow for the accumulation of "pipeline" evidence which has been received but not processed by the time of response, the effect of drift rate variability, and metacognitive noise. The expressions are valid for stimuli which change over the course of a trial with normally distributed fluctuations in the evidence they provide. A number of approximations are made to arrive at the final expressions, and we test all approximations via simulation. The derived expressions only contain a small number of standard functions, and only require evaluating once per trial, making trial-by-trial modelling of confidence data in dynamic stimuli tasks more feasible. We conclude by using the expressions to gain insight into the confidence of optimal observers, and empirically observed patterns.

1 Introduction

How animals and humans make perceptual decisions is of fundamental interest. It is increasingly recognised that decision confidence, an estimate of the probability a decision was correct, is both theoretically important and used by humans in a variety of ways (Bahrami et al., 2010; Desender, Boldt, Verguts & Donner, 2019; Desender, Boldt & Yeung, 2018; Drugowitsch, Mendonça, Mainen & Pouget, 2019; Sanders, Hangya & Kepecs, 2016; van den Berg, Zylberberg, Kiani, Shadlen & Wolpert, 2016). Confidence has also been linked to psychological disorder (Hauser, Allen, Rees & Dolan, 2017; Rouault, Seow, Gillan & Fleming, 2018). We aim to gain insight into the confidence of optimal observers. Specifically, we derive predictions for the probability distribution over confidence reports, given the response and response time on a trial. We aim for computationally cheap predictions which make trial-by-trial modelling of dynamic stimuli feasible. Working from the assumption of optimality in both decisions and confidence, we model confidence reports as noisy Bayesian readouts of the probability of being correct (Kiani & Shadlen, 2009; Moreno-Bote, 2010; Sanders et al., 2016), and work within the normative decision framework of the drift diffusion model (also known as the diffusion decision model; DDM; Ratcliff and McKoon, 2008).

The DDM is one of the most prominent models of two-alternative decisions, from a family of models in which observers receive noisy measurements of evidence for the two options (Bogacz, Brown, Moehlis, Holmes & Cohen, 2006; Green & Swets, 1966; Ratcliff & McKoon, 2008). In the DDM, observers track the difference in evidence measurements between the two options. That is, for each sample, observers subtract the measurement for option B from the measurement for option A, and add this to a running total (Ratcliff & McKoon, 2008). When the accumulator tracking this difference reaches a fixed threshold (positive or negative), a response is triggered. The DDM has successfully been used to model decisions in a wide range of tasks (Ratcliff, Smith, Brown & McKoon, 2016).

The DDM is also a normative model of decision making. If evidence measurements for the two options are equally reliable and signal strength is known, the DDM is equivalent to tracking the posterior prob-

ability of each option until a fixed threshold on these probabilities is reached (Bitzer, Park, Blankenburg & Kiebel, 2014; Gold & Shadlen, 2007; Moran, 2015). In such a context, this policy is optimal in the sense that, on average, no other policy can achieve the same level of performance with fewer evidence measurements (Wald & Wolfowitz, 1948). When signal strength is unknown, a time dependent threshold is required, but it remains optimal to track the difference between the two accumulators (Drugowitsch, Moreno-Bote, Churchland, Shadlen & Pouget, 2012; Tajima, Drugowitsch, Patel & Pouget, 2019). For an intuition of why this policy is optimal, consider a case in which the observer hasn't made a decision after lengthy deliberation. The observer must be accumulating evidence very slowly, suggesting to them that signal strength is very low. If the observer thinks signal strength is very low, there is almost nothing to gain from collecting more evidence measurements, so they should lower their decision threshold and make an immediate decision (Malhotra, Leslie, Ludwig & Bogacz, 2017).

While the DDM has optimal characteristics, and has been successfully applied to a wide range of decisions (Ratcliff et al., 2016), it is not clear the DDM provides an adequate account of confidence reports. Different ways of modelling confidence using the DDM have been proposed (Yeung & Summerfield, 2014). In one set of models, observers use some form of heuristic, based on variables which are directly accessible in the DDM, such as the state of the accumulator (Pleskac & Busemeyer, 2010), or the time taken to make a decision (Zylberberg, Barttfeld & Sigman, 2012). Another approach is to assume that observers map the state of the accumulator, and decision time, to the probability they are correct (Kiani, Corthell & Shadlen, 2014; Kiani & Shadlen, 2009; Moreno-Bote, 2010). A Bayesian readout of this kind could be learned over time, through the association of accumulator state and time with success or failure (Kiani & Shadlen, 2009). Alternatively, a Bayesian readout could reflect a probabilistic inference made using knowledge of the statistical structure of the task.

There are several techniques which can be used for calculating the probability, according to the DDM, of different responses and response times (Brown, Ratcliff & Smith, 2006; Chang & Cooper, 1970; Cox & Miller, 1965; Diederich & Busemeyer, 2003; Drugowitsch, 2016; Navarro & Fuss, 2009; Smith, 2000; Tuerlinckx, 2004; Tuerlinckx, Maris, Ratcliff & De Boeck, 2001; Voss & Voss, 2008). Importantly, approaches which can handle dynamic stimuli (stimuli which change over the course of a trial), and time-dependent thresholds, have been developed. One approach involves using finite difference methods to approximate the evolution of the probability distribution over accumulator state (which reflects the accumulated difference in evidence measurements; Chang and Cooper, 1970; Voss and Voss, 2008; Zylberberg, Wolpert and Shadlen, 2018). Time and space are discretised and, working forward from the first time step, we solve a set of simultaneous equations at each time step, to find the evolution of the probability distribution over accumulator state. If we are only interested in the probability distribution over response times and choices, we can use expressions described by Smith (2000). To evaluate these expressions we only need to discretise time, not space. Again working forward from the first time step, we can calculate the probability of deciding at each time step. Both approaches require that we discretise the time course of a trial into small time steps. Hence, in both approaches, we must perform a large number of computations, on the order of the number of time steps considered.

This computational cost becomes important if we want to leverage trial-by-trial variability in dynamic stimuli. Typically, computation time for calculating predictions is reduced by making predictions on a condition-by-condition basis (e.g. Kiani et al., 2014; Ratcliff & McKoon, 2008; van den Berg, Anandalingam et al., 2016; Zylberberg, Fetsch & Shadlen, 2016). We design the experiments so that there are a small number of different conditions (e.g. levels of contrast), then we treat all trials from a single condition as the same, and make predictions for behaviour in each condition, rather than for each stimulus individually. This approach does not capitalise on trial-by-trial variability of the dynamic stimuli which are often used. These random fluctuations in evidence make every trial unique. Hence, a method which reduced the computational cost of making trial-by-trial predictions for dynamic stimuli could be of great benefit. A simple mathematical expression would provide additional advantages, such as insight into the mechanisms and properties of confidence in DDM models.

Moreno-Bote (2010) derived expressions for confidence which take into account time-dependent thresholds, and which could be extended to account for dynamic stimuli of the kind we consider below. However, these derivations use two assumptions about the computation of confidence which are not in line with recent findings. First, Moreno-Bote (2010) made the intuitive assumption that decisions and confidence are based on the same information. However, as sensory and motor processing takes time, there will be stimulus information in these processing "pipelines" which does not contribute to the

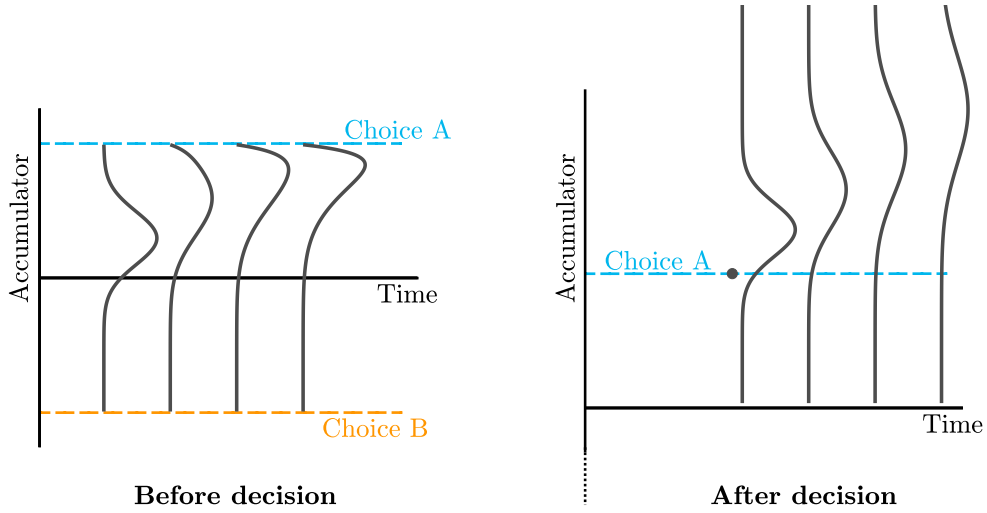

Figure 1: Probability distributions over accumulator state before and after a decision. Even if increments in the accumulator are normally distributed, prior to a decision, the probability distribution over the state of the accumulator quickly becomes non-normal. This is because, if we get to time t without a decision, we know the accumulator has not been beyond either decision threshold prior to t. Following a decision, normally distributed increments in the accumulator lead to a normal distribution over accumulator state, because there are no longer decision thresholds.

initial decision, but does inform confidence (Charles & Yeung, 2019; Moran, Teodorescu & Usher, 2015; Ratcliff & McKoon, 2008; Resulaj, Kiani, Wolpert & Shadlen, 2009; van den Berg, Anandalingam et al., 2016). Moreover, information in the processing pipeline is affected by trial-to-trial fluctuations in signal strength, and this may have important effects on confidence (Pleskac & Busemeyer, 2010). Second, there is now substantial evidence that the process which "reads out" confidence into a behavioural report is corrupted by "metacognitive noise" (Bang, Shekhar & Rahnev, 2019; De Martino, Fleming, Garrett & Dolan, 2013; Maniscalco & Lau, 2012; van den Berg, Yoo & Ma, 2017).

Our contribution is not to provide new expressions for the probability distribution over response times and decisions in diffusion models. Instead, we note that even if we cannot find simple expressions for the probability of decisions and response times, we may still be able to find such expressions for confidence, given a response and a response time. Prior to a decision, even if changes to the state of the accumulator are normally distributed over small intervals of time, the probability distribution over the state of the accumulator will not be normal. This is because, having reached a time t without a response, we know that the accumulator is not beyond either threshold, nor has it been at any point up to t (otherwise the observer would already have made a decision; Moreno-Bote, 2010). This constraint results in non-normal probability distributions over accumulator state (Fig. 1). Following a decision, decision boundaries are no longer relevant, hence normally distributed changes in the state of the accumulator lead to a normal distribution over this state (Fig. 1).

Using this insight we derive approximate expressions for the probability, according to the DDM, of different confidence reports, given the response and response time on a trial. As discussed, the framework of the DDM with possibly time-dependent thresholds, includes the optimal decision policy, whether or not signal strength is known by the observer. Confidence is allowed to be a noisy readout of the probability of being correct, and we account for the effects of pipeline evidence, and variability in signal-to-noise ratio. The derived expressions only need evaluating once per trial, instead of at each time step, and allow for dynamic stimuli of a certain form, thereby making trial-by-trial modelling of such stimuli feasible.

Symbol	Meaning
S	Stimulus (1 or 2)
R	Response (1 or 2)
C	Confidence
μ_i	Mean stimulus evidence for option i
μ	Difference between mean stimulus evidence for the two options
$\overline{\mu}$	Reference value (mid-point between mean stimulus evidence for the two options)
$\Delta\mu$	Magnitude of the difference between the two means
E_{ij}	Evidence for option i presented in stimulus frame j
δE_{ij}	Evidence presented for i in time step j. $E_{ik} = \sum_j \delta E_{ij}$ (sum over time steps in frame k)
E_i	Evidence for option 2 minus evidence for option 1 in frame i
E	Sum over all E_i
$\boldsymbol{E}$	Vector of all E_i
$\overline{E}$	Average difference in evidence between the two options, prior to decision
ΔE	Sum over $\delta E_{2i} - \delta E_{1i}$ for time steps following the decision
φ	Standardised drift rate (Quality of evidence processing)
δx_{ij}	Noisy measurement of option i in time step j
δx_i	Difference between the measurements for the two options ($\delta x_{2i} - \delta x_{1i}$)
$\delta \overline{x}_i$	Sum of the measurements for the two options ($\delta x_{2i} + \delta x_{1i}$)
$\boldsymbol{\delta x}$	Vector containing δx_i for every i
x	Accumulated difference in measurements, $\sum_i \delta x_i$
x_d	The value of x at t_d (i.e. height of relevant decision threshold at this time)
Δx	The change in x following a decision (at t_d)
x_{ll}	Log posterior ratio
x_c	Noisy measurement of x_{ll} which determines confidence
d_i	Boundaries on x_c which separate confidence reports falling into different bins
σ_E	Standard deviation of evidence in a frame
σ_{acc}	Standard deviation of accumulator noise
σ_φ	Standard deviation of φ
σ_m	Standard deviation of metacognitive noise
t_f	Duration of a frame
t_e	Duration of evidence presentation
t_d	Duration from onset of accumulation to first crossing of a decision threshold
t_r	Response time
I	The interval $t_e - t_d$, the duration of the evidence pipeline
ν	$\varphi(\mu_2 - \mu_1)/t_f$
$\overline{\nu}$	$\varphi(\mu_2 + \mu_1)/t_f$
ν_0	$\Delta\mu/t_f$
σ_ν	$\nu_0 \sigma_\varphi$
s^2	$\sigma_{acc}^2 + (\sigma_E^2/t_f)$
$\theta(t)$	$(s^2 + t\sigma_\nu^2)/2\nu_0$

Table 1: Symbols used in the derivations, along with key abbreviations.

2 Model

Overview

Our aim is to derive expressions for the probability distribution over confidence, given the response and response time on a trial. In order to derive these predictions we must first specify a model for decisions and confidence.

We consider a situation in which observers must make a choice between two alternatives. The presented stimulus provides two evidence signals, one for each option, and evidence provided by the stimulus can vary over time (Fig. 2; Bogacz et al., 2006; Moreno-Bote, 2010). For example, the observer might be presented with two clouds of dots, with the number of dots in each cloud constantly changing. Their task might be to determine which of the two clouds contains the most dots on average (Charles & Yeung, 2019). Here the dots in the two clouds would correspond to the two streams of evidence. We assume that the observer only receives noisy measurements of the presented evidence. In our model, consistent with the DDM, the observer takes the difference between the noisy measurements of evidence for the two options, and accumulates this difference (Fig. 2).

Typically, following stimulus onset, a participant can respond whenever they wish. Some instruction or incentive may be given to respond in a certain way, such as fast and accurately, but beyond this the participant is free to set the time of response. The stimulus continues to be presented until a response is made. We call this condition "free response" (McMillen & Holmes, 2006). We assume the observer sets two thresholds on the accumulator state, one for each choice (Bogacz et al., 2006; Ratcliff & McKoon, 2008). When the accumulator reaches one of these thresholds, the corresponding response is triggered (Fig. 2). As discussed in the introduction, at the time of response, measurements of evidence which has recently been presented will still be in sensory and motor processing pipelines, and hence will not contribute to a decision (Resulaj et al., 2009). These measurements will be processed immediately following a response, and be used to inform confidence (van den Berg, Anandalingam et al., 2016).

We also consider the "interrogation" condition (McMillen & Holmes, 2006), where the observer must respond at a time controlled by the researcher. Here we assume the stimulus is presented for a finite amount of time. Before the stimulus clears the participant cannot respond. Once the stimulus clears, the observer uses the final state of the accumulator to determine their response (Fig. 2). In all conditions, the observer uses a Bayesian readout of confidence which depends on the final state of the accumulator, and the time spent accumulating evidence (Kiani et al., 2014; Moreno-Bote, 2010).

In the remainder of this section we set out a model for confidence. First, we describe the observer's task, and the beliefs held by the observer. Second, we find the rule a Bayesian observer would use to map evidence measurements to a decision and confidence. Third, we test the effect of the beliefs we ascribe to the observer. Fourth, we describe the "read out" process which determines confidence. In section 3 we turn to the main aim of the paper, which is to use the drift diffusion framework to derive simple expressions for the probability distribution over confidence reports, given a response and response time. A summary of symbols used in the derivations can be found in Table 1.

Task and observer beliefs

Our first step is to describe the observer's task mathematically, and to describe beliefs about the task we assume the observer holds. The observer's task is to determine the correct response, by inferring which evidence stream is drawn from the distribution with the greater mean (Bogacz et al., 2006; Moreno-Bote, 2010). We consider the case where the two options are equally probable. Denote the mean evidence for the two options, μ_1 and μ_2, and the difference between these means, $\mu = \mu_2 - \mu_1$. $\Delta\mu$ is the magnitude of the difference between the two means. This value is fixed, but we incorporate variability in signal strength below (see discussion of standardised drift rate variability, φ). We also allow for the possibility that the sum of the means for the two streams, $\overline{\mu} = \mu_1 + \mu_2$, independently varies according to some

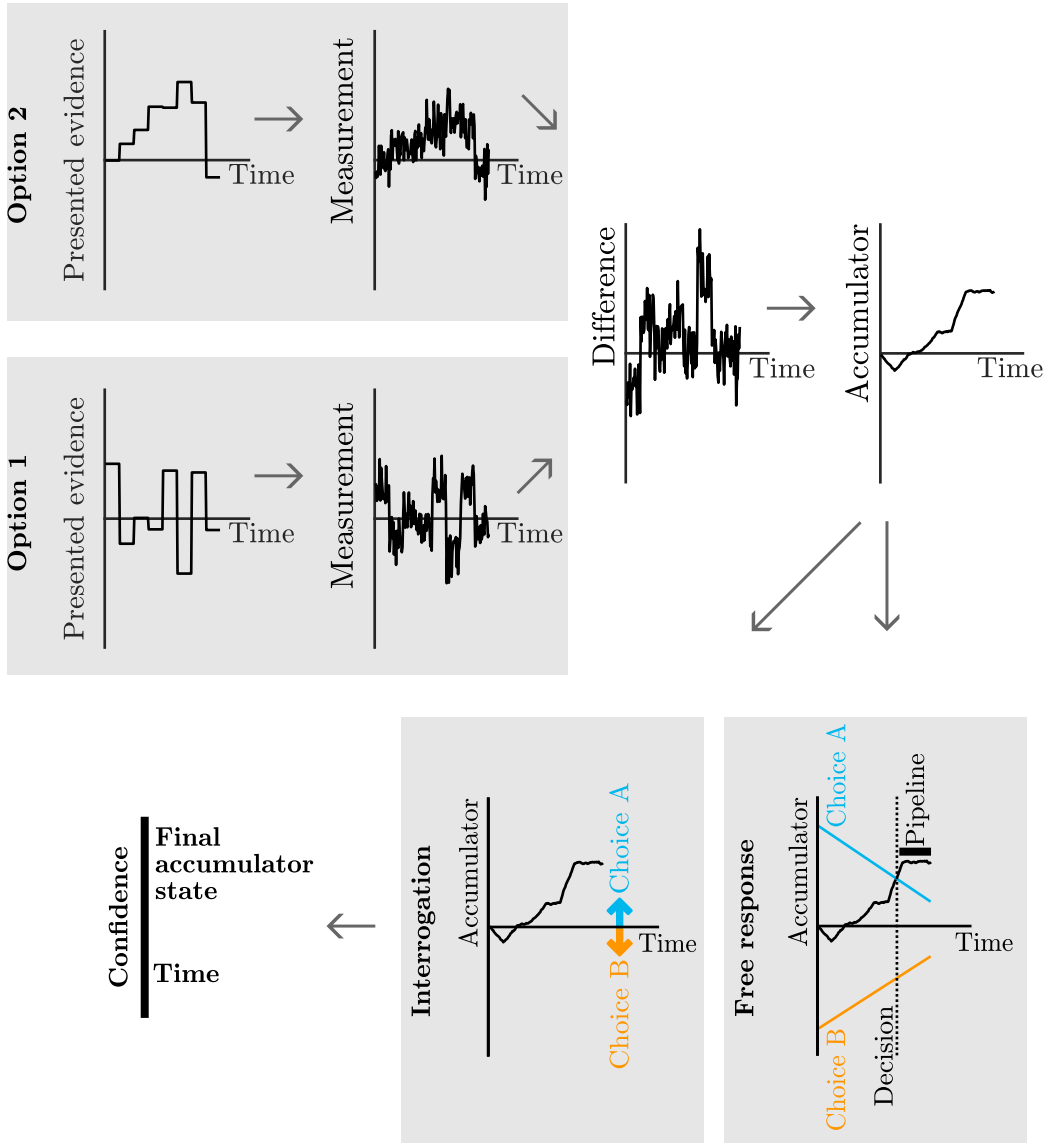

Figure 2: The model of confidence and decisions. Time varying evidence is presented for the two response options. The observer only receives noisy measurements of the evidence presented. The observer takes the difference between the evidence measurements and accumulates this difference. In the interrogation condition the duration of the stimulus is set by the researcher. The observer accumulates noisy measurements until the stimulus ends and all evidence is processed. Then the observer simply picks the option which is favoured by the accumulated measurements. In the free response condition the observer uses decision thresholds, one for each option, which trigger a response. Following a decision, evidence measurements in the processing pipeline are accumulated, and confidence is informed by the accumulator state at the time of threshold crossing, plus changes to the accumulator caused by evidence measurements from the pipeline.

distribution $p(\overline{\mu})$. These assumptions give us,

$$p(S=1) = p(S=2) = \frac{1}{2} \tag{1}$$
$$p(\mu|S=1) = \delta(\mu+\Delta\mu) \tag{2}$$
$$p(\mu|S=2) = \delta(\mu-\Delta\mu)\,. \tag{3}$$

S denotes the stimulus with the greater mean evidence, and hence the correct answer.

In the free response condition the observer sets decision thresholds to trigger a response. Denote the time spent accumulating measurements prior to the first crossing of a decision threshold, t_d, and the time of the response relative to the beginning of the trial t_r. At the time of response, recent evidence measurements are still in sensory and motor processing pipelines. Hence, information presented in the stimulus immediately prior to the response will not contribute to the decision. Denote the duration of the stimulus immediately prior to the response which does not contribute to the decision, because corresponding measurements are still being processed, I. In both the free response and interrogation conditions we denote the total time during which the stimulus is presented t_e. In the free response condition $t_e = t_r$, as a response triggers the end of the stimulus. In the models, observers use all the information the received in t_e to inform their confidence.

We consider here the general case of time-varying evidence in the stimulus. Our derivations also apply to time-constant evidence as a special case of time-varying. We consider evidence which is piecewise-constant within short stimulus "frames" of duration t_f. Consider a discretisation of time into very short time steps (much shorter than the duration of a frame). We use E_{ij} to denote the evidence presented for option i during frame j. If the evidence presented in each frame is normally distributed around the underlying mean, then we have,

$$p(E_{ij}|\mu_i) = N(E_{ij};\mu_i,\frac{\sigma_E^2}{2})\,, \tag{4}$$

where $\sigma_E^2/2$ is the variance over presented evidence. Note that μ_1 and μ_2 do not correspond to anything the observer directly observes. μ_1 and μ_2 are the underlying mean evidence for the two options, and are constant throughout a trial. The actual presented evidence, E_{ij}, is what varies over the course of a trial, and is drawn from a distribution centred on the underlying means.

While in principle observers could leverage knowledge that evidence is constant throughout each frame, we assume that observers ignore this additional structure. This will be a valid approximation when evidence frames used in an experiment are sufficiently short. In this case, the approximation will lead to little discrepancy between the observer's estimate of the probability of being correct, and the true probability. We test this claim below, once we have derived an expression for the probability of being correct (Fig. 3, and equation 27). Instead of accounting for the constant evidence over the course of a frame, we assume that observers treat the fraction of evidence received in a small time step, j, of duration δt, as generated according to,

$$p(\delta E_{ij}|\mu_i) = N(\delta E_{ij};\mu_i\frac{\delta t}{t_f},\frac{\sigma_E^2}{2}\frac{\delta t}{t_f})\,, \tag{5}$$

where, δE_{ij} indicates the evidence presented for option i in time step j only, not over the course of a frame ($E_{ik} = \sum_j \delta E_{ij}$ where the summation is taken over all time steps in frame k). Importantly, the observer incorrectly assumes that the δE_{ij} at different time steps, j, are independent (i.e. the observer ignores the fact that evidence is constant through all time steps in a single frame).

The presented evidence drives a noisy accumulation, such that over small time step, j, the incremental change in the state of the accumulator for option i, denoted δx_{ij}, can be described by,

$$p(\delta x_{ij}|\delta E_{ij},\varphi) = N(\delta x_{ij};\delta E_{ij}\varphi,\frac{\sigma_{acc}^2}{2}\delta t)\,. \tag{6}$$

where σ_{acc} is the variance of noise in the accumulation (Drugowitsch et al., 2012). φ is a random variable which accounts for variability in drift rate. "Drift rate" refers to the rate at which evidence presented in the stimulus drives the accumulation of evidence measurements (Ratcliff & McKoon, 2008). Where

a stimulus, and hence drift rate, is constant over the course of a trial, drift rate variability is trial-to-trial variability in this rate (Ratcliff & McKoon, 2008; Ratcliff & Smith, 2004; Voskuilen, Ratcliff & Smith, 2016). Here, where stimulus evidence, and hence drift rate, varies over the course of each trial, we operationalise drift rate variability as a multiplicative factor which determines how well stimulus information is processed. To distinguish this operationalisation from the usual operationalisation, we refer to the multiplicative factor as standardised drift rate, because the mean value of this variable will be 1, regardless of the strength of the presented evidence. When standardised drift rate is high, there is a strong signal from the stimulus, and evidence from the stimulus is accumulated rapidly. Noise in the accumulation is unaffected, hence, a higher standardised drift rate also leads to a higher signal-to-noise ratio.

It is usually assumed that drift rate variability follows a normal distribution (Ratcliff & McKoon, 2008). We make the same assumption here,

$$p(\varphi) = N(\varphi; 1, \sigma_\varphi^2)\,. \tag{7}$$

We assume that observers correctly model the effect of internal noise (accumulator noise, and standardised drift rate variability). Hence observers infer that increments in the accumulators are related to the underlying signal as follows,

$$\begin{aligned} p(\delta x_{ij}|\mu_i, \varphi) &= \int d(\delta E_{ij})\; p(\delta x_{ij}|\delta E_{ij}, \varphi) p(\delta E_{ij}|\mu_i) && (8)\\ &= \int d(\delta E_{ij})\; N(\delta x_{ij}; \delta E_{ij}\varphi, \frac{\sigma_{acc}^2}{2}\delta t) N(\delta E_{ij}; \mu_i \frac{\delta t}{t_f}, \frac{\sigma_E^2}{2}\frac{\delta t}{t_f}) && (9)\\ &= N(\delta x_{ij}; \mu_i \varphi \frac{\delta t}{t_f}, \frac{\delta t}{2}(\sigma_{acc}^2 + \varphi^2 \frac{\sigma_E^2}{t_f}))\,, && (10) \end{aligned}$$

To reach the final line we rearranged, and applied the result in appendix A. It is interesting to note that the standardised drift rate, φ, not only affects the mean increment, but also the variability of increments, through the term $\varphi^2\sigma_E^2/t_f$. This effect arises because high standardised drift rate increases the effect of the stimulus on the accumulation, increasing the effect of both the stimulus mean, and variability in the stimulus. We assume that participants ignore this difference between stimulus variability and internal variability (which is not affected by standardised drift rate). Hence we assume that observers believe,

$$p(\delta x_{ij}|\mu_i, \varphi) = N(\delta x_{ij}; \mu_i \varphi \frac{\delta t}{t_f}, \frac{\delta t}{2}(\sigma_{acc}^2 + \frac{\sigma_E^2}{t_f}))\,. \tag{11}$$

We test the effect of this assumption on the calibration of confidence below (Fig. 3).

It is simpler to perform calculations with transformed variables $\delta x_i = \delta x_{2i} - \delta x_{1i}$ and $\delta \overline{x}_i = \delta x_{2i} + \delta x_{1i}$. This is because the observer's task involves detecting the difference between the means of the stimuli, μ_1 and μ_2, so the difference between accumulator samples is likely to be the most important variable. As δx_{1i} and δx_{2i} are normally distributed, we can compute their sum using standard formulas. Using (11), this gives,

$$\begin{aligned} p(\delta x_i|\mu_1, \mu_2, \varphi) &= N(\delta x_i; \varphi(\mu_2 - \mu_1)\frac{\delta t}{t_f}, \delta t(\sigma_{acc}^2 + \frac{\sigma_E^2}{t_f})) && (12)\\ p(\delta \overline{x}_i|\mu_1, \mu_2, \varphi) &= N(\delta \overline{x}_i; \varphi(\mu_2 + \mu_1)\frac{\delta t}{t_f}, \delta t(\sigma_{acc}^2 + \frac{\sigma_E^2}{t_f})) && (13) \end{aligned}$$

To simplify these expressions we change variables to

$$\begin{aligned} \nu &= \frac{1}{t_f}\varphi(\mu_2 - \mu_1) = \frac{1}{t_f}\varphi\mu && (14)\\ \overline{\nu} &= \frac{1}{t_f}\varphi(\mu_2 + \mu_1) = \frac{1}{t_f}\varphi\overline{\mu} && (15)\\ s^2 &= \sigma_{acc}^2 + \frac{\sigma_E^2}{t_f}\,. && (16) \end{aligned}$$

This gives,

$$p(\delta x_i|\nu) = N(\delta x_i; \nu\delta t, s^2\delta t) \tag{17}$$
$$p(\delta \overline{x}_i|\overline{\nu}) = N(\delta \overline{x}_i; \overline{\nu}\delta t, s^2\delta t) \tag{18}$$

We can compute the probability distributions over ν and $\overline{\nu}$ given S using (2), (3) and standard formulas for distributions over products of random variables, giving,

$$p(\nu|S=1) = N(\nu; -\frac{\Delta\mu}{t_f}, \frac{\Delta\mu^2}{t_f^2}\sigma_\varphi^2) = N(\nu; -\nu_0, \sigma_\nu^2) \tag{19}$$
$$p(\nu|S=2) = N(\nu; \frac{\Delta\mu}{t_f}, \frac{\Delta\mu^2}{t_f^2}\sigma_\varphi^2) = N(\nu; \nu_0, \sigma_\nu^2) \tag{20}$$
$$p(t_f\overline{\nu}|S) = \int d\overline{\mu}\; p(\overline{\mu})p(\varphi = \frac{t_f\overline{\nu}}{\overline{\mu}})\frac{1}{|\overline{\mu}|} \tag{21}$$

where $\nu_0 = \frac{\Delta\mu}{t_f}$, and $\sigma_\nu = \frac{\Delta\mu}{t_f}\sigma_\varphi$. Note that $\overline{\nu}$ is independent of S.

In summary, observers are aiming to infer S, which affects ν via,

$$p(\nu|S=1) = N(\nu; -\nu_0, \sigma_\nu^2) \tag{22}$$
$$p(\nu|S=2) = N(\nu; \nu_0, \sigma_\nu^2)\,. \tag{23}$$

They receive two independent streams of evidence which they assume are related to ν via,

$$p(\delta x_i|\nu) = N(\delta x_i; \nu\delta t, s^2\delta t) \tag{24}$$
$$p(\delta \overline{x}_i|\overline{\nu}) = N(\delta \overline{x}_i; \overline{\nu}\delta t, s^2\delta t)\,. \tag{25}$$

As $\delta\overline{x}$ only depends on $\overline{\nu}$, which is independent of S, $\delta\overline{x}$ provides no information regarding S, and can be ignored by the observer without any loss of relevant information. Hence, as described in the model overview, the observer only tracks the difference between evidence measurements, not their sum.

The Bayesian observer

Having specified the properties of the task and the beliefs of the observer, we can now infer the rule used by a Bayesian observer to translate all evidence measurements gathered into a decision and confidence. In both the free and interrogation conditions, a Bayesian observer uses all measurements, δx_i, received over the duration of stimulus presentation, t_e, to determine their response and confidence. We denote the vector of all evidence measurements, $\delta x_1, \delta x_2, ..., \delta x_N$ as $\boldsymbol{\delta x}$. Inferring the posterior probability over S, given $\boldsymbol{\delta x}$, in the environment described by (22), (23), and (24), is an inference problem which has been considered before. Moran (2015), using a result from Drugowitsch et al. (2012), derived the posterior probability over the two options $S = 1$ and $S = 2$. We also provide a derivation in appendix B, and simply state the result here. Using Bayes rule with all evidence samples, the log posterior ratio is given by (appendix B; Moran, 2015),

$$x_{ll} = \log\frac{p(S=2|\boldsymbol{\delta x})}{p(S=1|\boldsymbol{\delta x})} = \frac{2x\nu_0}{s^2 + t\sigma_\nu^2} = \frac{x}{\theta(t)}\,, \tag{26}$$

where $x = \sum_{i=1}^{N}\delta x_i$, while x_{ll} and $\theta()$ provide abbreviations, and t is the time spent accumulating evidence.

A Bayesian observer would report whichever option is more likely. Hence, they will report $S = 1$ when $x_{ll} < 0$, which is the same as $x < 0$. Denote this report $R = 1$, and a report for $S = 2$ as $R = 2$. In the interrogation condition, the observer simply has to wait for the stimulus to end at t_e. Once the observer has accumulated all evidence measurements, taking t_e, the observer can respond according to the sign of the accumulator state, x (Fig. 2). In the free response condition, the observer uses a decision threshold for triggering a response (Bogacz et al., 2006; Ratcliff & McKoon, 2008). This threshold describes an accumulator magnitude, $|x|$, which when reached, triggers a response. We allow the threshold to vary

with time (Drugowitsch et al., 2012). As discussed, at the time of response, measurements corresponding to recently presented evidence will still be in the processing pipeline (Resulaj et al., 2009). The response will be based on x at the time of the decision, t_d, while confidence will incorporate additional pipeline evidence measurements (Fig. 2). Processing will continue until measurements from the full duration of stimulus presentation, t_e, have been processed.

The observer can use the log posterior ratio in the computation of confidence, as it is monotonically related to the probability they are correct through,

$$p(\text{correct}|\boldsymbol{\delta x}) = p(S = 1|\boldsymbol{\delta x}) = \frac{1}{1 + e^{x_{ll}}} = \frac{1}{1 + e^{\frac{x}{\theta(t_e)}}}\,, \tag{27}$$

if the observer reports $R = 1$. A similar expression holds for $R = 2$, the only difference being that x is replaced with $-x$, and x_{ll} with $-x_{ll}$.

Testing the observer's beliefs

Above, we assumed that observers approximate the true generative model for the evidence measurements. Specifically we assumed, (a) observers ignore the fact that evidence is constant within each frame, and (b) observers ignore the increased effect of variability in the stimulus, when standardised drift rate is high, and the decreased effect when standardised drift rate is low. While the assumptions may be plausible, it is also plausible that, if given feedback, observers could learn to closely approximate the mapping from x and t_e to probability correct (Kiani & Shadlen, 2009). To adequately capture this situation we need to ensure that, under our assumptions about the observer, confidence remains closely related to performance.

To test the two approximations we simulated the diffusion process using small time steps. At each time step the accumulator increment, δx_i, was determined by drawing a value consistent with (6), until a decision threshold was crossed (Tuerlinckx et al., 2001). Accumulation continued until all evidence measurements had been processed, at which point a simulated confidence was determined in accordance with (27). Full details of the simulation can be found in appendix C.

Fig. 3 shows accuracy against confidence for two cases, one in which there is no variability in standardised drift rate, φ, and one with variability. Full details of the process used to plot the data is also provided in appendix C. Only one approximation is relevant when there is no variability in standardised drift rate, approximation (a). This approximation appears to have little effect on the mapping between accuracy and confidence. An additional approximation is relevant when there is variability in standardised drift rate, assumption (b). We can see that this causes a moderate mismatch between accuracy and confidence (Fig. 3).

It is interesting to note the latter approximation, that observers ignore the effect of drift rate on stimulus variability, is implicitly present in much previous work. This is because dynamic stimuli are often used, but stimulus variability is not treated separately from accumulator variability in the computational models applied (e.g. Kiani et al., 2014; Ratcliff and McKoon, 2008; van den Berg, Anandalingam et al., 2016, although see Zylberberg et al., 2016). Fig. 3 suggests this approximation does not cause large issues.

Confidence reporting model

Having described the task, observer beliefs, tested those beliefs, and found the confidence of a Bayesian observer, we turn to the question of what exact quantity confidence reports reflect. While the log-posterior ratio, x_{ll} is monotonically related to the probability of being correct, we do not assume that confidence is a direct readout. Instead, we allow the possibility that metacognitive noise corrupts this estimate (Maniscalco & Lau, 2012), and hence that confidence is based on a noisy representation of x_{ll}, denoted x_c,

$$p(x_c|x_{ll}) = N(x_c; x_{ll}, \sigma_m^2)\,, \tag{28}$$

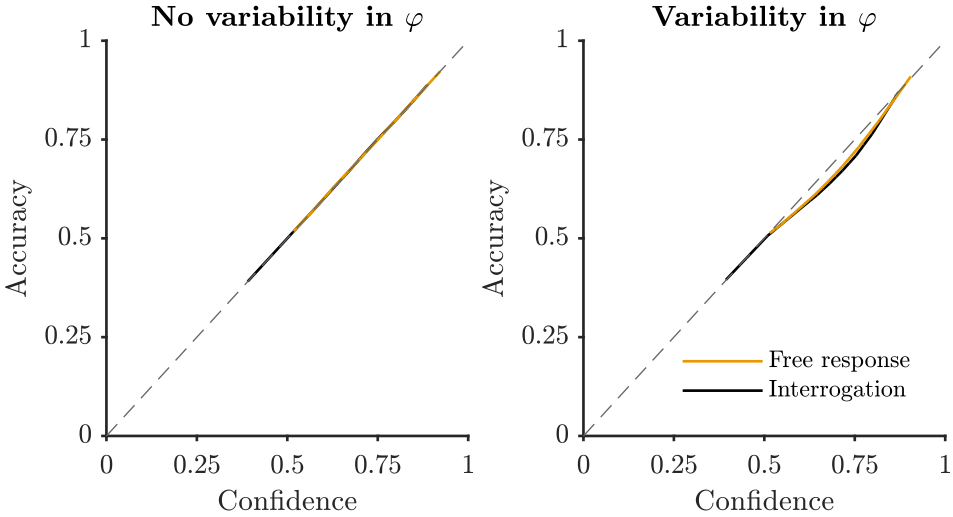

Figure 3: Accuracy as a function of confidence, where confidence is calculated using the observer's generative model of their evidence measurements. This model only approximates the true generative model. When there is variability in standardised drift rate, the approximations cause the observer's confidence to show some deviations from calibration.

where σ_m is the standard deviation of metacognitive noise.

We would also like to minimise the number of assumptions we make about how x_c is transformed into a confidence report. There is evidence that different people use confidence scales in different ways (Ais, Zylberberg, Barttfeld & Sigman, 2016; Navajas et al., 2017). To make minimal assumptions about how people view and treat confidence scales, we treat confidence reports, C, as ordinal data only. Confidence reports on a continuous scale can be analysed by binning them first.

If people report greater confidence when x_c favours their decision to a greater extent, then all confidence reports falling into a higher confidence bin will have come from further along the x_c scale (in the direction that favours the choice made). Using d_i we denote the boundary on x_c which separates the confidence reports which fall into confidence category $C = i - 1$ from $C = i$, when the observer reports $R = 2$. When $R = 1$, the boundary applies to $-x_c$, or equivalently, a boundary of $-d_i$ is applied to x_c.

3 Results

We now have a complete description of the model, and everything we need to derive the probability distribution over confidence reports in both the interrogation and free response conditions. We would like to find the probability distribution over confidence, given the evidence presented, $\boldsymbol{E}$, the response, R, and in the free response condition, the amount of time the observer monitors the stimulus before making a response, t_r. ($\boldsymbol{E}$ is a vector containing every $E_i = E_{2i} - E_{1i}$.) A key variable is the observer's log posterior ratio after they have seen all evidence, x_{ll}. Our general strategy will be to find a probability distribution over this variable. From this distribution we will be able to infer a distribution over the noisy readout of the log-posterior ratio, x_c. As described in the previous section, on a trial with a response $R = 2$, if x_c falls between d_i and d_{i+1} the observer reports confidence $C = i$. If $R = 1$, the boundaries are $-d_{i+1}$ and $-d_i$. The probability of a confidence report $C = i$ will be given by the probability that x_c falls between the corresponding boundaries (Fig. 4).

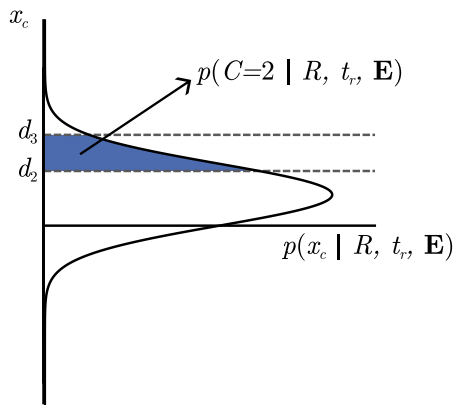

Figure 4: Probability of a confidence report, based on a distribution over x_c. x_c is a noisy representation of the log-posterior ratio. We make no specific assumptions about how observers use confidence scales apart from assuming that, if the observer reports higher confidence C, then the underlying variable x_c has a greater absolute value, in the direction corresponding to the response made. We use d_i to denote the boundaries between values of x_c that lead to different confidence reports. If we know the distribution over x_c, then the probability of a specific confidence report can be found by integrating x_c or $-x_c$ between the corresponding boundaries. Whether we integrate over x_c or $-x_c$ depends on the response made.

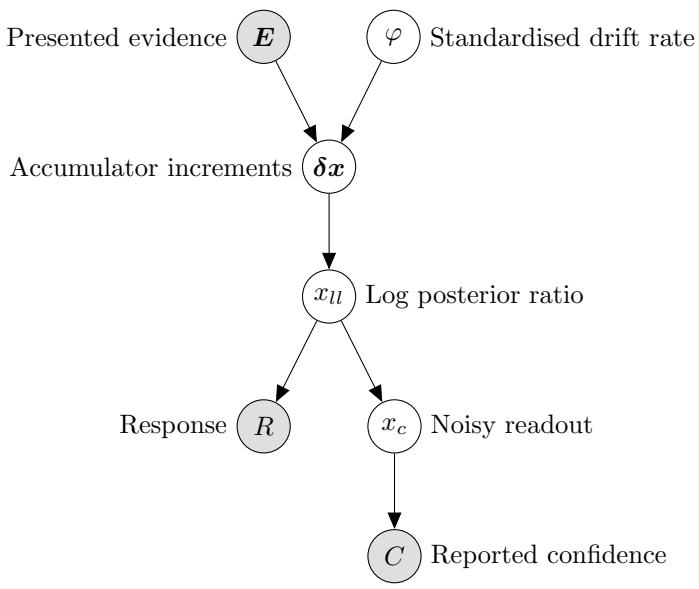

Figure 5: Representation of the generative model for confidence reports in the interrogation condition, from the perspective of the researcher. We want to infer the probability of reported confidence, using our knowledge of the evidence presented and the response given.

Interrogation condition

In a trial from the interrogation condition, the stimulus is presented for some amount of time, t_e. The observer can only respond after the end of the stimulus. We aim to find the probability of confidence reports, given the response and evidence presented. Assuming the response occurs at some fixed amount of time following t_e, the response time provides us with no information. This is because t_e is set by the researcher, and hence unaffected by processes internal to the observer. A summary of the generative model for interrogation condition confidence reports, from the perspective of the researcher, is shown in Fig. 5.

We start by integrating x_c over the region which leads to a confidence report $C = i$. When the response is $R = 2$, this is the region between d_i and d_{i+1} (Fig. 4). The case where $R = 1$ is identical except the limits become $-d_{i+1}$ and $-d_i$. Also marginalising over, x_{ll} and using Bayes rule,

$$p(C = i|R, \boldsymbol{E}) = \int_{d_i}^{d_{i+1}} dx_c \int dx_{ll}\ p(x_c|x_{ll})p(x_{ll}|R, \boldsymbol{E}) \tag{29}$$

$$= \int_{d_i}^{d_{i+1}} dx_c \int dx_{ll}\ \frac{p(x_c|x_{ll})p(R|x_{ll})p(x_{ll}|\boldsymbol{E})}{p(R|\boldsymbol{E})}\,. \tag{30}$$

We want to find,

$$I(x_c) = \int dx_{ll}\ p(x_c|x_{ll})p(R|x_{ll})p(x_{ll}|\boldsymbol{E})\,, \tag{31}$$

and then to use,

$$p(C = i|R, \boldsymbol{E}) = \int_{d_i}^{d_{i+1}} dx_c\ \frac{I(x_c)}{p(R|\boldsymbol{E})}\,. \tag{32}$$

For now just consider $p(x_{ll}|\boldsymbol{E})$. Let's marginalise over φ,

$$p(x_{ll}|\boldsymbol{E}) = \int d\varphi\ p(x_{ll}|\varphi, \boldsymbol{E})p(\varphi|\boldsymbol{E})\,. \tag{33}$$

$\boldsymbol{E}$ and φ are independent of each other (when not conditioned on other variables; see Fig. 5). Hence,

$$p(\varphi|\boldsymbol{E}) = p(\varphi) = N(\varphi; 1, \sigma_\varphi^2)\,, \tag{34}$$

where we have used (7).

Recall from (6) that,

$$p(\delta x_{ij}|\delta E_{ij}, \varphi) = N(\delta x_{ij}; \delta E_{ij}\varphi, \frac{\sigma_{acc}^2}{2}\delta t)\,. \tag{35}$$

Using standard formulae for the sum of normally distributed random variables, we have that $x = \sum_i \delta x_{2i} - \delta x_{1i}$ will be given by,

$$p(x|\varphi, \boldsymbol{E}) = N(x; E\varphi, t_e\sigma_{acc}^2)\,. \tag{36}$$

where $E = \sum_i \delta E_{2i} - \delta E_{1i}$, and $t_e = \sum_i \delta t$. Using (26) we have,

$$p(x_{ll}|\varphi, \boldsymbol{E}) = N(x_{ll}; \frac{E\varphi}{\theta(t_e)}, \frac{t_e\sigma_{acc}^2}{\theta^2(t_e)})\,. \tag{37}$$

Substituting these results into (33) gives us,

$$p(x_{ll}|\boldsymbol{E}) = \int d\varphi\ N(x_{ll}; \frac{E\varphi}{\theta(t_e)}, \frac{t_e\sigma_{acc}^2}{\theta^2(t_e)})N(\varphi; 1, \sigma_\varphi^2) \tag{38}$$

$$= \frac{\theta(t_e)}{E}\int d\varphi\ N(\varphi; \frac{x_{ll}\theta(t_e)}{E}, \frac{t_e\sigma_{acc}^2}{E^2})N(\varphi; 1, \sigma_\varphi^2)\,. \tag{39}$$

We can apply the result in appendix A. Simplifying, we find,

$$p(x_{ll}|\boldsymbol{E}) \quad = \quad N(x_{ll}; \frac{E}{\theta(t_e)}, \frac{E^2\sigma_\varphi^2 + t_e\sigma_{acc}^2}{\theta^2(t_e)}) \,. \tag{40}$$

Returning to (31), we now have,

$$I(x_c) = \int dx_{ll}\ p(x_c|x_{ll})p(R|x_{ll})N(x_{ll}; \frac{E}{\theta(t_e)}, \frac{E^2\sigma_\varphi^2 + t_e\sigma_{acc}^2}{\theta^2(t_e)}) \,. \tag{41}$$

In the case in which we have no metacognitive noise, $x_c = x_{ll}$. We can express this using the Dirac delta function as $p(x_c|x_{ll}) = \delta(x_c - x_{ll})$. Then we have,

$$I(x_c) \quad = \quad \int dx_{ll}\ \delta(x_c - x_{ll})p(R|x_{ll})N(x_{ll}; \frac{E}{\theta(t_e)}, \frac{E^2\sigma_\varphi^2 + t_e\sigma_{acc}^2}{\theta^2(t_e)}) \tag{42}$$

$$= \quad p(R|x_{ll} = x_c)N(x_c; \frac{E}{\theta(t_e)}, \frac{E^2\sigma_\varphi^2 + t_e\sigma_{acc}^2}{\theta^2(t_e)}) \,, \tag{43}$$

Consider the term $p(R|x_{ll} = x_c)$, which describes the probability of a response, given a final log-posterior x_{ll}. The Bayesian observer's decision rule is deterministic, and was described in section 2. If $x_{ll} < 0$ the observer reports that $R = 1$, and reports $R = 2$ if $x_{ll} > 0$. In all cases, the observer makes the response which is most likely to be correct. Hence, the log-posterior ratio at the time of the decision, x_{ll}, always favours the response made, and the probability of x_{ll} and R being inconsistent is zero. Due to the absence of metacognitive noise, $x_{ll} = x_c$. Hence, $p(R|x_{ll} = x_c)$ will also be one when x_c and R are consistent, and zero when x_c is inconsistent with R, i.e. when x_c suggests a different response should have been made. Because x_c and R are always consistent, the observer will never report a confidence of less than 50%.

In the no metacognitive noise case we have, using (32) and our expression for $I(x_c)$,

$$p(C = i|R = 2, \boldsymbol{E}) \quad = \quad \frac{1}{p(R = 2|\mathbf{E})} \int_{d_i}^{d_{i+1}} dx_c\ p(R = 2|x_{ll} = x_c)N(x_c; \frac{E}{\theta(t_e)}, \frac{E^2\sigma_\varphi^2 + t_e\sigma_{acc}^2}{\theta^2(t_e)}) \tag{44}$$

$$= \quad \frac{1}{p(R = 2|\mathbf{E})} \int_{\max\{0,d_i\}}^{\max\{0,d_{i+1}\}} dx_c\ N(x_c; \frac{E}{\theta(t_e)}, \frac{E^2\sigma_\varphi^2 + t_e\sigma_{acc}^2}{\theta^2(t_e)}) \,, \tag{45}$$

where we have used that $p(R = 2|x_{ll} = x_c)$ is zero anywhere x_{ll} is inconsistent with the response $R = 2$ (i.e. $x_{ll} < 0$), and one elsewhere. In the case where $R = 1$, an identical expression holds, except the limits become $\min\{0, -d_{i+1}\}$ and $\min\{0, -d_i\}$.

We can compute the probability of the response, $p(R|\mathbf{E})$, using,

$$p(R|\boldsymbol{E}) \quad = \quad \int_{-\infty}^{\infty} dx_{ll}\ p(R|x_{ll})p(x_{ll}|\boldsymbol{E}) \tag{46}$$

$$= \quad \int_a^b dx_{ll}\ N(x_{ll}; \frac{E}{\theta(t_e)}, \frac{E^2\sigma_\varphi^2 + t_e\sigma_{acc}^2}{\theta^2(t_e)}) \tag{47}$$

Where $a = -\infty, b = 0$ for $R = 1$ or $a = 0, b = \infty$ for $R = 2$. These limits again come from using that $p(R|x_{ll}) = 1$ when R and x_{ll} are consistent, and $p(R|x_{ll}) = 0$ otherwise.

Let's now consider what happens in the presence of metacognitive noise, when x_c no longer equals x_{ll}, but is a noisy version of it as in (28). Returning to (41), in this case $I(x_c)$ becomes,

$$I(x_c) = \int dx_{ll}\ N(x_c; x_{ll}, \sigma_m^2)p(R|x_{ll})N(x_{ll}; \frac{E}{\theta(t_e)}, \frac{E^2\sigma_\varphi^2 + t_e\sigma_{acc}^2}{\theta^2(t_e)}) \,. \tag{48}$$

Because $p(R|x_{ll})$ is 1 when R is consistent with the log-posterior ratio, and 0 otherwise, the product of this term and the normal distribution over x_{ll} is a normal distribution truncated to the region where R and x_{ll} are consistent. Note that the product of these two terms is not a probability distribution

over x_{ll} because it is not normalised. However, we can write this product as a scaled truncated normal distribution,

$$
\begin{aligned}
J(x_{ll}) &= p(R|x_{ll})N(x_{ll}; \frac{E}{\theta(t_e)}, \frac{E^2\sigma_\varphi^2 + t_e\sigma_{acc}^2}{\theta^2(t_e)}) && (49)\\
&= TN(x_{ll}; \mu_{ll}, \sigma_{ll}^2, a, b) \int_a^b dx'_{ll}\ N(x'_{ll}; \frac{E}{\theta(t_e)}, \frac{E^2\sigma_\varphi^2 + t_e\sigma_{acc}^2}{\theta^2(t_e)}) && (50)\\
&= TN(x_{ll}; \mu_{ll}, \sigma_{ll}^2, a, b)p(R|\boldsymbol{E})\,. && (51)
\end{aligned}
$$

We have used (47) to get the final line. As before $a = -\infty, b = 0$ or $a = 0, b = \infty$ depending on the response made, and TN indicates a truncated normal distribution, truncated at a and b. The first and second parameters, μ_{ll} and σ_{ll}^2, are the mean and variance of the distribution prior to truncation, and therefore are,

$$
\begin{aligned}
\mu_{ll} &= \frac{E}{\theta(t_e)} && (52)\\
\sigma_{ll}^2 &= \frac{E^2\sigma_\varphi^2 + t_e\sigma_{acc}^2}{\theta^2(t_e)} && (53)
\end{aligned}
$$

We have then,

$$
I(x_c) = p(R|\boldsymbol{E}) \int_{-\infty}^{\infty} dx_{ll}\ N(x_c; x_{ll}, \sigma_m^2) TN(x_{ll}; \mu_{ll}, \sigma_{ll}^2, a, b)\,. \tag{54}
$$

Consider a change of variables $y = x_c - x_{ll}$,

$$
\begin{aligned}
I(x_c) &= -p(R|\boldsymbol{E}) \int_{\infty}^{-\infty} dy\ N(y; 0, \sigma_m^2) TN(x_c - y; \mu_{ll}, \sigma_{ll}^2, a, b) && (55)\\
&= p(R|\boldsymbol{E}) \int_{-\infty}^{\infty} dy\ N(y; 0, \sigma_m^2) TN(x_c - y; \mu_{ll}, \sigma_{ll}^2, a, b)\,. && (56)
\end{aligned}
$$

This expression is the convolution of a normal distribution, and a truncated normal distribution. We can perform the convolution by writing the distributions out in full, collecting all terms containing y into a single exponential, and integrating (S. Turban, personal communication, December, 2019):

$$
\begin{aligned}
I(x_c) &= p(R|\boldsymbol{E}) \int_{-\infty}^{\infty} dy\ N(y; 0, \sigma_m^2) TN(x_c - y; \mu_{ll}, \sigma_{ll}^2, a, b) && (57)\\
&= p(R|\boldsymbol{E}) \int_{-\infty}^{\infty} dy \frac{1}{2\pi\sqrt{\sigma_m^2\sigma_{ll}^2}} \frac{1}{\Phi(\frac{b-\mu_{ll}}{\sigma_{ll}}) - \Phi(\frac{a-\mu_{ll}}{\sigma_{ll}})} e^{-\frac{y^2}{2\sigma_m^2}} e^{-\frac{(x_c-y-\mu_{ll})^2}{2\sigma_{ll}^2}} \mathbb{1}_{y\in[x_c-b, x_c-a]} && (58)\\
&= p(R|\boldsymbol{E}) \frac{1}{\Phi(\frac{b-\mu_{ll}}{\sigma_{ll}}) - \Phi(\frac{a-\mu_{ll}}{\sigma_{ll}})} N(x_c; \mu_{ll}, \sigma_m^2 + \sigma_{ll}^2) \Big[\Phi(\frac{x_c - a - \alpha}{\beta}) - \Phi(\frac{x_c - b - \alpha}{\beta})\Big] && (59)
\end{aligned}
$$

where $\Phi()$ indicates the standard cumulative normal distribution, $\mathbb{1}$ is an indicator function (which is one when its argument is true, and zero otherwise), $\alpha = \frac{\beta^2(x_c-\mu_{ll})}{\sigma_{ll}^2}$ and $\beta^2 = \frac{\sigma_m^2\sigma_{ll}^2}{\sigma_m^2+\sigma_{ll}^2}$.

In our case, a and b take very specific values. Let,

$$
L = \begin{cases} -1 & \text{if } a = -\infty, b = 0;\ \text{i.e. R=1} \\ 1 & \text{if } a = 0, b = \infty;\ \text{i.e. R=2}\,, \end{cases}
$$

then we can write,

$$
I(x_c) = p(R|\boldsymbol{E}) \frac{1}{\Phi(L\frac{\mu_{ll}}{\sigma_{ll}})} N(x_c; \mu_{ll}, \sigma_m^2 + \sigma_{ll}^2) \Phi\left(L \frac{x_c\sigma_{ll}^2 + \mu_{ll}\sigma_m^2}{\sigma_m\sigma_{ll}\sqrt{\sigma_m^2 + \sigma_{ll}^2}}\right), \tag{60}
$$

and using (32) our final result for confidence is,

$$
p(C = i|R, t_e, \boldsymbol{E}) = L \int_{Ld_i}^{Ld_{i+1}} dx_c\ \frac{1}{\Phi(L\frac{\mu_{ll}}{\sigma_{ll}})} N(x_c; \mu_{ll}, \sigma_m^2 + \sigma_{ll}^2) \Phi\left(L \frac{x_c\sigma_{ll}^2 + \mu_{ll}\sigma_m^2}{\sigma_m\sigma_{ll}\sqrt{\sigma_m^2 + \sigma_{ll}^2}}\right). \tag{61}
$$

We have also used L in the above expression to account for the different limits of integration when $R = 1$ and $R = 2$. We can solve the integral in (61) via numerical integration. It is also possible to rearrange the expression to a form which is faster to evaluate numerically. We rearrange the expression in appendix D, and state the result here. The expression in (61) becomes,

$$p(C = i|R, t_e, \boldsymbol{E}) = \frac{L}{\Phi(L\frac{\mu_{ll}}{\sigma_{ll}})}\Bigg[\quad BvN\Big(\frac{g}{\sqrt{1+h^2}}, e_{i+1}; \frac{-h}{\sqrt{1+h^2}}\Big) \tag{62}$$

$$- \quad BvN\Big(\frac{g}{\sqrt{1+h^2}}, e_i; \frac{-h}{\sqrt{1+h^2}}\Big)\Bigg], \tag{63}$$

where,

$$BvN(l_1, l_2, \rho) = \int_{-\infty}^{l_1}\int_{-\infty}^{l_2} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2xy\rho)} dxdy\,, \tag{64}$$

is the bivariate cumulative normal distribution, corresponding to a distribution with mean and covariance,

$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{65}$$

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \tag{66}$$

Additionally, g, h, and e_i denote

$$g = L\frac{\mu_{ll}\sigma_{ll}^2 + \mu_{ll}\sigma_m^2}{\sigma_m\sigma_{ll}\sqrt{\sigma_m^2+\sigma_{ll}^2}} \tag{67}$$

$$h = \frac{L\sigma_{ll}^2}{\sigma_m\sigma_{ll}} \tag{68}$$

$$e_i = \frac{Ld_i - \mu_{ll}}{\sqrt{\sigma_m^2+\sigma_{ll}^2}}. \tag{69}$$

This integral can be numerically evaluated using standard functions.

Free response condition

In the free response condition we again want to find the probability distribution over confidence reports, but now have an additional piece of information to incorporate into our predictions, the time of the response t_r. Response time will be determined by the evolution of the accumulator, and specifically, by the first time the accumulator reaches a decision threshold (Fig. 6).

Integrating x_c over the region which leads to a confidence report $C = i$ after $R = 2$, and marginalising over x_{ll},

$$p(C = i|R, t_r, \boldsymbol{E}) = \int_{d_i}^{d_{i+1}} dx_c \int dx_{ll}\; p(x_c|x_{ll})p(x_{ll}|R, t_r, \boldsymbol{E})\,. \tag{70}$$

(As before, for $R = 1$ we use integration limits $-d_{i+1}$ and $-d_i$ instead.) The second distribution in this expression can be obtained by marginalising over φ,

$$p(x_{ll}|R, t_r, \boldsymbol{E}) = \int d\varphi\; p(x_{ll}|\varphi, R, t_r, \boldsymbol{E})p(\varphi|R, t_r, \boldsymbol{E})\,. \tag{71}$$

The final term is a distribution over φ, the standardised drift rate. Observers often need to calculate the distribution over drift rate, given the evidence received (Drugowitsch et al., 2012; Moran, 2015; Moreno-Bote, 2010). As researchers, we can infer the value of φ in a similar way.

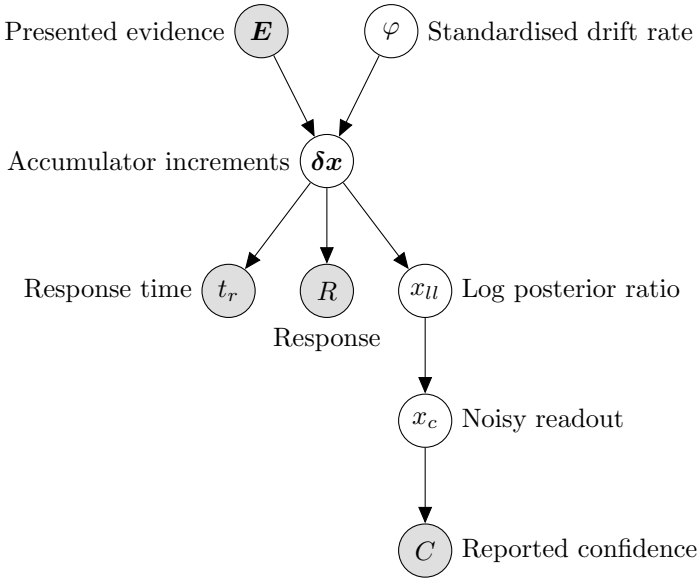

Figure 6: Representation of the generative model for confidence reports in the free response condition, from the perspective of the researcher. We want to infer the probability of reported confidence, using our knowledge of the evidence presented, the response given, and the response time.

Evidence presented immediately prior to a response does not contribute to the response itself, as it is still being processed (Resulaj et al., 2009). If, for an observer, the interval of stimulus in this processing pipeline is I, we can infer that the amount of time they spent accumulating evidence prior to a decision, t_d, is $t_d = t_e - I = t_r - I$. Additionally, if an observer uses a decision threshold, $f(t)$, to trigger responses $R = 2$, and $-f(t)$ to trigger responses $R = 1$, then we know that at the time they made their decision t_d,

$$x = \begin{cases} f(t_d) & \text{if } R = 2 \\ -f(t_d) & \text{if } R = 1\,, \end{cases} \tag{72}$$

because the time they made their decision was the time the accumulator hit the decision threshold (Moreno-Bote, 2010). Denote the value of x at t_d by x_d. Hence for known (or hypothesised) values of I and $f(t)$, we can infer t_d and x_d from R and t_r. Note that in this case we can also infer R and t_r from t_d and x_d. Hence, these quantities are interchangeable, i.e.

$$p(\varphi|R, t_r, \boldsymbol{E}) = p(\varphi|x_d, t_d, \boldsymbol{E}) \tag{73}$$

The probability distribution over φ will depend on the entire stream of evidence up to the time of the decision (i.e. all elements of $\boldsymbol{E}$ which correspond to evidence received before a decision). This is because measurements of all evidence prior to a decision, in conjunction with φ, determine changes in the accumulator, which in turn determines the time of the response, and response itself (see Fig. 6). However, for the purpose of inferring φ, we approximate the evidence stream by its average prior to the time of the decision, $\overline{E}$,

$$p(\varphi|R, t_r, \boldsymbol{E}) = p(\varphi|x_d, t_d, \overline{E})\,. \tag{74}$$

Precisely, $\overline{E}$ is given by $\overline{E} = \sum_i (\delta E_{2i} - \delta E_{1i})/t_d$, where the summation is taken over time steps prior to the decision. We test this approximation, in conjunction with other approximations, once the derivation is complete. Using Bayes rule,

$$\begin{aligned} p(\varphi|R, t_r, \boldsymbol{E}) &= p(\varphi|x_d, t_d, \overline{E}) && (75)\\ &\propto p(x_d, t_d|\varphi, \overline{E})p(\varphi|\overline{E}) && (76)\\ &\propto p(\varphi|\overline{E}) \sum_{\omega\in\Omega} p(\boldsymbol{\delta x}^{(\omega)}|\varphi, \overline{E}) && (77)\\ &\propto p(\varphi|\overline{E}) \sum_{\omega\in\Omega} \prod_i p(\delta x_i^{(\omega)}|\varphi, \overline{E})\,. && (78) \end{aligned}$$

In this equation, Ω is the set of all paths which do not cross the decision threshold prior to t_d, and arrive at the threshold at t_d (Moreno-Bote, 2010). $\boldsymbol{\delta x}^{(\omega)}$ is the $\boldsymbol{\delta x}$ vector for the particular path ω.

We approximate $p(\varphi|\overline{E}) \approx p(\varphi)$. φ is independent of $\boldsymbol{E}$, but could depend on $\overline{E}$, because the average evidence is related to the time of the response, which φ also affects (Fig. 6). Again, this approximation will be tested once we have derived predictions for confidence. The approximation gives,

$$p(\varphi|R, t_r, \boldsymbol{E}) \quad \propto \quad p(\varphi) \sum_{\omega \in \Omega} \prod_i p(\delta x_i^{(\omega)}|\varphi, \overline{E}) \,. \tag{79}$$

Using (6) we have that $\delta x_i = \delta x_{2i} - \delta x_{1i}$ is given by

$$\begin{aligned} p(\delta x_i|\varphi, \overline{E}) &= N(\delta x_i; (\delta E_{2i} - \delta E_{1i})\varphi, \sigma_{acc}^2 \delta t) && (80) \\ &= N(\delta x_i; \varphi \overline{E} \delta t, \sigma_{acc}^2 \delta t) && (81) \\ &\propto N(\varphi; \frac{\delta x_i}{\overline{E}\delta t}, \frac{\sigma_{acc}^2}{\overline{E}^2 \delta t}) && (82) \end{aligned}$$

where we have again used our approximation replacing the full evidence stream with the average evidence. The proportionality is with respect to φ.

Using standard results for the product of normal distributions we find

$$\begin{aligned} \prod_i p(\delta x_i|\varphi, \overline{E}) &\propto \prod_i N(\varphi; \frac{\delta x_i}{\overline{E}\delta t}, \frac{\sigma_{acc}^2}{\overline{E}^2 \delta t}) && (83) \\ &\propto N(\varphi; \frac{x_d}{\overline{E} t_d}, \frac{\sigma_{acc}^2}{\overline{E}^2 t_d}) && (84) \\ &= K^{(\omega)} N(\varphi; \frac{x_d}{\overline{E} t_d}, \frac{\sigma_{acc}^2}{\overline{E}^2 t_d}) \,, && (85) \end{aligned}$$

where we have used $t_d = \sum \delta t$, $x_d = \sum \delta x_i$, and $K^{(\omega)}$ is the constant of proportionality. This constant will be different for different paths, so we explicitly indicate the path, ω. Hence, we have in (79)

$$\begin{aligned} p(\varphi|R, t_r, \boldsymbol{E}) &\propto p(\varphi) \sum_{\omega \in \Omega} K^{(\omega)} N(\varphi; \frac{x_d}{\overline{E} t_d}, \frac{\sigma_{acc}^2}{\overline{E}^2 t_d}) && (86) \\ &\propto p(\varphi) N(\varphi; \frac{x_d}{\overline{E} t_d}, \frac{\sigma_{acc}^2}{\overline{E}^2 t_d}) && (87) \\ & && (88) \end{aligned}$$

Recalling (7), φ is distributed as,

$$p(\varphi) = N(\varphi; 1, \sigma_\varphi^2) \,, \tag{89}$$

giving

$$\begin{aligned} p(\varphi|R, t_r, \boldsymbol{E}) &\propto N(\varphi; 1, \sigma_\varphi^2) N(\varphi; \frac{x_d}{\overline{E} t_d}, \frac{\sigma_{acc}^2}{\overline{E}^2 t_d}) && (90) \\ &= N\Big(\varphi; \frac{\sigma_{acc}^2 + x_d \sigma_\varphi^2 \overline{E}}{\sigma_{acc}^2 + t_d \sigma_\varphi^2 \overline{E}^2}, \frac{\sigma_{acc}^2 \sigma_\varphi^2}{\sigma_{acc}^2 + t_d \sigma_\varphi^2 \overline{E}^2}\Big) \,, && (91) \end{aligned}$$

where we have used standard formulae for the product of normal distributions.

Returning to (71), we see that in addition to an expression for $p(\varphi|R, t_r, \boldsymbol{E})$, we need an expression for $p(x_{ll}|\varphi, R, t_r, \boldsymbol{E})$. As discussed, from the response and response time, we can infer the time spent accumulating evidence prior to the decision, and the state of the accumulator at the time a decision was made (Fig. 2). Following a decision, evidence accumulation continues in the same manner as in the forced case (Pleskac & Busemeyer, 2010), until all evidence measurements have been processed. Therefore, the expression in (36) is valid for predicting the accumulation between the time of the decision and the end

of stimulus processing (t_d, t_e). Denote the accumulation in this time Δx, and the sum over evidence presented in this interval $\Delta E = \sum_i \delta E_{2i} - \delta E_{1i}$ (the summation is taken over all i which correspond to time steps following a decision). Using (36),

$$p(\Delta x|\varphi, R, t_r, \boldsymbol{E}) = N(\Delta x; \varphi\Delta E, \sigma_{acc}^2 I)\,, \tag{92}$$

where I denotes the duration of the pipeline, $t_e - t_d$. Using our knowledge of the location of x at t_d, denoted x_d, and that the final state of x is given by $x_d + \Delta x$, the distribution over the final state is given by,

$$p(x|\varphi, R, t_r, \boldsymbol{E}) \quad = \quad N(x; x_d + \varphi\Delta E, \sigma_{acc}^2 I) \tag{93}$$

Using (26) we have,

$$p(x_{ll}|\varphi, R, t_r, \boldsymbol{E}) \quad = \quad N(x_{ll}; \frac{x_d + \varphi\Delta E}{\theta(t_e)}, \frac{\sigma_{acc}^2 I}{\theta^2(t_e)}) \tag{94}$$

We have now derived expressions for both the distributions in (71). Returning to this equation we have,

$$\begin{aligned}
p(x_{ll}|R, t_r, \boldsymbol{E}) &= \int d\varphi\; p(x_{ll}|\varphi, R, t_r, \boldsymbol{E}) p(\varphi|R, t_r, \boldsymbol{E}) && (95)\\
&= \int d\varphi\; N(x_{ll}; \frac{x_d + \varphi\Delta E}{\theta(t_e)}, \frac{\sigma_{acc}^2 I}{\theta^2(t_e)}) N\Big(\varphi; \frac{\sigma_{acc}^2 + x_d\sigma_\varphi^2\overline{E}}{\sigma_{acc}^2 + t_d\sigma_\varphi^2\overline{E}^2}, \frac{\sigma_{acc}^2\sigma_\varphi^2}{\sigma_{acc}^2 + t_d\sigma_\varphi^2\overline{E}^2}\Big) && (96)\\
&= \frac{\theta(t_e)}{\Delta E}\int d\varphi\; N(\varphi; \frac{\theta(t_e)x_{ll} - x_d}{\Delta E}, \frac{\sigma_{acc}^2 I}{\Delta E^2}) N\Big(\varphi; \frac{\sigma_{acc}^2 + x_d\sigma_\varphi^2\overline{E}}{\sigma_{acc}^2 + t_d\sigma_\varphi^2\overline{E}^2}, \frac{\sigma_{acc}^2\sigma_\varphi^2}{\sigma_{acc}^2 + t_d\sigma_\varphi^2\overline{E}^2}\Big) && (97)
\end{aligned}$$

Applying the result in appendix A and rearranging we find,

$$p(x_{ll}|R, t_r, \boldsymbol{E}) = N(x_{ll}; \mu_{ll}, \sigma_{ll}^2) \tag{98}$$

Where,

$$\begin{aligned}
\mu_{ll} &= \frac{1}{\theta(t_e)}\left(x_d + \Delta E\frac{\sigma_{acc}^2 + x_d\sigma_\varphi^2\overline{E}}{\sigma_{acc}^2 + t_d\sigma_\varphi^2\overline{E}^2}\right) && (99)\\
\sigma_{ll}^2 &= \frac{\sigma_{acc}^2}{\theta^2(t_e)}\left(\frac{\Delta E^2\sigma_\varphi^2}{\sigma_{acc}^2 + t_d\sigma_\varphi^2\overline{E}^2} + I\right) && (100)
\end{aligned}$$

Using this result in (70), and allowing for normally distributed metacognitive noise as in (28),

$$\begin{aligned}
p(C = i|R, t_r, \boldsymbol{E}) &= \int_{d_i}^{d_{i+1}} dx_c \int dx_{ll}\; p(x_c|x_{ll}) p(x_{ll}|R, t_r, \boldsymbol{E}) && (101)\\
&= \int_{d_i}^{d_{i+1}} dx_c \int dx_{ll}\; N(x_c; x_{ll}, \sigma_m^2) N(x_{ll}; \mu_{ll}, \sigma_{ll}^2) && (102)\\
&= \int_{d_n}^{d_{n+1}} dx_c\; N(x_c; \mu_{ll}, \sigma_{ll}^2 + \sigma_m^2)\,. && (103)
\end{aligned}$$

This expression applies for $R = 2$. For the case of $R = 1$ the limits change to $-d_{i+1}$ and $-d_i$.

Testing the approximations

We made several approximations in the derivations above, so it is important to check that our predictions for confidence closely match confidence reports, when these are simulated. We simulated the diffusion process using small time steps, and produced confidence reports in accordance with the model (see (27), (28) and Fig. 2; for details of the simulations see appendix C). We then took each trial and computed predictions for the probability of each confidence report using the derived expressions, before randomly

drawing a confidence report in accordance with the probability assigned to it. This allowed us to plot confidence simulated from the model, and confidence reports which match the derived predictions.

Fig. 7 shows simulations of confidence using the model (error bars), and confidence based on the derived predictions (error shading). No approximations were made in deriving confidence predictions in the interrogation condition. Consistent with this, simulated confidence, and the variance of simulated confidence, closely match predicted confidence and predicted confidence variance, as functions of response time and average evidence, both with and without variability in standardised drift rate.

For the free response derivations, we approximated evidence prior to a decision by its average, for the purpose of estimating standardised drift rate, and approximated standardised drift rate as independent of the average evidence prior to a decision. These approximations are only relevant when standardised drift rate is variable. Consistent with this, simulated and predicted confidence closely match in plots corresponding to no variability in standardised drift rate φ (Fig. 7). When variability in standardised drift rate is present, we can see that the approximations introduce some small discrepancies between simulations and predictions. For example, the predictions appear to over estimate the variability in confidence reports, in trials with long response times.

4 Discussion

Using the normative framework of the DDM and the idea of a Bayesian readout for confidence, we derived predictions for the probability distribution over confidence reports, given the response, response time, and stimulus presented on a trial. We considered both the typical case, where response time is under the control of the participant (free response), and the less common case in which the observer has to respond at a particular time (interrogation). In the free response case, where the observer must set decision thresholds to trigger a response, we allow for the use of decision thresholds of arbitrary shape. These results build on the work of Moreno-Bote (2010), by including features which are important in the construction of confidence. Namely, the derivations account for accumulation of pipeline evidence (Moran et al., 2015), the effect of drift rate variability on pipeline evidence (Pleskac & Busemeyer, 2010), and metacognitive noise (Maniscalco & Lau, 2012).

Importantly, the derivations cover both static stimuli, and dynamic stimuli which generate normally distributed fluctuations in evidence signal each frame. The derived expressions only require one evaluation per trial, in contrast to previous approaches that could handle dynamic stimuli, which required the evaluation of some function at every time step prior to a decision (Chang & Cooper, 1970; Smith, 2000; Zylberberg et al., 2018). Reducing computational cost is crucial for making trial-by-trial modelling of dynamic stimuli feasible. Trial-by-trial modelling may provide a stronger constraint when fitting models, than predictions made for large groups of trials at once, which has until now been the standard approach (see Introduction). Computationally cheap predictions may also allow us to use techniques which require predictions to be evaluated many times, such as cross-validation and Markov chain Monte Carlo (MCMC; Bishop, 2006).

At the outset, we noted that simple expressions may also provide additional insight into the mechanisms responsible for confidence. It is instructive to consider a situation in which observers can report their confidence on a very fine grained scale. In this case, the most likely confidence report corresponds to the most likely value of x_c, the observer's noisy, continuous readout of confidence. In the interrogation condition without metacognitive noise, the most likely value of x_c is given by the mean of the distribution over x_c in (45). Substituting in all abbreviations, we have that the most likely confidence report is

$$\frac{E}{\theta(t_e)} = \frac{2\nu_0 E}{s^2 + t_e\sigma_\nu^2} = \frac{2t_f\Delta\mu E}{t_f^2\sigma_{acc}^2 + t_f\sigma_E^2 + t_e\Delta\mu^2\sigma_\varphi^2}\,. \tag{104}$$

As might be expected, the most likely confidence report is proportional to the total evidence presented, E. We also see that any form of noise, whether internal accumulator noise, variability in the stimulus, or drift rate variability, reduces the most likely confidence. Note that these effects apply once the observer has learned the statistics of the task. Hence, this result does not conflict with the finding that when

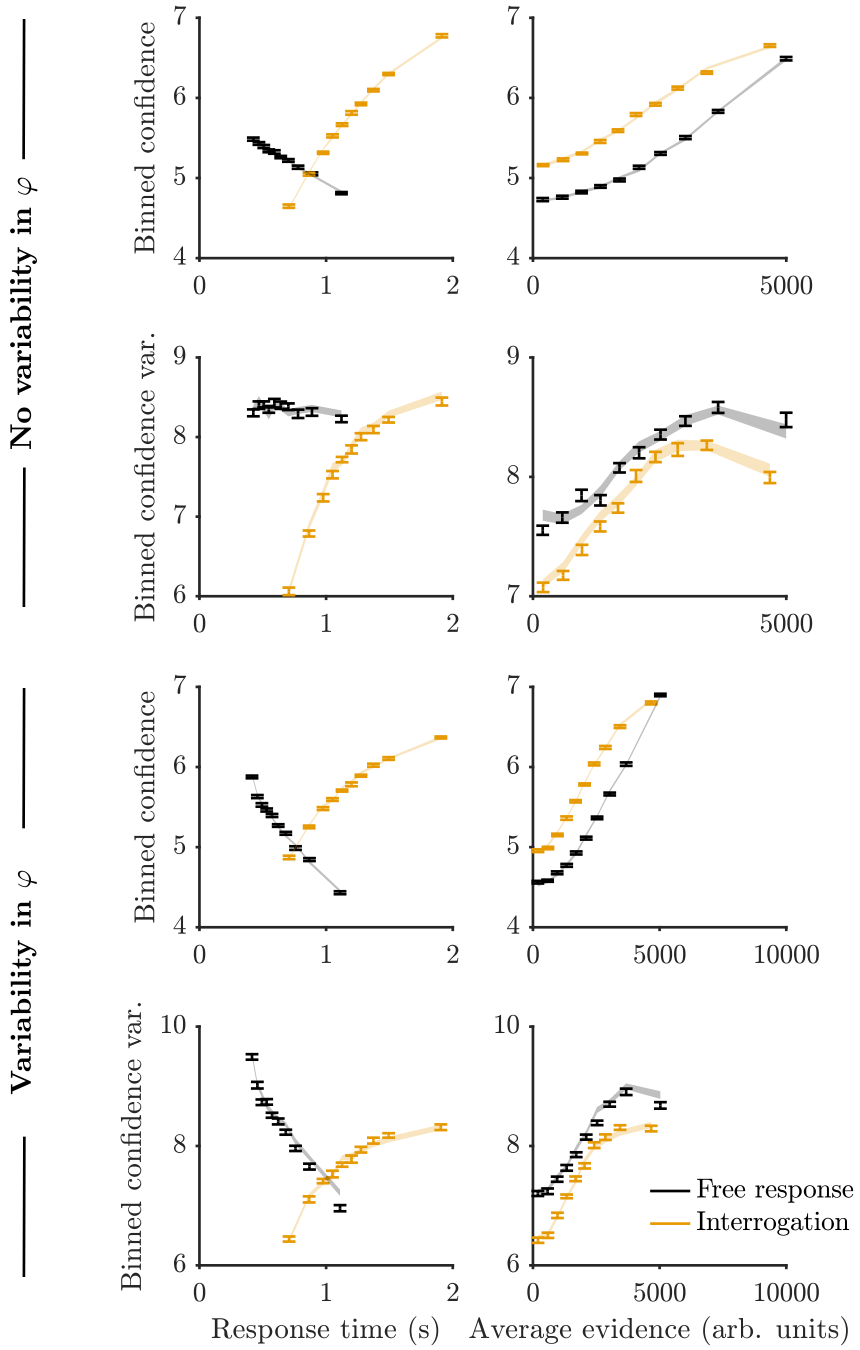

Figure 7: Mean and variance of binned confidence, produced via simulation of the model (error bars), and through the derived predictions (shading). Details of the simulation and plotting are provided in appendix C. Predictions matched the simulation closely. There were some signs that approximations used to derive the predictions lead to a small overestimation of variability in confidence, when there is variability in standardised drift rate.

stimulus variability is uncued, greater variability causes increases in confidence (Zylberberg et al., 2016). It would be simple to extend the derivations to account for the case where observers incorrectly estimate variance in the stimulus. We would only need to change the generative model used by the observer, that determines how the observer transforms the time and accumulator state, x, into an estimate of the log posterior ratio, x_{ll}. Such changes would affect the function $\theta()$ but would leave the dependence on total evidence, E, unchanged.

We can take the same approach for the free response condition, and consider the most likely confidence report if a fine grained confidence scale is used. In the absence of metacognitive noise, the most likely confidence report is given by (99),

$$\frac{1}{\theta(t_e)}\left(x_d + \Delta E \frac{\sigma_{acc}^2 + x_d \sigma_\varphi^2 \overline{E}}{\sigma_{acc}^2 + t_d \sigma_\varphi^2 \overline{E}^2}\right) = \frac{2 t_f \Delta\mu}{t_f^2 \sigma_{acc}^2 + t_f \sigma_E^2 + t_e \Delta\mu^2 \sigma_\varphi^2}\left(x_d + \Delta E \frac{\sigma_{acc}^2 + x_d \sigma_\varphi^2 \overline{E}}{\sigma_{acc}^2 + t_d \sigma_\varphi^2 \overline{E}^2}\right). \tag{105}$$

Breaking this expression down, we see that the prefactor which multiplies the term in brackets is the same as the prefactor which multiplies E in the interrogation case, hence the same dependencies from this prefactor will carry over. It is interesting to note that if there is no variability in standardised drift rate ($\sigma_\varphi = 0$), then the only evidence used for predicting confidence is evidence from the processing pipeline, ΔE. It seems counterintuitive that the evidence on which the decision was based adds nothing to our prediction for confidence. This occurs because, at the time of the decision, we know that the state of the accumulator is at the decision boundary corresponding to the response made (Fig. 2; Pleskac and Busemeyer, 2010). Given knowledge of the state of the accumulator at the time of the decision, we do not need to know the evidence presented up to this point.

Aside from the prefactor, the most likely level of confidence is determined by the state of the accumulator at decision time x_d, plus an additional term which captures the path of the accumulator following the decision. Note that if standardised drift rate variability, σ_φ, is zero, this second term,

$$\Delta E \frac{\sigma_{acc}^2 + x_d \sigma_\varphi^2 \overline{E}}{\sigma_{acc}^2 + t_d \sigma_\varphi^2 \overline{E}^2}\,, \tag{106}$$

simply reduces to ΔE. In this case, the most probable level of confidence is determined by the state of the accumulator at decision time, plus the evidence presented following a decision. This consideration also shows that the fraction multiplying ΔE, which is 1 when $\sigma_\varphi = 0$, is a term which accounts for the effect of drift variability on confidence. This term provides a mathematical description of the fact that, if lots of evidence has been gathered by the time of the decision, relative to the time spent deliberating, the standardised drift rate is likely to be high (Moreno-Bote, 2010), and pipeline evidence will have a big impact on decision confidence (Pleskac & Busemeyer, 2010). On the other hand, if at the time of decision, little evidence has been gathered relative to the time spent deliberating, evidence is accumulating slowly, suggesting a low standardised drift rate. In turn, this suggests that pipeline evidence will be processed poorly and will have a small effect on confidence. This is why the fraction contains decision time in the denominator, reducing the effect of pipeline evidence, and the height of the threshold at decision time is in the numerator, increasing the effect of pipeline evidence. We are not the first to describe this effect of drift rate variability; it was a central idea in the model for confidence proposed by Pleskac and Busemeyer (2010). Our contribution is to derive an expression for this effect.

If we assume that the decision threshold, x_d, does not depend on time, then decision time and stimulus duration only appear in denominators in (105). Hence, in the absence of other changes, decision time will decrease confidence. This is consistent with the findings and conclusions drawn by Kiani et al. (2014). Kiani et al. (2014) increased decision time by using stimuli which, for 360 ms, provided no net evidence for either choice. This manipulation decreased confidence, without clear changes in accuracy, suggesting that the effect of decision time on confidence is direct, and not mediated by changes in evidence.

Expressions for confidence may also help us understand confidence findings which previously appeared difficult to explain. One confusing feature of the literature has been inconsistency in the observed relationship between confidence and signal strength, on error trials. Sanders et al. (2016) found confidence on error trials decreased as signal strength increased. However, Kiani et al. (2014) found that confidence on error trials increased with signal strength. Sanders et al. (2016) collected confidence reports following a decision, suggesting that pipeline evidence would have contributed to confidence. In contrast, Kiani

et al. (2014) asked participants to simultaneously report decisions and confidence using a saccade. The stimulus turned off as soon as a saccade began, minimising the chance that confidence was based on any more information than the decision, i.e. $\Delta E = 0$.

Kiani et al. (2014) suggested that simultaneous decisions and confidence may have been key to the pattern they observed. Our derivations support this idea. Looking again at our expression for most likely confidence, (105), but setting $\Delta E = 0$, we have,

$$\frac{2t_f \Delta\mu}{t_f^2 \sigma_{acc}^2 + t_f \sigma_E^2 + t_e \Delta\mu^2 \sigma_\varphi^2} x_d \,. \tag{107}$$

If we have a flat threshold, x_d is constant, and the only variable which changes between trials is t_e, the duration of evidence presentation. Duration of evidence presentation is the same as response time in the free response task. Consider a set of trials with one shared value for signal strength. For these trials, a flat decision threshold, which is symmetric for the two options, generates response times for correct and error trials which are on average identical (Shadlen, Hanks, Churchland, Kiani & Yang, 2006). Additionally, a greater signal strength will lead to faster responses. Hence, we predict that higher signal strength leads to both faster correct responses and faster error responses, i.e. lower t_e for correct and error trials. As we can see from the expression, the most likely confidence report will also be higher, due to the lower t_e, when signal strength is greater.

We can also explain why confidence on error trials decreases with signal strength, when the observer has time to process pipeline evidence. The processing pipeline contains a considerable amount of information from the stimulus (approximately 400ms of the stimulus prior to response; Ratcliff and McKoon, 2008; Resulaj et al., 2009). On error trials, this evidence will tend to favour the alternative (correct) option, decreasing confidence, and this effect will be stronger when signal strength is high. The implication is that confidence will decrease as signal strength increases.

To test the intuitions gained from studying the equations, we simulated response times, decisions, and confidence reports, under two conditions. In one condition, we simulated the task designed by Kiani et al. (2014), setting pipeline evidence to zero. In the other condition, we simulated a processing pipeline containing (a conservative) 100ms of the stimulus prior to response (see appendix C for details of the simulations). Confidence in errors increased with signal strength when decisions and confidence reports were simultaneous, while confidence in errors decreased with signal strength when observers received pipeline evidence (Fig. 8).

An additional strength of the derivations is the treatment of metacognitive noise. If we fit the predictions to data, we will get an estimate for metacognitive noise from the value of the fitted metacognitive noise parameter. Existing methods for inferring the level of metacognitive noise are based on signal detection frameworks (Maniscalco & Lau, 2012). Therefore, these methods implicitly assume that signal detection theory is an adequate description of decision making mechanisms. The DDM can be viewed as an extension of signal detection theory to the case where response time, and hence the amount of information obtained, is under the control of the observer (Smith, 2000). This is a common situation in empirical studies. Therefore, more accurate estimates of metacognitive noise may be obtainable using derivations in which the DDM is the assumed framework, such as here.

There are a number of limitations to this work which can be divided into those which affect the scope of the derivations themselves, and those which concern the practical application of the predictions in model fitting. Regarding the former, limits to the derivations themselves, we made two particularly important assumptions. Namely, we used a DDM framework, and assumed stimuli of a specific kind. A family of alternatives to the DDM assume not just one accumulator, but two (Bogacz et al., 2006; Moreno-Bote, 2010). Each accumulator corresponds to one of the choices, and the accumulators may be more or less correlated or anti-correlated with each other. It may be possible to use recently derived expressions for the state of the second accumulator at decision time, to extend the approach developed here, although it may be necessary to approximate stimuli as static prior to a decision (Shan, Moreno-Bote & Drugowitsch, 2019). Notwithstanding this consideration, we believe the DDM to be a particularly interesting case due to the normative properties of the diffusion mechanism (see Introduction). Given the evolutionary age of perceptual decision making (500 million years ago animals had complex eyes; Lee et al., 2011), it seems plausible that the algorithm used closely approximates the normative mechanism of the DDM.

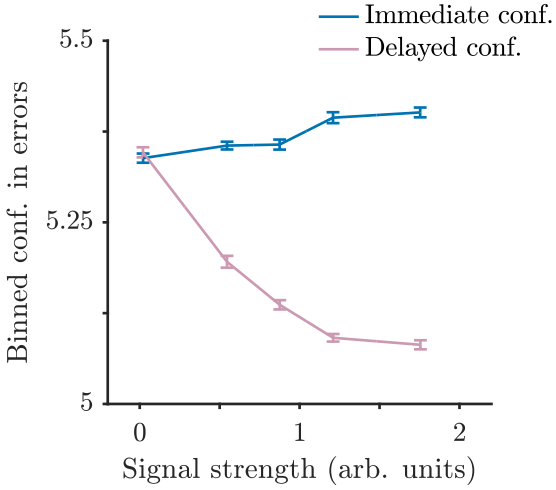

Figure 8: Confidence on error trials as a function of signal strength. When confidence is reported immediately, and therefore does not reflect evidence measurements in the processing pipeline, confidence on error trials increases with signal strength. On the other hand, when confidence reports are made after a decision, and include 100ms of evidence measurements from the processing pipeline, confidence on error trials decreases with signal strength.

The second limitation to the scope of the derivations is that we assumed stimuli of a specific kind. It may be possible to extend the derivations to more general dynamic stimuli, rather than just those with normally distributed evidence fluctuations. Properties of the stimuli are only used to derive confidence of the optimal observer, given an accumulator state and deliberation time. Properties of the stimuli are not important for deriving predictions for confidence, once the observer's internal mapping from accumulator and time, to confidence, is known. Therefore, any stimulus for which we can find the optimal observer's decision and confidence rules could be modelled using the approach we have described. One of the most common stimuli, the random dot motion stimulus, contains dots which move randomly over the course of a trial, creating random fluctuations in evidence for the prevailing motion direction (Kiani, Hanks & Shadlen, 2008; Pilly & Seitz, 2009). If it was possible to characterise the nature of these fluctuations, and derive the optimal observer's decision and confidence rule, a very large quantity of data could be analysed on a trial-by-trial basis using the approach set out here.

Turning to limitations which affect the applicability of these derivations to modelling, the fact that we only make predictions for confidence, not for response times and decisions may be a concern, along with the fact that we have not yet accounted for lapses. It is certainly the case that we can only fit to confidence given decisions, and response times, rather than fitting to decisions and response times themselves. However, as discussed in the introduction, only predicting confidence is key for avoiding difficulties in deriving computationally cheap expressions for responses and response times. Additionally, when fitting confidence reports, we will still model all aspects of the decision mechanism, in the sense that we will generate estimates for all parameters of the decision mechanism. Using these parameters we will be able to make predictions for decisions and response times, providing an additional check of the model and its assumptions.

Regarding lapses, it would not be difficult to include a lapse rate parameter which describes some probability that a random confidence report is given. It would be trickier to model lapses if one had specific hypotheses about the process leading to lapses (e.g. those in Ratcliff and Tuerlinckx, 2002). The probability that a given response and response time resulted from a lapse would be difficult to calculate because, whilst we might know the probability distribution over responses and response times generated by lapses, we do not have expressions for the probability distribution over decisions and response times generated by the non-lapse diffusion process. For similar reasons, it would be difficult to incorporate the idea of variability in the start point of the accumulator, and variability in the duration of the pipeline (Ratcliff & McKoon, 2008; Ratcliff & Tuerlinckx, 2002).

Notwithstanding the limitations discussed, we hope these derivations will prove useful, both for gaining insight into the model of confidence implied by our current knowledge, and for model fitting. Specifically, by permitting analyses which would otherwise be too computationally intensive, and by making it feasible to capitalise on the trial-by-trial variability in dynamic stimuli.

Code availability

All code used in the study will be made publicly available online upon publication.

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A Product of normal distributions

We will repeatedly encounter a situation where we are interested in the probability distribution over z in the following equation,

$$p(z) \propto \int N(y; z, \sigma_A^2) N(y; \mu, \sigma_B^2) dy \,, \tag{108}$$

where N denotes the normal distribution. The product of two normal distributions is a scaled normal distribution (over y; Bromiley, 2018). The scaling is given by,

$$S = N(z; \mu, \sigma_A^2 + \sigma_B^2) \,. \tag{109}$$

Once we have integrated over y, only the scaling is left, as the normal distribution over y integrates to one. Hence,

$$p(z) = N(z; \mu, \sigma_A^2 + \sigma_B^2) \,. \tag{110}$$

B Bayesian observer's policy

At some time t, the observer has gathered N sensory samples, $\delta x_1, \delta x_2, ..., \delta x_N$, collectively denoted $\boldsymbol{\delta x}$. The samples are generated according to (22), (23), and (24). Using Bayes rule the observer can infer the probability of the two possible values for S. Moran (2015), using a result from Drugowitsch et al. (2012), derived the posterior probability over the two options $S = 1$ and $S = 2$, for exactly the case we have. For completeness, we rederive the posterior here.

Given all the evidence measurements so far,

$$p(S|\boldsymbol{\delta x}) \propto p(\boldsymbol{\delta x}|S)p(S) \tag{111}$$

$$\propto \int \prod_{i=1}^{N} p(\delta x_i|\nu)p(\nu|S)d\nu\,, \tag{112}$$

where we have used the flat prior over S in (1). Looking at the product in (112), using (24), and using standard formulae for the product of normal distributions (e.g. see Bromiley, 2018) we have,

$$\prod_{i=1}^{N} p(\delta x_i|\nu) = \prod_{i=1}^{N} N(\delta x_i; \nu\delta t, s^2\delta t) \tag{113}$$

$$\propto N(\nu; \frac{x}{t}, \frac{s^2}{t})\,, \tag{114}$$

where the proportionality sign indicates proportionality with respect to ν, $x = \sum_{i=i}^{N} \delta x_i$, and we have used that $t = \sum_{i=1}^{N} \delta t$. Hence in (112) we have,

$$p(S|\boldsymbol{\delta x}) \propto \int N(\nu; \frac{x}{t}, \frac{s^2}{t})N(\nu; \pm\nu_0, \sigma_\nu^2)d\nu\,, \tag{115}$$

where the sign on ν_0 is determined by S. Using the result in appendix A,

$$p(S|\boldsymbol{\delta x}) \propto N(\pm\nu_0; \frac{x}{t}, \frac{s^2}{t} + \sigma_\nu^2)\,. \tag{116}$$

Rearranging, we find the log-posterior ratio is given by,

$$x_{ll} = \log \frac{p(S=2|\boldsymbol{\delta x})}{p(S=1|\boldsymbol{\delta x})} = \frac{2x\nu_0}{s^2 + t\sigma_\nu^2} = \frac{x}{\theta(t)}\,. \tag{117}$$

Where x_{ll}, and $\theta()$ provide abbreviations.

C Simulation and plotting details

We simulated the task described in the main text with the parameters shown in Table 2. The only difference between the simulation and the setup described in the main text is that evidence in a frame could not go below 0 or above 3096. Evidence within a frame was resampled until it met these constraints. The reference value, $\overline{\mu}$, the midpoint between the means of the two evidence signals, was sampled each trial. It was sampled from a normal distribution centred on 1000, with standard deviation of 100, that was truncated at 500 and 1500.

Variable	Expression	Simulation value
Duration of a frame	t_f	50ms
Max evidence in frame		3096
Min evidence in frame		0
Reference value	$p(\overline{\mu})$	See text
Mean difference between evidence signals	$\Delta\mu$	90
Standard deviation of evidence in a frame	σ_E	311

Table 2: Parameters for the task used in the simulation.

The parameters used for the observer are shown in Table 3. All participants were simulated using the same parameters. Half of the trials were simulated for the free response condition, and half for the interrogation condition. A linear decision threshold, symmetric for both options, was used in the free response condition. At the onset of evidence accumulation the threshold was at the value given in Table 3 for threshold height, and it subsequently changed at the rate given by threshold slope. In the interrogation condition, stimulus presentation duration was drawn from a normal distribution of mean 0.8 s and standard deviation 0.3 s, truncated at 0.2 and 4 s.

Variable	Expression	Fig. 3	Fig. 7	Fig. 8
Simulated participants		40	40	40
Trials per participant		128000	12800	128000
Pipeline duration	I	350 ms	350 ms	100ms
Standard deviation of accumulator noise	σ_{acc}	2860	2860	2860
Standard deviation of standardised drift rate	σ_φ	1 or 0	1 or 0	0.6
Standard deviation of metacognitive noise	σ_m	0	$1500/\theta(t=1)$	$1500/\theta(t=1)$
Threshold height		2000	2000	2000
Threshold slope		0	$-800\ s^{-1}$	0

Table 3: Parameters of the simulated observer.

The evolution of the observer's accumulator was simulated using small time steps of duration 0.1 ms. Each accumulator increment was determined by sampling from the distribution in (6). Simulation of the accumulation continued until a decision threshold was crossed, and all remaining evidence in the pipeline had been processed. The only exception was the simulation for Fig. 8. For the "Immediate conf." condition in this plot, accumulation terminated as soon as the decision threshold was reached, consistent with the idea that the manipulation used by Kiani et al. (2014) ensured confidence was not based on pipeline evidence.

Confidence reports were produced in accordance with the model (see (27), (28) and Fig. 2). (The only difference to the model was that simulated confidence reports were scaled by multiplication with a constant, $\theta(t = 1)$.) Where a plot refers to binned confidence, the confidence values produced by the model have been divided into 10 bins of equal numbers of data points on a participant-by-participant basis.

In order to make predictions for confidence using the derived expressions, we used the true parameters for σ_{acc}, σ_{φ}, σ_m, I, threshold height and threshold slope. For the bin edge parameters, d_i, we used the bin edges which were used for dividing the simulated confidence reports into 10 bins. (The code used the fact that simulated confidence reports were scaled by $\theta(t = 1)$.)

Prior to plotting, data for the x-axis was binned. For each participant, and each bin, the average value of the variable on the x-axis was computed. Using these averages, the average value across participants was computed. 10 bins were used apart from in Fig. 8 where 5 bins were used. The y-variable was calculated separately for each participant and bin, before averaging across participants. Error bars, and the width of error shading, represent ± 1 standard error of the mean, across participants.

D Rearranging interrogation condition result

For the probability distribution over C in the forced condition, we found in (61) that

$$p(C=i|R,t_e,\boldsymbol{E}) = L\int_{Ld_i}^{Ld_{i+1}} dx_c\ \frac{1}{\Phi(L\frac{\mu_{ll}}{\sigma_{ll}})}N(x_c;\mu_{ll},\sigma_m^2+\sigma_{ll}^2)\Phi\left(L\frac{x_c\sigma_{ll}^2+\mu_{ll}\sigma_m^2}{\sigma_m\sigma_{ll}\sqrt{\sigma_m^2+\sigma_{ll}^2}}\right). \tag{118}$$

It is possible to rearrange this expression into a form which is easier to evaluate numerically.

Consider a change of variables,

$$z = \frac{x_c-\mu_{ll}}{\sqrt{\sigma_m^2+\sigma_{ll}^2}}\,. \tag{119}$$

Then we have,

$$dz = \frac{dx_c}{\sqrt{\sigma_m^2+\sigma_{ll}^2}} \tag{120}$$

and,

$$N(x_c;\mu_{ll},\sigma_m^2+\sigma_{ll}^2) = \frac{1}{\sqrt{\sigma_m^2+\sigma_{ll}^2}}N(z;0,1) \tag{121}$$

$$= \frac{1}{\sqrt{\sigma_m^2+\sigma_{ll}^2}}\phi(z)\,, \tag{122}$$

where $\phi()$ indicates the standard normal distribution. Also denote,

$$g = \frac{L\mu_{ll}\sqrt{\sigma_m^2+\sigma_{ll}^2}}{\sigma_m\sigma_{ll}} = L\frac{\mu_{ll}\sigma_{ll}^2+\mu_{ll}\sigma_m^2}{\sigma_m\sigma_{ll}\sqrt{\sigma_m^2+\sigma_{ll}^2}} \tag{123}$$

$$h = \frac{L\sigma_{ll}}{\sigma_m} = \frac{L\sigma_{ll}^2}{\sigma_m\sigma_{ll}}\,. \tag{124}$$

Then,

$$g+hz = L\frac{x_c\sigma_{ll}^2+\mu_{ll}\sigma_m^2}{\sigma_m\sigma_{ll}\sqrt{\sigma_m^2+\sigma_{ll}^2}}\,. \tag{125}$$

Hence, in (118) we have,

$$p(C=i|R,t_e,\boldsymbol{E}) = L\frac{1}{\Phi(L\frac{\mu_{ll}}{\sigma_{ll}})}\int_{e_i}^{e_{i+1}} dz\ \phi(z)\Phi(g+hz)\,, \tag{126}$$

where,

$$e_i = \frac{Ld_i-\mu_{ll}}{\sqrt{\sigma_m^2+\sigma_{ll}^2}}\,. \tag{127}$$

Using result 10010.1 from Owen (1980), gives us,

$$p(C=i|R,t_e,\boldsymbol{E}) = L\frac{1}{\Phi(L\frac{\mu_{ll}}{\sigma_{ll}})}\Bigg[\quad BvN\Big(\frac{g}{\sqrt{1+h^2}},e_{i+1};\frac{-h}{\sqrt{1+h^2}}\Big) \tag{128}$$

$$-\ BvN\Big(\frac{g}{\sqrt{1+h^2}},e_i;\frac{-h}{\sqrt{1+h^2}}\Big)\Bigg]\,, \tag{129}$$

where,

$$BvN(l_1,l_2,\rho) = \int_{-\infty}^{l_1}\int_{-\infty}^{l_2}\frac{1}{2\pi\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2xy\rho)}dxdy\,, \tag{130}$$

is the bivariate cumulative normal distribution, corresponding to a distribution with mean and covariance,

$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{131}$$

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \tag{132}$$

This integral can then be numerically evaluated using standard functions.