Dynamic analysis and control of a rice-pest system under transcritical bifurcations

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Abstract

World-wide rice consummation constantly grows constantly with non-proportional yield as large amounts of rice are lost due to pest infestations. Cultural methods were widely applied at an early stage of agricultural pest management but then replaced over time through insecticides. To describe a rice-pest system and to control the corresponding pests applying cultural methods and/or insecticides, statistical analyses have been used, and also other mathematical models using an Integrated Pest Management (IPM) strategy considering long time period and more parametric values. Considering the limitations of IPM, we have developed a mathematical model for a rice-pest system found in agricultural management. The mathematical model consists of two non-linear differential equations (NDEs) to illustrate the interrelation of rice and its corresponding agricultural pests. This model is extended to become an optimal control problem (minimization problem), considering both, cultural methods and pesticides, to minimize the density of agricultural pests and to increase the production of rice, reducing gross annual losses. Pesticides have been applied only in emergencies to reduce environmental pollution and damage to nearby ecosystems such as aquatic ecosystems, and a decision model has also been developed to mitigate potential risks. To compare the effectiveness of the considered controls, the ratio of annual production of rice is studied for both controls and without control. This study contributes to building a relationship between NDEs and agricultural management as well as connecting mathematically rice-pest relationships to global food security.

Keywords: Lotka-Volterra model; Pest control; Nonlinear dynamics; Pontryagin's maximum principle; Optimal control.

MSC: 93A30; 49K15; 81T80.

1. Introduction

The production of rice increases annually along with a growing human population worldwide. Improper irrigation, environmental or climatic factors, and the overuse of pesticides [1,2], hamper the rice production [3,4]. Due to high yield losses, costly research and development and their application of disease and pest resistant crops becomes increasingly popular in the recent years [5,6]. Agricultural pests are a driving factor behind those efforts. Animals (insects, birds, rodents) pathogens, or weeds, are commonly considered as rice pests. Without pest control the combined annual losses of rice would exceed 70% [7] mostly due to infestations of flies of the species *Gonatocerus* spp. (Mymaridae), or *Megaselia* spp. (Phoridae) [8]. Despite applying about three billion tons of pesticides each year in the world, approximately 37% to 40% of the total production of rice is lost [9-11].

Control methods. To avoid large and unforeseen losses, agricultural pests need to be controlled. Choosing the best variety of local growing conditions and maintaining a healthy crop is the first line of defence. The first prevention strategy then usually consists of applying cultural methods [12]. Cultural methods e.g., crop rotation, soil enhancement, removal of diseased plants and proper irrigation do not require chemical substances, are less invasive and naturally contribute to higher productivity [13]. Pest species of certain crops decline due to the deficiency of food when a different crop species becomes the focus of cultivation. On the other hand, when plants have enough space so that the sunlight can reach the ground some insect pests can be decreased by more than 96% [14] facilitating visibility to predatory species [15-18].

Conventionally, chemical controls are used in modern agriculture. Many chemicals such as nicotoide or glyphosate are applied annually [3,4] which, however, are increasingly under scrutiny due to their long-lasting negative effects on the environment, especially insect populations – essential for pollinating more than 80% of crop [19,20]. Other chemical controls include the use of DDT, BHC, 2-D, or 4-D, etc. [3,4] which are banned in many countries of this world. Because of the problems with chemicals alternative treatment and control methods have been developed including e.g. cultural methods (field sanitation, crop rotation, tillage, etc.), biological controls, or combinations thereof [5,13].

Biological controls constitute effective methods including natural enemies of insect pests as control agents such as parasitoids (e.g., Diapriidae for flies) [21], predators (e.g., *Ophionea nigrofasciata*) [22], genetic sterilisation [23], pathogens (viral infections) [24,25], or the exploitation of competitor relationships (e.g. a competition for preying between the protist *Didinium* and *Paramecium*) [26]. Further, the application of artificial signals such as that mimicking a female pheromone can be applied to confuse heterospecifics or to trap reproductives [27] which seems a valid alternative as no harmful chemicals are emitted [22].

Predator-prey relationship. Owing to growing concern about the environment worldwide including the study of species and their interactions, modelling of population dynamics using complex predator-prey systems is increasingly found its application [28-30]. Drechsler and Settele [18] statistically describe predator-prey interactions in the cultivation of rice in Ifugao (Philipines) and highlighted the importance of natural enemies to the agricultural system. Nurrohman et al. [28] developed a predator-prey model considering the barn owl (*Tyto Alba*) as a predator to the rice field rat (*Rattus argentiventer*); the presence of the owl showed a significant positive effect on the yield per annum. Kiritani and Kakiya [29] developed a predator-prey model to describe the growth of agricultural pests population on food (rice) deficiency. Adequate nutrition at the beginning of the cultivation cycle results in significant pest growth which provides abundent recources predators to multiply in the middle of cultivation, but which decreases towards the end of the cultivation. Arditi and Ginzburg [30] developed a predator-prey model based on the trophic function considering the ratio of prey (barnacles *Balanus balanoides*) and predator (snails *Urosalpinx cinerea*). The study is appropriate for homogeneous systems (e.g. Oat populations, *Avena sativa*) [31], but also accounts for several types of heterogenities (e.g. Rice farming systems) [32] in large-scale systems. As the crop grows, the density of insects increases, but in the first 40 days of cultivation, their density increases significantly [8].

Role of modelling. Applications of optimised integrated pest management (IPM) strategies consisting of a set of impulsive differential equations (IDEs) are becoming more popular [33-35]. Sun et al. [5] formulated an optimal control strategy approaching a predator-prey model considering biological control and chemical control variables indicating that chemical act faster. By incorporating different regulatory methods, the *model interplay*, itself between pesticides and natural enemies including factors such as residual effects and spray times of pesticides, or release time of natural enemies has shown to have a significant impact on pest species populations [36-38].

Including biological controls implies using mathematical predator-prey models including the periodic release of predators [1,39,40], the periodic application of pesticides to the pest species [41], and pests infected by bacterial diseases or viruses [42,43]. The Lotka-Volterra model is commonly found as mathematical predator-prey model which consists of a pair of first order nonlinear ordinary differential equations (NODEs) [44,45]. Yet, biological controls are expensive to impose and may require a long-term process, and a pest density at a maintainable threshold [46,47]. Therefore, IPM control strategies focusing only on biological controls are limited and usually require several input factors such as host-parasite ratios, starting and ending point population densities, parasite population or insecticide dosage as well as release duration, and levels of parasitism and host-feeding.

Motivated by five previous studies [5,28,29,36,37], we formulate here a novel mathematical model configuration by adopting the Lotka-Volterra model to set up the interrelationship between the production of rice and corresponding pests. This system is validated by analysis and investigated by bifurcation analysis and then applied to an optimal control problem to reduce the annual losses and to increase the annual production of rice by incorporating chemicals with the purpose to evaluate a hybrid cultural/chemical IPM strategy. When the density of pests in the cultivation becomes so high that it could not possibly be anymore controlled by cultural methods, the application of pesticides as a last resort of controlling the agricultural pests instantly is considered. In this regard, a decision model is also developed to take decision when the pesticides should be applied and stopped. The decision model aims to control the adverse effects of the long-term use of pesticides.

2. Modelling of the rice-pests dynamic system and its biological control

2.1 Lotka-Volterra Model in an Optimal Control Problem Setting

The general form of the Lotka-Volterra model can be written in terms of a pair of autonomous differential equations

$$\begin{cases} \dot{x} = xf(x,y) \\ \dot{y} = yg(x,y) \end{cases} \tag{1}$$

with x and y being the prey and predator populations and $f(x,y)$ and $g(x,y)$ being the species' growth rates.

An optimal control problem (OCP) is concerned with the state and the control variable. Consider $x(t)$ and $u(t)$ are these two variables of an OCP respectively, then $x(t)$ satisfies

$$\dot{x}(t) = g(t, x(t), u(t)) \tag{2}$$

with g being continuously differentiable. The objective of this OCP is to find a piecewise continuous control function $u(t)$ and its corresponding state variable $x(t)$ to optimize (either maximize or minimize) a specific objective function [48]. An OCP can be described in the following way

$$\begin{aligned} &\text{Maximize or minimize} \quad J(x,u) = \int_a^b L\big(t, x(t), u(t)\big)\, dt \\ &\text{Subject to} \quad \dot{x}(t) = g\big(t, x(t), u(t)\big) \quad a.e.\ t \in [a,b] \\ &\qquad u(t) \in U \quad a.e.\ t \in [a,b] \\ &\qquad x(a) \in x_0 \quad a.e.\ t \in [a,b] \\ &\qquad \text{and } x(b) \in \mathbb{R}^+ \text{ is free} \end{aligned} \tag{3}$$

with $u(t)$ and $x(t)$ both being piecewise C^1 differentiable with t in $[a,b]$ being the time interval where $a, b \in \mathbb{R}^+$ and $a < b$. In the problem, $u(t)$ belongs to a certain space U that may be a piecewise continuous function or a space of measurable function which satisfies all the constraints of the problem. Therefore, (x^*, u^*) constitutes the optimal solution of the OCP in case the costs can be minimized over all admissible processes [2].

2.2 Assumption and model formulation

Let $x_1(t)$ be the annual production of rice and $x_2(t)$ be the corresponding pest population at a time t where the annual production of rice acts as the prey population, and the corresponding pest population acts as the predator population. In this regard, the pest population consumes and damages the rice growth which declines the annual production of rice. The relationships between the production of rice and the density of pests are as follows [44,45,49]:

(i) At the initial state, let the reproduction rate of rice be α_1. For the consumption of rice by the pest population, a part of the annual production of rice is lost which let's β_1 be the loss rate of $x_1(t)$ due to the consumption of $x_2(t)$. Then the reproduction rate of rice can be represented by

$$\frac{d}{dt} x_1(t) = \big(\alpha_1 - \beta_1 x_2(t)\big) x_1(t) \tag{4}$$

where $\beta_1 x_1(t) x_2(t)$ presents the annual loss of rice due to pests.

(ii) On the other hand, the pest population in the system depends on rice for feeding. Therefore, the pest population proportionally increases with the increasing rice production. Let β_2 be the energy gain rate of pest population by consuming rice. Again, since the pest population depends on the production of rice for feeding, their growth and population number will be reduced when the production of rice declines. Therefore, α_2 is taken as the decline rate of the pest's population proportionally with the decline of rice production. Then the growth rate of the pest population can be represented by

$$\frac{d}{dt} x_2(t) = \big(\beta_2 x_1(t) - \alpha_2\big) x_2(t) \tag{5}$$

where $\beta_2 x_1(t) x_2(t)$ presents the annual consumption of rice by the pest population.

The rice-pests system can be written in terms of a pair of Nonlinear Ordinary Differential Equations (NODEs) in form of a Lotka-Volterra model:

$$\begin{cases} \dfrac{d}{dt} x_1(t) = \big(\alpha_1 - \beta_1 x_2(t)\big) x_1(t) \\ \dfrac{d}{dt} x_2(t) = \big(\beta_2 x_1(t) - \alpha_2\big) x_2(t) \\ x(t) = (x_1(t), x_2(t)), \quad \forall t \\ x(t) = \big(x_{10},\ x_{20}\big),\ when \quad t = 0 \end{cases} \tag{6}$$

where $x_1(t)$ and $x_2(t)$ respectively present the annual production of rice and the density of the rice pests at time t, and x_{10} and x_{20} respectively show the initial conditions of $x_1(t)$ and $x_2(t)$ i.e. the values of $x_2(t)$ and $x_2(t)$ at $t=0$.

A schematic diagram is shown in Fig. 1 which represents the dynamic relations of this model.

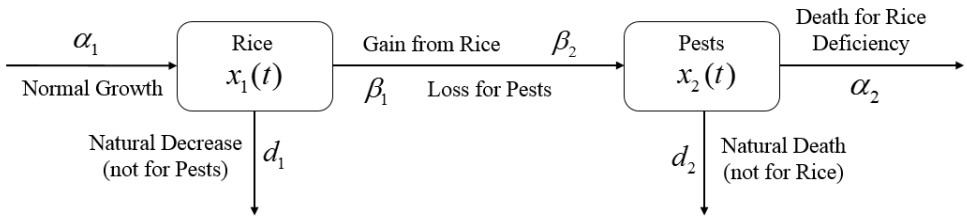

Fig. 1. A schematic diagram of the rice-pests system (7) disclosing the interrelationship between rice and corresponding rice pests along with the impact of adverse environment.

The production rate of rice does not damage only for the attack of pests, but also it may be damaged owing to environmental impacts such as drought, global warming, etc. Similarly, the pest population may be reduced due to natural causes such as floods, droughts, etc. Therefore, two parameters i.e., d_1 (decrease rate of rice production due to natural causes not related to the pest population), and d_2 (death rate of pest population due to natural causes not related to the deficiency of rice) need to be considered in the model as well:

$$\begin{cases} \frac{d}{dt}x_1(t) = \left(\alpha_1 - \beta_1 x_2(t)\right)x_1(t) - d_1 x_1^2(t) \\ \frac{d}{dt}x_2(t) = \left(\beta_2 x_1(t) - \alpha_2\right)x_2(t) - d_2 x_2^2(t) \\ x(t) = (x_1(t), x_2(t)), \quad \forall t \\ x(t) = \left(x_{10},\ x_{20}\right),\ when \quad t=0 \end{cases} \tag{7}$$

Here, the term $d_1 x_1^2(t)$ presents the decrease rate due to intraspecific competition in species $x_1(t)$ due to natural causes that are not related to $x_2(t)$ e.g., viral infections, droughts or floods [50,51]. Similarly, the term $d_2 x_2^2(t)$ shows the intraspecific competition between $x_2(t)$ due to natural causes that not related to $x_1(t)$, e.g. viral infection and heavy rains [50,52].

2.3 Positivity Analysis

In the following we will show that the rice-pests system (7) is bounded in a positive region and the dynamic species of the system (7) consist of non-negative values at any time [53].

Theorem 2.1. *The rice-pests system (7) is bounded in a positive region* $\Psi = \left\{(x_1(t), x_2(t)) \in \mathbb{R}_+^2\right\}$.

Proof: Consider $N(t) = x_1(t) + x_2(t)$ and $\mu \in \mathbb{R}^+$, then we can write

$$\frac{dN(t)}{dt} + \mu N(t) = \frac{d\,x_1(t)}{dt} + \frac{dx_2(t)}{dt} + \mu x_1(t) + \mu x_2(t)$$

$$\Rightarrow \frac{dN(t)}{dt} + \mu N(t) = (\alpha_1 - \beta_1 x_2(t))x_1(t) - d_1 x_1^2(t) - (\alpha_2 - \beta_2 x_1(t))x_2(t) - d_2 x_2^2(t) + \mu x_1(t) + \mu x_2(t) \tag{8}$$

Since the consumption of pests and the losses of rice are approximately equal at the equilibrium point i.e., $\beta_1 \approx \beta_2$, Eq. (8) takes the following form

$$\frac{dN(t)}{dt} + \mu N(t) = (\alpha_1 + \mu)x_1(t) - d_1 x_1^2(t) + (\mu - \alpha_2)x_2(t) - d_2 x_2^2(t)$$

$$\Rightarrow \frac{dN(t)}{dt} + \mu N(t) \le \frac{1}{4d_1}(\alpha_1 + \mu)^2 + \frac{1}{4d_2}(\mu - \alpha_2)^2 \tag{9}$$

Applying the principle of differential inequalities in Eq. (9), we get

$$0 \le N(t)\left(x_1(t), x_2(t)\right) \le \frac{\xi}{\mu}\left(1 - e^{-\mu t}\right) + (x_{10} + x_{20})e^{-\mu t}, \text{ where } \xi = \frac{1}{4d_1}(\alpha_1 + \mu)^2 + \frac{1}{4d_2}(\mu - \alpha_2)^2$$

Taking the limit as $t \to \infty$ we obtain $0 \le N(t) \le \frac{\xi}{\mu}$, hence the region of attraction for the system can be formulated as $\Psi = \left\{ (x_1(t), x_2(t)) \in \mathbb{R}_+^2 : N(t) = x_1(t) + x_2(t),\ 0 \le N(t) \le \frac{\xi}{\mu} \right\}$ with $\xi, \mu \in \mathbb{R}^+$. This indicates that the system is positively bounded in Ψ.

Theorem 2.2. *Each species of the model (7) contains a non-negative real value for all* $t \ge 0$.
Proof: To estimate the solution, consider the first equation of the model (7) given as

$$x_1'(t) = (\alpha_1 - \beta_1 x_2(t))x_1(t) - d_1 x_1^2(t) \qquad \text{where } \frac{dx_1(t)}{dt} = x_1'(t) \tag{10}$$

According to the condition of positivity, Eq. (10) takes the following form

$$x_1'(t) \le \alpha_1 x_1(t) - d_1 x_1^2(t) \ \Rightarrow \frac{1}{x_1(t)} \frac{dx_1(t)}{dt} \le \alpha_1 - d_1 x_1(t) \tag{11}$$

After solving Eq. (11), the solution of $x_1(t)$ becomes $x_1(t) \le \frac{\alpha_1 x_{10}}{(\alpha_1 - d_1 x_{10})e^{-\alpha_1 t} + d_1 x_{10}}$. Therefore, $\lim_{t\to\infty} x(t) \le \frac{\alpha_1}{d_1}$ i.e. $\frac{\alpha_1}{d_1}$ is the upper limit of $x_1(t)$ which implies that $x_{10} \le x_1(t) \le \frac{\alpha_1}{d_1}$ i.e., $x_1(t)$ is non-negatively bounded. Similarly, it can be shown that $x_{20} \le x_2(t) \le \frac{\alpha_2}{d_2}$. Hence the species of the system are positively bounded.

2.4 Equilibrium Points

To conduct the stability analysis of the rice-pests system (7), we first obtain the equilibria of the system, considering three different situations [54]:

(i). The *trivial* or *pre-cultivation* case refers to the situation prior to cultivation. No plants and pests exists yet. Therefore, the pre-cultivating equilibrium point is $(x_{10}, x_{20}) = (0, 0)$.

(ii). The *pest-free* case refers to the pre-existing state of rice pests, indicating that there are rice plants, but no pests. Here, the pest-free equilibrium point is $(\bar{x}_1, 0) = \left(\frac{\alpha_1}{d_1}, 0\right)$.

(iii). A *co-existence* or *farming* situation refers to a competitive case where both rice and pests coexist [55]. The co-existing equilibrium point of the rice-pest system (7) is

$$(\hat{x}_1, \hat{x}_2) = \left(\frac{\alpha_1 d_2 + \alpha_2 \beta_1}{d_1 d_2 + \beta_1 \beta_2}, \frac{\alpha_1 \beta_2 - \alpha_2 d_1}{d_1 d_2 + \beta_1 \beta_2} \right).$$

2.5 Stability Analysis

To illustrate the nature of the equilibrium points of the model (7), a stability analysis has been conducted. In this case, let's consider system (7) in the following vectorial form [53]

$$\begin{cases} x'(t) = f(\bar{x}, t) \\ f = (f_1, f_2) \end{cases} \tag{12}$$

where $f_1(x(t), t) = (\alpha_1 - \beta_1 x_2(t))x_1(t) - d_1 x_1^2(t)$ and $f_2(x(t), t) = (\beta_2 x_1(t) - \alpha_2)x_2(t) - d_2 x_2^2(t)$

After evaluating the vector field, the Jacobian matrix is obtained as the following

$$J = \begin{bmatrix} \alpha_1 - \beta_1 x_2(t) - 2d_1 x_1(t) & -\beta_1 x_1(t) \\ \beta_2 x_2(t) & -\alpha_2 + \beta_2 x_1(t) - 2d_2 x_2(t) \end{bmatrix} \tag{13}$$

Theorem 2.3. *The system (7) approaches a saddle-node at the pre-cultivation equilibrium.*
Proof: To prove the theorem, evaluate the Jacobian matrix (13) at (x_{10}, x_{20}) which becomes of the following form

$$J_{(x_{10}, x_{20})} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & -\alpha_2 \end{bmatrix} \leftrightarrow \det\left(J_{(x_{10}, x_{20})} - I\lambda\right) = \begin{vmatrix} \alpha_1 - \lambda & 0 \\ 0 & -\alpha_2 - \lambda \end{vmatrix} \tag{14}$$

The characteristic equation of Eq. (14) is

$$(\alpha_1-\lambda)(-\alpha_2-\lambda)=\lambda^2-(\alpha_1-\alpha_2)\lambda-\alpha_1\alpha_2=0 \tag{15}$$

Hence the eigenvalues of Eq. (15) are $\lambda_1=\alpha_1$ and $\lambda_2=-\alpha_2$. Since the eigenvalues are real and are opposite in sign, the *pre-cultivating equilibrium* represents a saddle point.

For an autonomous system, an equilibrium point is a saddle point if the characteristic equation consists of opposite signed (one positive and one negative) real eigenvalues [56]. A stable point exhibits a mean position between the stability and instability of a point. For a biological example in two dimensions, the dynamic species appear to have reached a stable equilibrium situation but, their trajectory bends to the other side instead.

Theorem 2.4. *The system described by Eq. (7) at pest-free equilibrium approaches a saddle point.*

Proof: To test the stability at *pest-free equilibrium*, evaluate the Jacobian matrix (13) at $(\bar{x}_1, 0)$ through its characteristic equation by calculating $\det\left(J_{(\bar{x}_1,0)}-I\lambda\right)=0$

$$\therefore J_{(\bar{x}_1,0)}=\begin{bmatrix}-\alpha_1 & -\dfrac{\beta_1\alpha_1}{d_1}\\ 0 & \dfrac{\alpha_1\beta_2}{d_1}-\alpha_2\end{bmatrix} \longleftrightarrow \det\left(J_{(\bar{x}_1,0)}-I\lambda\right)=\begin{vmatrix}-\alpha_1-\lambda & -\dfrac{\beta_1\alpha_1}{d_1}\\ 0 & \dfrac{\alpha_1\beta_2}{d_1}-\alpha_2-\lambda\end{vmatrix}. \tag{16}$$

Therefore, the characteristic equation of Eq. (16) is

$$(-\alpha_1-\lambda)\left(\frac{\alpha_1\beta_2}{d_1}-\alpha_2-\lambda\right)=\lambda^2-\left(\alpha_1+\alpha_2-\frac{\alpha_1\beta_2}{d_1}\right)\lambda-\frac{\alpha_1^2\beta_2}{d_1}+\alpha_1\alpha_2=0 \tag{17}$$

Similarly, the eigenvalues of Eq. (17) become $\lambda_1=-\alpha_1$, and $\lambda_2=\dfrac{\alpha_1\beta_2-\alpha_2 d_1}{d_1}$ with λ_2 being strictly positive since $\alpha_1\beta_2>\alpha_2 d_1$ which indicates that the *pest-free equilibrium* represents a saddle point.

Theorem 2.5. *The rice-pests system (7) approaches a spiral node at the co-existing equilibrium point.*

Proof: Evaluating the Jacobian matrix (13) at the co-existing equilibrium point, it takes the following form

$$J_{(\hat{x}_1,\hat{x}_2)}=\begin{bmatrix}-2\dfrac{(d_2\alpha_1+\alpha_2\beta_1)d_1}{d_1d_2+\beta_1\beta_2}+\dfrac{(\alpha_2d_1-\alpha_1\beta_2)\beta_1}{d_1d_2+\beta_1\beta_2}+\alpha_1 & -\dfrac{\beta_1(d_2\alpha_1+\alpha_2\beta_1)}{d_1d_2+\beta_1\beta_2}\\ -\dfrac{\beta_2(\alpha_2d_1-\alpha_1\beta_2)}{d_1d_2+\beta_1\beta_2} & \dfrac{(d_2\alpha_1+\alpha_2\beta_1)\beta_2}{d_1d_2+\beta_1\beta_2}+2\dfrac{(\alpha_2d_1-\alpha_1\beta_2)d_2}{d_1d_2+\beta_1\beta_2}-\alpha_2\end{bmatrix}. \tag{18}$$

The characteristic equation of Eq. (18) is again calculated via $\det\left(J_{(\hat{x}_1,\hat{x}_2)}-I\lambda\right)=0$

$$\left|J_{(\hat{x}_1,\hat{x}_2)}-I\lambda\right|=\begin{vmatrix}-2\dfrac{(d_2\alpha_1+\alpha_2\beta_1)d_1}{d_1d_2+\beta_1\beta_2}+\dfrac{(\alpha_2d_1-\alpha_1\beta_2)\beta_1}{d_1d_2+\beta_1\beta_2}+\alpha_1-\lambda & -\dfrac{\beta_1(d_2\alpha_1+\alpha_2\beta_1)}{d_1d_2+\beta_1\beta_2}\\ -\dfrac{\beta_2(\alpha_2d_1-\alpha_1\beta_2)}{d_1d_2+\beta_1\beta_2} & \dfrac{(d_2\alpha_1+\alpha_2\beta_1)\beta_2}{d_1d_2+\beta_1\beta_2}+2\dfrac{(\alpha_2d_1-\alpha_1\beta_2)d_2}{d_1d_2+\beta_1\beta_2}-\alpha_2-\lambda\end{vmatrix}=0$$

with the eigenvalues becoming

$$\lambda_1=\frac{1}{2}\frac{1}{d_1d_2+\beta_1\beta_2}\left[-w_1+\sqrt{w_2+2w_3-4w_4}\right] \text{ and } \lambda_2=-\frac{1}{2}\frac{1}{d_1d_2+\beta_1\beta_2}\left[w_1+\sqrt{w_2+2w_3-4w_4}\right]$$

with $w_1=\alpha_1d_2(\beta_2+d_1)+d_1\alpha_2(\beta_1-d_2)$, $w_2=\alpha_1^2d_2^2\left(\beta_2^2+d_1^2\right)+\alpha_2^2d_1^2\left(\beta_1^2+d_2^2\right)$,

$w_3=d_1d_2\left(\alpha_1\alpha_2d_1(\beta_1+d_2)+\alpha_2\beta_1(\alpha_2d_1+\alpha_1\beta_2)-\alpha_1\beta_2d_2(\alpha_1+\alpha_2)\right)$, and

$w_4=\beta_1\beta_2\left(\alpha_1\alpha_2\beta_1\beta_2+d_2\alpha_1^2\beta_2-d_1\alpha_2^2\beta_1\right)$.

Both the eigenvalues are complex numbers which indicate that the *co-existing equilibrium* approaches a spiral node.

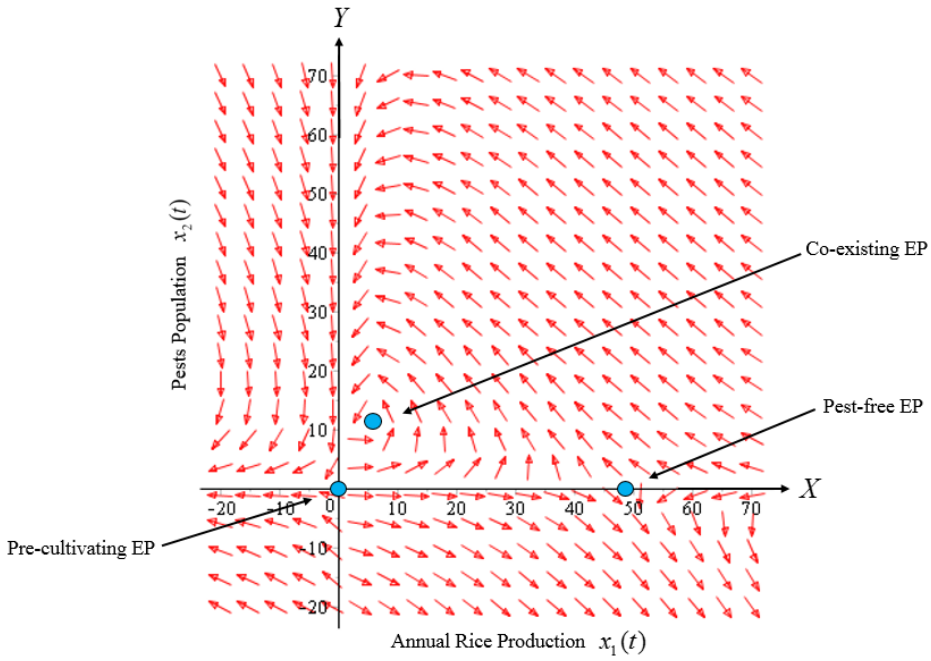

Fig. 2. Phase portrait discloses the characteristics of three equilibrium points (EP) of the rice-pests system (7). Here, the co-existing equilibrium point is a spiral node, and the others, pre-cultivating, and pest-free equilibrium points, are saddle points.

3. Bifurcation analysis for the rice-pests system

In this section, we investigate the rice-pests system (7) through transcritical bifurcation analysis [57]. For a transcritical bifurcation exists a non-destructible fixed point over the whole bifurcation parameter range, which, however, changes its stability characteristic for altered bifurcation parameter values [57,58]. For transcritical bifurcation analysis of rice-pests system (7), it is more convenient to make the rice-pests system (7) dimensionless to reduce the number of parameters [57]. In this case, we introduce the dimensionless variables $x_1(t) = x^*.\hat{x}$ and $x_2(t) = y^*.\hat{y}$ and consider the dimensionless time $t = \tau^*\sigma$. Removing the symbols '$*$' and '^', the system (7) becomes dimentionless given as

$$\begin{cases} \dfrac{dx}{d\tau} = x(\alpha - x) - xy \equiv f_1(x,y) \\ \dfrac{dy}{d\tau} = \beta xy - y - \gamma y^2 \equiv f_2(x,y) \end{cases} \tag{19}$$

where $\sigma = \dfrac{1}{\alpha_2}$, $\alpha = \dfrac{\alpha_1}{\alpha_2}$, $\beta = \dfrac{\beta_2}{d_1}$, $\gamma = \dfrac{d_2}{\beta_1}$.

The biologically meaningful equilibria of the system (19) are the non-negative solutions of $f_1(x,y) = 0$ and $f_2(x,y) = 0$. The rice (prey) isocline consists of the axis $x = 0$ and the straight line $y = \alpha - x$ and the pests (predator) isocline consists of the axis $y = 0$ and the line $x = \dfrac{1+\gamma y}{\beta}$ [57]. Correspondingly, the system (19) is bounded by two equilibrium point, the trivial/pre-cultural equilibrium $E_0(x,y) = (0,0)$ and pest-free equilibrium $E_1(x,y) = (\alpha, 0)$. Besides, an interior equilibrium $E_*(x,y) = (x_*, y_*)$ where $(x_*, y_*) = \left(\dfrac{1+\gamma y_*}{\beta}, \alpha - x_*\right) = \left(\dfrac{\alpha\gamma+1}{\beta+\gamma}, \dfrac{\alpha\beta-1}{\beta+\gamma}\right)$ can be found at the intersection of the two isoclines. For the existence of E_*, the value of the parameters must follow the conditions $\alpha\gamma + 1 \geq 0$, $\beta + \gamma > 0$ and $\alpha\beta - 1 \geq 0$.

Theorem 3.1: *The system (19) experiences transcritical bifurcation at the pest-free equilibrium point $E_1(\alpha, 0)$ as the growth parameter α passes through the critical value α^*.*

Proof: At the pest-free equilibrium point $E_1(\alpha, 0)$, the associated Jacobian matrix of the system (19) takes the form:

$$J_{E_1}(\alpha^*) = \begin{bmatrix} -\alpha & 0 \\ 0 & 0 \end{bmatrix} \tag{20}$$

The set of eigenvalues of $J_{E_1}(\alpha^*)$ is $\lambda = \begin{bmatrix} -\alpha \\ 0 \end{bmatrix}$ i.e. one eigenvalue is zero and other is negative since $\alpha > 0$. Therefore, to examine the nature of the system at E_1, we have applied *Sotomayor's theorem* [58,59]. For this purposet, we consider the system (19) as

$$f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} \tag{21}$$

Let the eigenvectors corresposponding to the zero eigenvalues of $J_{E_1}(\alpha^*)$ and $J_{E_1}{}^T(\alpha^*)$ be V and W, respectively, where $V = W = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. From Eq. (21), we have $f_\alpha(E_1;\alpha^*) = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$, $Df_\alpha(E_1;\alpha^*) = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$ and $D^2 f(E_1;\alpha^*)(V,V) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$. Here, Df denotes the partial derivative of f with respect to x and y and Df_α denotes the partial derivative of f with respect to parameter α. Therefore,

$$W^T f_\alpha(E_1;\alpha^*) = \alpha \neq 0$$
$$W^T \left[Df_\alpha(E_1;\alpha^*)V \right] = \alpha \neq 0$$
$$W^T \left[D^2 f(E_1;\alpha^*)(V,V) \right] = -2 \neq 0$$

Hence, there is a saddle node bifurcation at the nonhyperbolic equilibrium point $E_1(\alpha,0)$ at the bifurcation value α. For $\alpha < 0$, there is no equilibrium point. For $\alpha = 0$, the $f_1(x,y) = -x^2$ is structually unstable and the bifurcation value $\alpha = 0$. Therefore, there is a transcritical bifurcation at the origin for $\alpha = 0$. There are two equilibria at origin $(0,0)$ and $E_1(\alpha,0)$ [59].

Theorem 3.2: *The system (19) experiences transcritical bifurcation at the equilibrium point $E_*(x_*, y_*)$ as the growth parameter of pest population β passes through the critical value β^*.*

Proof: At the equilibrium point $E_*(x_*, y_*)$, the associated Jacobian matrix of the system (19) takes the form:

$$J_{E_*}(\beta^*) = \begin{bmatrix} -\dfrac{\alpha\gamma+1}{\beta+\gamma} & -\dfrac{\alpha\gamma+1}{\beta+\gamma} \\ \dfrac{\alpha\beta^2-\beta}{\beta+\gamma} & \dfrac{\gamma(1-\alpha\beta)}{\beta+\gamma} \end{bmatrix} \tag{22}$$

It can been shown that one eigenvalue of $J_{E_*}(\beta^*)$ is negative and other is zero for the condition $\alpha\beta = 1$ [60]. To investigate the nature of the system at E_*, we have applied *Sotomayor's theorem* [58,59]. Let the eigenvectors corresposponding to the zero eigenvalues of $J_{E_*}(\beta^*)$ and $J_{E_*}{}^T(\beta^*)$ be V and W, respectively, where $V = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $W = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. From Eq. (21), we have $f_\beta(E_*;\beta^*) = \begin{bmatrix} 0 \\ x_* y_* \end{bmatrix}$, $Df_\beta(E_*;\beta^*) = \begin{bmatrix} 0 & 0 \\ y_* & x_* \end{bmatrix}$ and $D^2 f(E_*;\beta^*)(V,V) = \begin{bmatrix} 0 \\ -2\beta - 2\gamma \end{bmatrix}$. Therefore,

$$W^T f_\beta(E_*;\beta^*) = 0$$
$$W^T \left[Df_\beta(E_*;\beta^*)V \right] = y_* - x_* \neq 0$$
$$W^T \left[D^2 f(E_*;\beta^*)(V,V) \right] = -2\beta - 2\gamma \neq 0$$

Hence the system (19) satisfies all the necessary conditions of *Sotomayor's theorem* and thus the system (19) experiences a transcritical bifurcation at the coexistance equilibrium point E_* for the bifurcation parameter β [59].

3.1 Significance of β as a transcritical bifurcation paremeter

We have numerically investigated the dynamic behaviour of the system (19) for the variation in the growth of pest populations (β). Let β_0 be the initial condition for the existence of E_*. The parameters must follow the conditions $\alpha\gamma+1\geq 0$, $\beta+\gamma>0$ and $\alpha\beta-1\geq 0$ and to estimate β_0, we consider $\alpha=1$ and $\gamma=0.001$ which satisfy all the conditions. Therefore, we get $\beta_0=1$ after calculating $\det\left(J_{E_*}(\beta^*)\right)=0$ [61].

For $\beta<1$, there is no intersection between the rice isocline and pests isocline as represented in Fig. 3(a). In this case, the system (19) experiences unstable and no equilibrium point. For $\beta=1$, the isoclines of rice and pests intersect at the pest-free equilibrium point $E_1(1,0)$ as shown in Fig. 3(b). For $\beta>1$, the system (19) experiences an interior point $E_*(x_*,y_*)$ between the origin (0, 0) and $E_1(1,0)$ as represented in Fig. 3(c). The nature of the system (19) is illustrated at the interior equilibrium point $E_*(x_*,y_*)$ for the different values of $\beta>1$ which is described in Fig. 4. The rice-pests system (19) is stable at the interior critical point E_* for the bifurcation parameter $\beta\in[1,30]$; e.g., the system is stable for choices of β = 2, β = 8 and β = 30 as shown in Fig. 4(a)-4(c), respectively. But for $\beta=31$, there is a limit cycle at E_* which describes that the system (19) remains in ocillation with respect to time and becomes unstable as represented Fig. 4(d). Hence, the maximum value of β is $\beta^*=30$. Moreover, the transcritical bifurcation diagram with respect to bifurcation parameter β is represented in Fig. 5. The diagram also examines that the rice-pests system (19) is stable only for $\beta\in[1,30]$ and gradually tends to an imbalance situation with the increase of the growth of the pest populations (β) and even the system will be defeated for high growth rate ($\beta>30$).

Hence the bifurcation analysis suggests to control the pest populations for the sustainable management of rice-pests system (7), otherwise the high pest populations will destroy system.

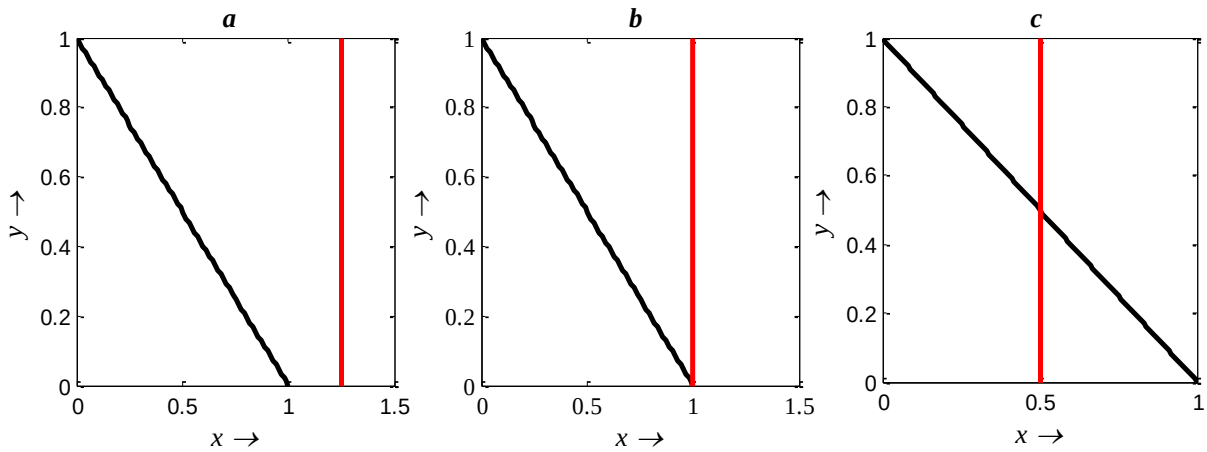

Fig. 3. Diagram for the isocline of the rice (vertical) and the pest's isocline (inclined): (a) $\alpha=1$, $\beta=0.8$, $\gamma=0.001$; (b) $\alpha=1$, $\beta=1$, $\gamma=0.001$; (c) $\alpha=1$, $\beta=2$, $\gamma=0.001$.

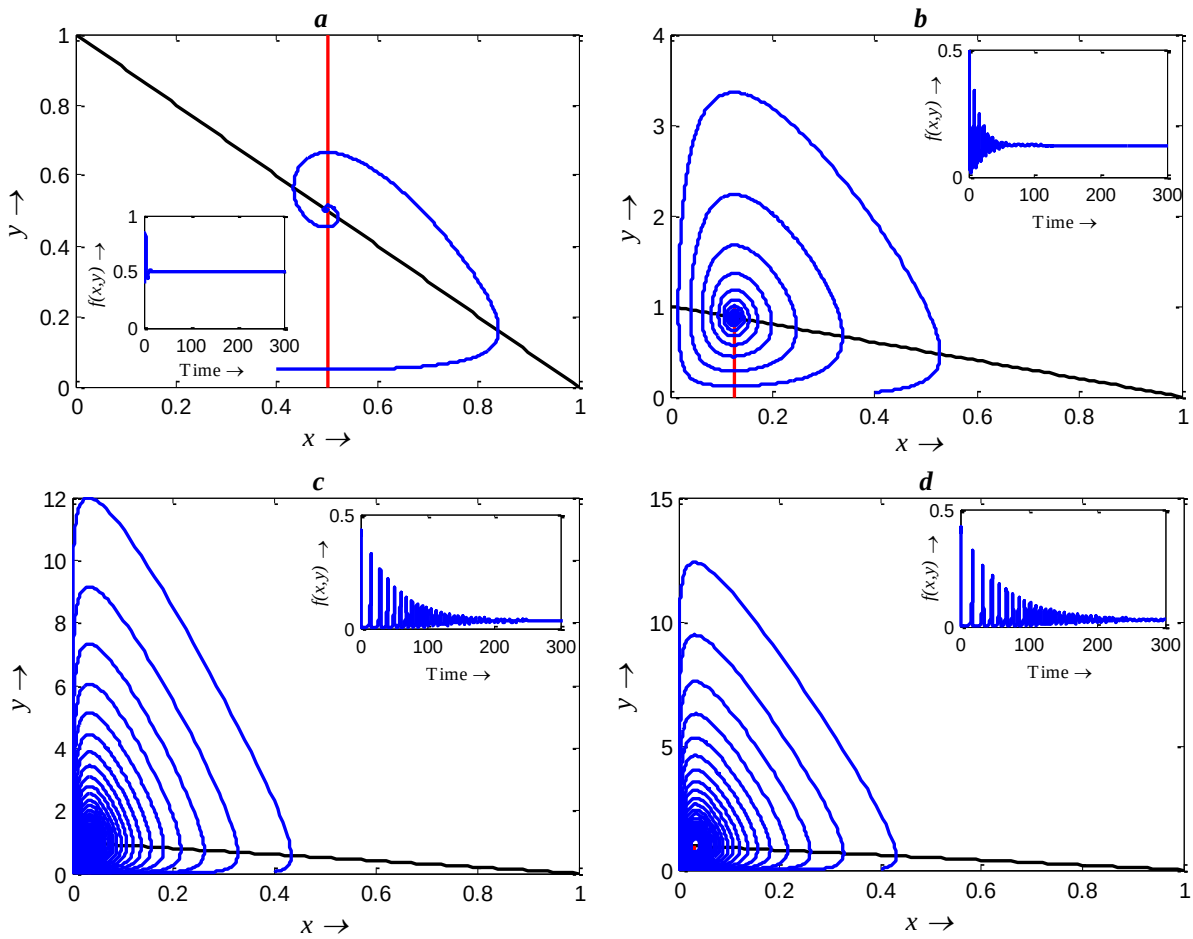

Fig. 4. Phase plane describing the nature of the system (19) at the interior point for the variation of β : (a) $\beta = 2$, (b) $\beta = 8$, (c) $\beta = 30$ and (d) $\beta = 31$ where $\alpha = 1$ and $\gamma = 0.001$ remain same.

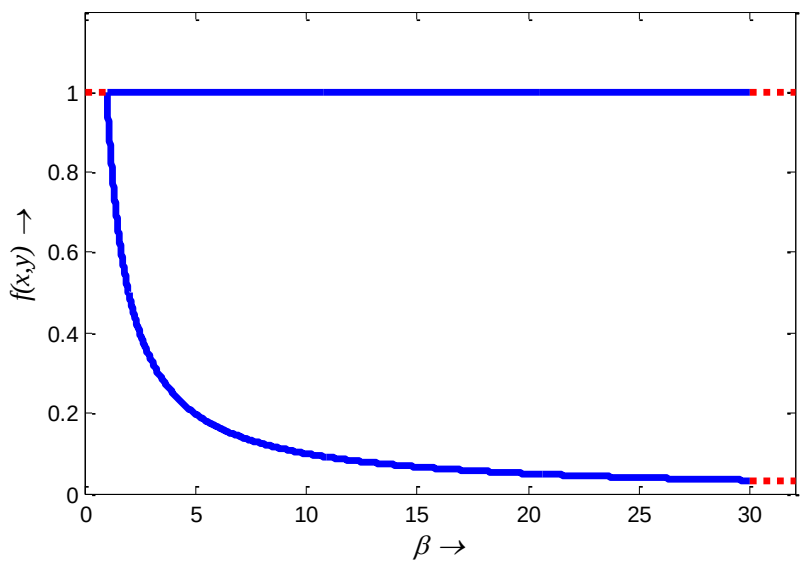

Fig. 5. Transcritical bifurcation diagram of the rice-pests system (19) for the bifurcation parameter β showing the system (19) stable for $\beta \in [1,30]$ (continuous blue line) and unstable for $0 \le \beta < 1$ and $\beta > 30$ (dash red line). The bifurcation diagram reveals that the rice-pests system (19) is present within the acceptable thresholds of the pests population and is destroyed for above and below the acceptable thresholds.

4. Dynamical modelling in form of a hybrid natural/chemical IPM strategy

4.1 Assumption and formulation of the optimal control problem

Millions of metric tons (MMT) of rice are lost annually only due to pest infestations amounting to about 37% of the annual production in world. With the increase in the annual rice yield, the number of annual production and losses gradually increase as shown in Fig. 6. Fig. 6 depicts the annual losses from 1960 (~10.6 MMT) to 2013 (~41.05 MMT). On the other hand, the mathematical model of the rice-pest system (7) shows that the production of rice inversely changes with the pest population density.

Cultural methods consist of some methods or techniques that arise crop production naturally by controlling pests. Good soil preparation, use of indigenous varieties, use of mesh screen, pruning, encouraging insect predators, and crop rotation are the most used cultural methods [62]. Chemical control is the most effective strategy for instantaneous pest control when cultural methods cannot be used. However the overuse of pesticides and insecticides seriously hampers the food quality and impact aquatic ecosystems [63].

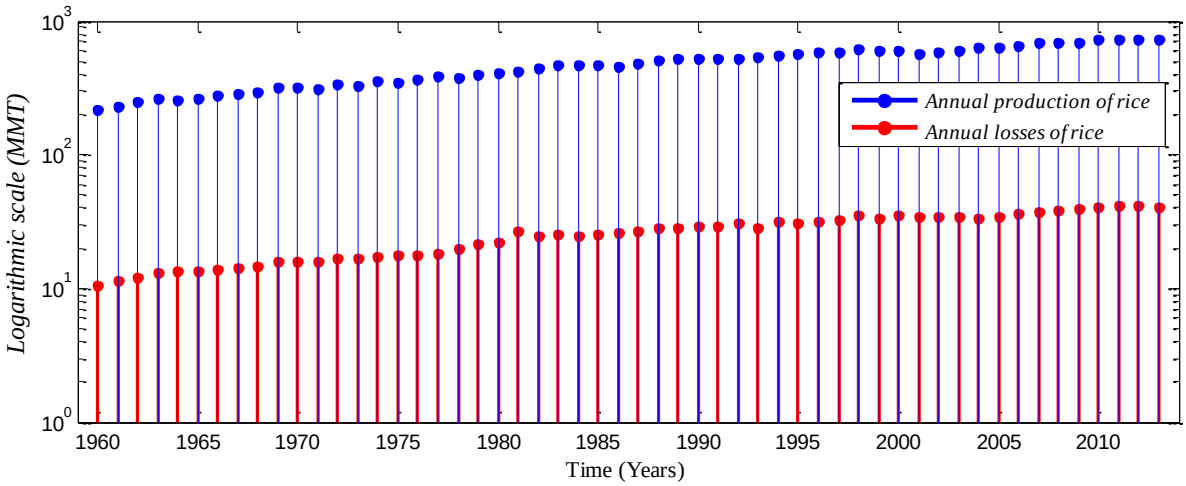

Fig. 6. Annual production and losses (due to the attack of pests) of rice from 1960-2013 [9] represented under a logarithmic scale with MMT measurement unit. In comparison to the production, we see that losses decrease but that their absolute number is very high and that losses increased over the years.

Wiping out the entire pest population from the agricultural field is impossible, and an attempt can be unsafe, expensive, and may lead to a rebound of pest numbers [12]. To control the pest population, we first apply cultural methods that are safe, cheap, and easily applicable. We then model the application of chemical controls (pesticides) only when the unacceptable action thresholds are crossed [12]. The application of pesticides depends on the species of pest and their density.

An important question is the determination of the point in time when to apply pesticides and for how long it should continue. Let u_1 denote cultural methods which are used incessantly throughout the entire cultivation period because they are natural and safe. Let u_2 denote chemical controls (pesticides) which are used only when the pest level crosses the acceptable threshold. The controls u_1 and u_2 both take values between 0 and 1. Here, $u_1, u_2 = 0$ denote that no control is applied and $u_1, u_2 = 1$ indicate controls are applied with full effort. Let A_T denote the unacceptable threshold of the pest population. The unacceptable threshold can be defined as: $A_T = \mu\Delta$, where μ denotes an unacceptable pest population per unit area and Δ represents the total area of a field. The threshold can depend on the pest variety, pest size, and site/region [12]. Let $D(x_2, u_2, E)$ denote the function of the nearest ecosystem which is potentially subject to pesticide damage, with x_2 being the pest population and E represents environmental damage. E is defined in the interval [0, 1] with $E = 1$ denoting that the pollution reaches the most acceptable level ($E = 0$ denotes zero pollution). The damage function, D, is increased in the rise of x_2 and E but decreased in the application of u_2, with $D \in [0,1]$ [64,65]. $D = 1$ represents the upper level of acceptable damage threshold, meaning the application of pesticides should be stopped i.e., $u_2 = 0$. Conversely, $D = 0$ denotes that there is no damage, meaning pesticides can be applied with full effort i.e., $u_2 = 1$. Here E is proportionately increased with the application of u_2. If $E \to 1$ as $u_2 \to 1$ which leads

$D \to 1$ that means the use of pesticides should be stopped immediately. To reflect this, the following decision model has been developed:

$$F(D,t) = \begin{cases} \left.\begin{matrix} u_1(t) \in [0,1] \\ u_2(t) = 0 \end{matrix}\right\}, & \forall x_2(t) \le A_T \\ \left.\begin{matrix} u_1(t), u_2(t) \in [0,1] \\ u_2(t) = 1 - D\left(x_2(t), u_2(t), E(t)\right) \end{matrix}\right\}, & \forall x_2(t) > A_T \end{cases} \tag{23}$$

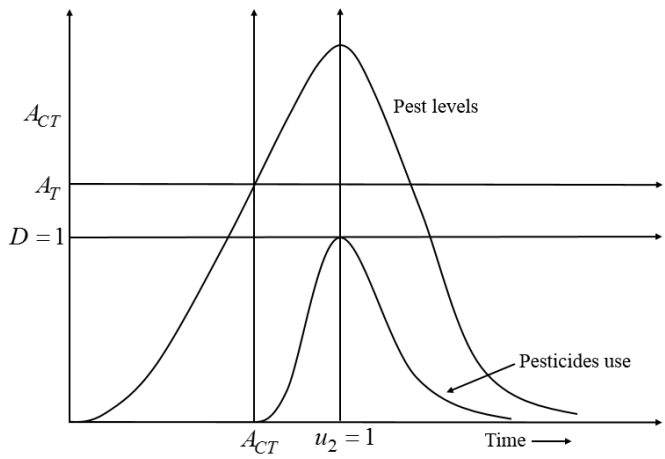

Fig. 7: A decision-making diagram of the model expressed by Eq. (23) describing the best time of pesticides application. Here, $D = 1$ represents the acceptable damage threshold reaches the maximum level, which means environmental pollution reaches the most acceptable level; A_T denotes the action threshold to determine the use of pesticides to control the pest population; A_{CT} represents the interval in which pesticides should be applied.

A control model has been proposed where cultural methods and chemical controls are applied as two control variables with the aim to minimize the density of the rice pests. The dynamical relationships between the annual production of rice and the pest population under controls is described below:

(i) When the cultural methods are applied during cultivation, the production rate of rice increases rapidly. Since the use of cultural methods does not depend on the density of rice pests, let $u_1 x_1$ be the increment in the annual production of rice. Therefore, the first equation of Eq. (7) can be represented by including control as

$$\frac{d}{dt} x_1(t) = \left(\alpha_1 - \beta_1 x_2(t)\right) x_1(t) - d_1 x_1^2(t) + u_1(t) x_1(t) \tag{24}$$

(ii) When cultural methods are applied in rice cultivation, several rice pests die off because of soil rotation and the presence of predators. Therefore, the cultural methods decline the density of rice pests and let $u_1 x_1 x_2$ be the declining number of pests due to the adoption of cultural methods. On the other hand, when emergency situations, pesticides are applied according to Eq. (23), which significantly reduces the pest population. Since the application of pesticides in the cultivation of rice depends on the amount of cultivated area, let $u_2 x_1 x_2$ be the decline in density of pest population after the use of pesticides. Hence, the second equation of Eq. (7) can be represented considering the decision model (23) as in the following

$$\frac{d}{dt} x_2(t) = \left(\beta_2 x_1(t) - \alpha_2\right) x_2(t) - d_2 x_2^2(t) - F(D,t) x_1(t) x_2(t) \tag{25}$$

The modified rice-pests system (7) under all controls can be represented by arranging Eq. (24) and Eq. (25) as in the following:

$$\begin{cases} \frac{d}{dt}x_1(t) = \left(\alpha_1 - \beta_1 x_2(t)\right)x_1(t) - d_1 x_1^2(t) + u_1(t)x_1(t) \\ \frac{d}{dt}x_2(t) = \left(\beta_2 x_1(t) - \alpha_2\right)x_2(t) - d_2 x_2^2(t) - F(D,t)x_1(t)x_2(t) \\ x(t) = (x_1(t), x_2(t)), \quad \forall t \\ x(t) = \left(x_{10},\ x_{20}\right),\ when \quad t = 0 \end{cases} \tag{26}$$

The characteristics of the controls are represented in the following measurable control set

$$U = \left\{(u_1(t), u_2(t)) : 0 \le u_i(t) \le 1),\ i = 1,2 \ \text{ at } t \in [0,T]\right\} \tag{27}$$

Where T is a preselected period for applied controls. The objective functional of the control model (26) becomes

$$\text{Minimize } J(u_1,u_2) = \int_0^T \left(x_2(t) + \frac{A}{2}u_1^2 + \frac{B}{2}u_2^2 \right) dt \tag{28}$$

The optimal control model which approximates model (26) can be represented [48] as:

$$\begin{cases} \text{Minimize } J(x,u) = \int_0^T L(t, x(t), u(t))dt \\ \text{Subject to } \quad x' = g(t, x(t), u(t)) \\ \qquad\qquad u(t) \in U, \quad \forall t \in [0,T] \\ \qquad\qquad x(0) = x_0 \end{cases} \tag{29}$$

where $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $g(t, x(t), u(t)) = \begin{pmatrix} \left(\alpha_1 - \beta_1 x_2(t)\right)x_1(t) - d_1 x_1^2(t) + u_1(t)x_1(t) \\ \left(\beta_2 x_1(t) - \alpha_2\right)x_2(t) - d_2 x_2^2(t) - F(D,t)x_1(t)x_2(t) \end{pmatrix}$, $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$, and the integrand of the performance index is $L(t, x(t), u(t)) = x_2(t) + \frac{A}{2}u_1^2 + \frac{B}{2}u_2^2$

here A and B are used for cost balancing weight parameters for the control variables; the function g is continuously differentiable; and functions $u(t)$ and $x(t)$ are piecewise continuous differentiable. In this problem, $u(t)$ belongs to a certain space U that may be a piecewise continuous function or a space of measurable functions which satisfy all constraints of the problem. The main goal of the objective functional is to increase the annual rice yield by minimizing the pest population by simultaneously considering the controls with the lowest costs. A schematic diagram of the rice-pest-control system (26) is shown in Fig. 8 to illustrate the dynamic behaviour of the species under control.

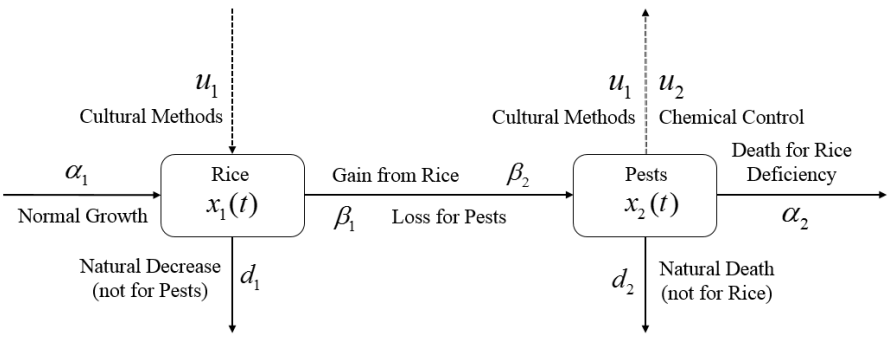

Fig. 8. Schematic diagram of the rice-pest-control system (26) describes the rice-pests system (7) under control. The diagram also shows that the control strategies, cultural methods and chemical control (pesticides), increase the production of rice and decline the corresponding pests population.

4.2 Characterization of the optimal control

To estimate the necessary conditions for the optimality of an optimal control problem (29), *Pontryagin's maximum principle* has been imposed in terms of the *Hamiltonian H(t)* defined as [66]

$$H(t,x,u,\lambda) = L(t,x,u) + \sum_{i=1}^{2} \lambda_i\, g(t,x,u)$$

$$= x_2(t) + \frac{A}{2}u_1^2(t) + \frac{B}{2}u_2^2(t) + \lambda_1\left[\left(\alpha_1 - \beta_1 x_2(t)\right)x_1(t) - d_1 x_1^2(t) + u_1(t)x_1(t)\right] +$$
$$\lambda_2\left[\left(\beta_2 x_1(t) - \alpha_2\right)x_2(t) - d_2 x_2^2(t) - F(D,t)x_1(t)x_2(t)\right]$$

where λ_i, $i = 1,2$ is the co-state variable which satisfies the following adjoint equations

$$\frac{d\lambda_1}{dt} = -\frac{dH(t,x,u,\lambda)}{dx_1(t)} = -\lambda_1\left[(\alpha_1 - \beta_1 x_2(t)) - 2d_1 x_1(t) + u_1(t)\right] - \lambda_2\left[\beta_2 x_2(t) - F(D,t)x_2(t)\right], \text{ and}$$

$$\frac{d\lambda_2}{dt} = -\frac{dH(t,x,u,\lambda)}{dx_2(t)} = -1 + \lambda_1\beta_1 x_1(t) + \lambda_2\left[(\alpha_2 - \beta_2 x_1(t)) + 2d_2 x_2(t) + F(D,t)x_1(t)\right]$$

as well as the *transversality conditions* $\lambda_1(T) = 0$ and $\lambda_2(T) = 0$.

Now, to obtain the optimal solution of the controls, Theorem 4.1 and Theorem 4.2 have to be proven as shown below by applying *Pontryagin's maximum principle*.

Theorem 4.1. *The control variables for the acceptable damage threshold attain the optimal solutions* $(u_{1*}, u_{2*}) = \left(\max\left\{0, \min\left(1, \frac{\lambda_2 x_{1*}x_{2*} - \lambda_1 x_{1*}}{A}\right)\right\}, 0\right)$ *for which the objective function* J *over* U *is minimized.*

Proof: At the acceptable damage threshold, there is no use of pesticides i.e. $u_2 = 0$. Therefore, let's differentiate the *Hamiltonian* (*H*) with respect to the control variable u_1 only, then it becomes as

$$\frac{dH}{du_1} = Au_1 + \lambda_1 x_1 - \lambda_2 x_1 x_2 \tag{30}$$

By applying the conditions of optimality in Eq. (30), the characterization of the control variable u_1

(i) when $\frac{dH}{du_1} > 0$ then $u_1 > \frac{\lambda_2 x_1 x_2 - \lambda_1 x_1}{A}$ but for the minimization problem $u_1 = 0$

(ii) when $\frac{dH}{du_1} = 0$ then $u_1 = \frac{\lambda_2 x_1 x_2 - \lambda_1 x_1}{A}$

(iii) when $\frac{dH}{du_1} < 0$ then $u_1 < \frac{\lambda_2 x_1 x_2 - \lambda_1 x_1}{A}$ but for the minimization problem $u_1 = 1$

Therefore $u_1^* = \begin{cases} 1 & \text{when } dH/du_1 < 0 \\ \frac{\lambda_2 x_1 x_2 - \lambda_1 x_1}{A} & \text{when } dH/du_1 = 0 \\ 0 & \text{when } dH/du_1 > 0 \end{cases}$, which can be written in the following compact form

$$u_{1*} = \max\left\{0, \min\left(1, \frac{\lambda_2 x_{1*}x_{2*} - \lambda_1 x_{1*}}{A}\right)\right\}$$

Since $u_2 = 0$ at the acceptable threshold, the compact form of u_2 is $u_{2*} = 0$. Then, the optimal solutions of the control variables for the acceptable threshold are

$$(u_{1*}, u_{2*}) = \left(\max\left\{0, \min\left(1, \frac{\lambda_2 x_{1*}x_{2*} - \lambda_1 x_{1*}}{A}\right)\right\}, 0\right)$$

Hence it completes the proof.

Theorem 4.2. *The control variables for the action threshold attain the optimal solutions* $(u_1^*, u_2^*) = \left(\max\left\{0, \min\left(1, \frac{\lambda_2 x_1^* x_2^* - \lambda_1 x_1^*}{A}\right)\right\}, \max\left\{0, \min\left(1, \frac{\lambda_2 x_1^* x_2^*}{B}\right)\right\}\right)$ *for which the objective function* J *over* U *is minimized.*

Proof: For the action threshold, the decision model is fully active i.e. $F(D,t) = u_1(t) + u_2(t)$. Therefore, let's differentiate the *Hamiltonian* (*H*) with respect to the control variables, then it becomes as

$$\frac{dH}{du_1} = Au_1 + \lambda_1 x_1 - \lambda_2 x_1 x_2 \tag{31}$$

$$\frac{dH}{du_2} = Bu_2 - \lambda_2 x_1 x_2 \tag{32}$$

By applying the conditions of optimality in Eq. (31), the characterization of the control variable u_1

(i) when $\frac{dH}{du_1} > 0$ then $u_1 > \frac{\lambda_2 x_1 x_2 - \lambda_1 x_1}{A}$ but for the minimization problem $u_1 = 0$

(ii) when $\frac{dH}{du_1} = 0$ then $u_1 = \frac{\lambda_2 x_1 x_2 - \lambda_1 x_1}{A}$

(iii) when $\frac{dH}{du_1} < 0$ then $u_1 < \frac{\lambda_2 x_1 x_2 - \lambda_1 x_1}{A}$ but for the minimization problem $u_1 = 1$

Therefore $u_1^* = \begin{cases} 0 & when\ \ dH/du_1 > 0 \\ \frac{\lambda_2 x_1 x_2 - \lambda_1 x_1}{A} & when\ \ dH/du_1 = 0 \\ 1 & when\ dH/du_1 < 0 \end{cases}$, which can be written in the following compact form

$$u_1^* = \max\left\{0, \min\left(1, \frac{\lambda_2 x_1^* x_2^* - \lambda_1 x_1^*}{A}\right)\right\}$$

Similarly, by applying the conditions of optimality in Eq. (32), the characterization of the control variable u_2 becomes

$u_2^* = \begin{cases} 0 & when\ \ dH/du_2 > 0 \\ \frac{\lambda_2 x_1 x_2}{B} & when\ \ dH/du_2 = 0 \\ 1 & when\ dH/du_2 < 0 \end{cases}$, which can be written in the following compact form

$$u_2^* = \max\left\{0, \min\left(1, \frac{\lambda_2 x_1^* x_2^*}{B}\right)\right\}$$

Then, the optimal solutions of the control variables are

$$(u_1^*, u_2^*) = \left(\max\left\{0, \min\left(1, \frac{\lambda_2 x_1^* x_2^* - \lambda_1 x_1^*}{A}\right)\right\}, \max\left\{0, \min\left(1, \frac{\lambda_2 x_1^* x_2^*}{B}\right)\right\}\right)$$

Hence the theorem completes the proof.

5. Results

Numerical simulations have been carried out for the rice-pests system (7) and rice-pest-control system (26) using the ode45 solver in MATLAB. This section aims to illustrate the dynamic change of the annual production of rice and the growth of the rice pests before and after adopting the control methods. In this case, the numerical values of parameters used in the simulations are taken from Table 1 and the initial conditions of the state variables chosen are $x_{10} = 4.679,\ x_{20} = 0.05085$. We consider 12 months (1 year) for the simulations.

Table 1
A description of parameters used in this study, including the numerical value. All of these are secondary parameters (derived/estimated or collected from other sources), two of which are estimated. We have conducted a statistical analysis for parametric estimation after collecting and observing the corresponding data collecting from different research [9,67,68].

Symbol	Values	Descriptions	References
α_1	4.679 metric tons/hector	Reproduction rate of rice	Estimated [9]
α_2	94.915%	Mortality rate of the pest population	Estimated [67,68]
β_1	37% of the total production	Losing rate of rice due to consumption of pests	[6,39]
β_2	37% of the total production	Consumption rate of the pest population	[6,39]
d_1	10% of the total production	Natural decrease rate of rice	[9]
d_2	5.085%	Death rate of pests due to natural causes	[9]

Both the time series and phase portrait of the considered dynamics are shown in Fig. 6. Initially, the growth of the rice-plants sharply increases due to low numbers in the pest population. When the pest population increases by getting sufficient food, the growth of the rice-plants is hampered so that eventually the pest population subdues (which in turn allows rice to grow again). The phase portrait is shown in Fig. 9 (b) spiralling out to a stable equilibrium at $(\hat{x}_1(t), \hat{x}_2(t)) = (4.12, 11.54)$.

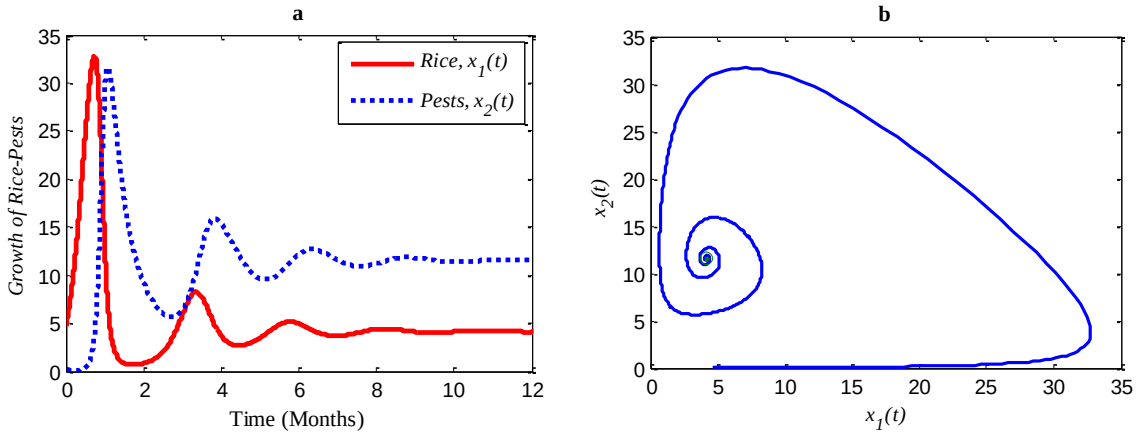

Fig. 9. Time series of (a) Rice-pest population, and (b) phase portrait of the rice-pest system (7) with the solution $(\hat{x}_1(t), \hat{x}_2(t)) = (4.12, 11.54)$

Fig. 10 (a) and (b) show (i) when the consumption rate β_2 of the pest population is reduced from 37% to 25%, the density of pest population decreases so that the rice is increasing again. Similarly, in (ii): when the consumption rate declines from 25% to 15%, the density of pests declines sharper than before, and the rice grows better again.

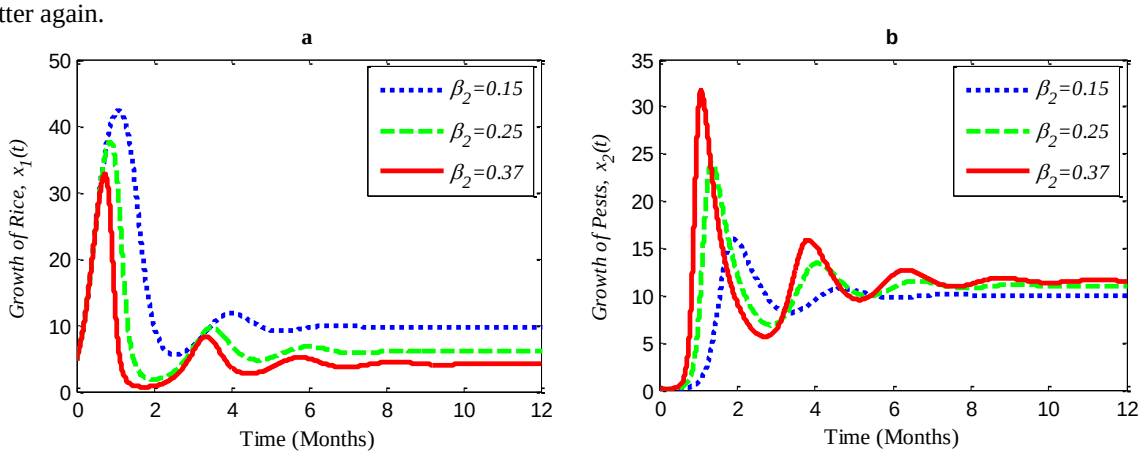

Fig. 10. Time series of (a) annual rice production, and (b) rice pest population for different consumption rates of the pests.

According to the result of Fig. 9 (a), the annual production of rice approaches 4.12 Metric tons/hector (Mt/h) whereas the growth rate of pests approaches 11.54%. The changes in the growth of rice for different consumption rates of pests are described in Fig. 10 which let us conclude that the production of rice can be increased if the consumption rate of pests is controlled.

When controls are applied, the results change. The results of the rice-pest-control system (26) are described for the following three scenarios designed on the efficacy of the control variable u_1 (cultural methods) and u_2 (chemical controls): (i) $u_1 \neq 0$ and $u_2 = 0$, (ii) $u_1 = 0$ and $u_2 \neq 0$, (iii) $u_1 \neq 0$ and $u_2 \neq 0$. The simulations are performed distinguishing the cases “without control” and “with control”. Here, "without control" represents the results of the rice-pests system (7) meaning there is no control strategy ($u_1 = 0$, $u_2 = 0$).

(i) When only the cultural methods are implemented to the system as a control variable, the annual losses of rice declined from 41.05 to 38.39 MMT approximately, at the same time, the production rate of rice increases from 4.12 to 8.81 Mt/h approximately which are shown in Fig. 11 (a). As a result of adopting cultural methods, the density of rice pests declines from 11.54% to 10.79% approximately over one period due to e.g., the enhancement of soil and rotation of crops as shown in Fig. 11 (b). Fig. 11 (c) shows that the system under the application of cultural methods converges to the stable equilibrium point at $(\hat{x}_1(t), \hat{x}_2(t)) = (8.81, 10.79)$.

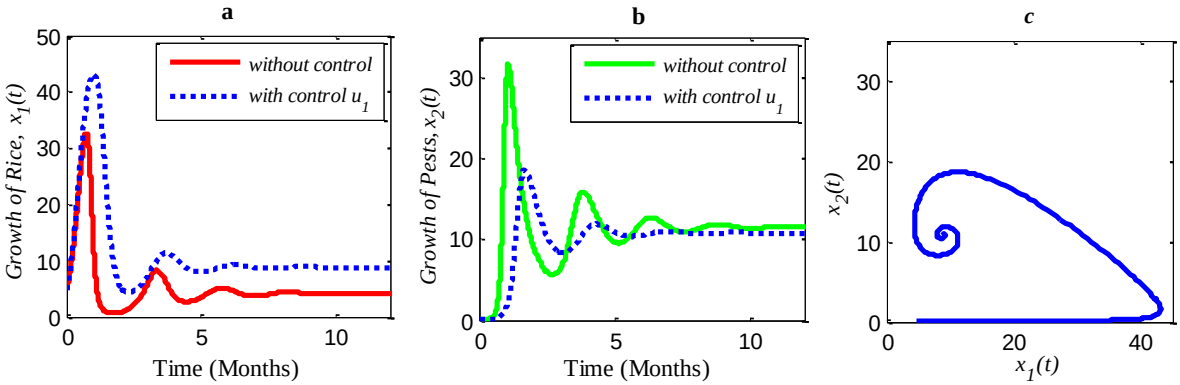

Fig. 11. Time series of (a) annual rice production under control u_1 only, (b) pests population under control u_1 only, (c) phase portrait of rice-pests under control u_1, where the solution is $(\hat{x}_1(t), \hat{x}_2(t)) = (8.81, 10.79)$.

(ii) When only the chemical control is implemented according to the condition expressed in Eq. (23), the pests population comes down from 11.54% to 9.17% approximately that's just over one-fifth of the total population shown in Fig. 12 (b). Because of the decreasing pest population, the annual losses of rice drop sharply from 41.05 to 32.62 MMT approximately with a decrease approximating 25%. As a result, the annual rice production rate substantially grows from 4.12 to 12.86 Mt/h approximately which is shown in Fig. 12 (a). The system converges to a stable equilibrium point at $(\hat{x}_1(t), \hat{x}_2(t)) = (12.86, 9.17)$, presented in Fig. 12 (c).

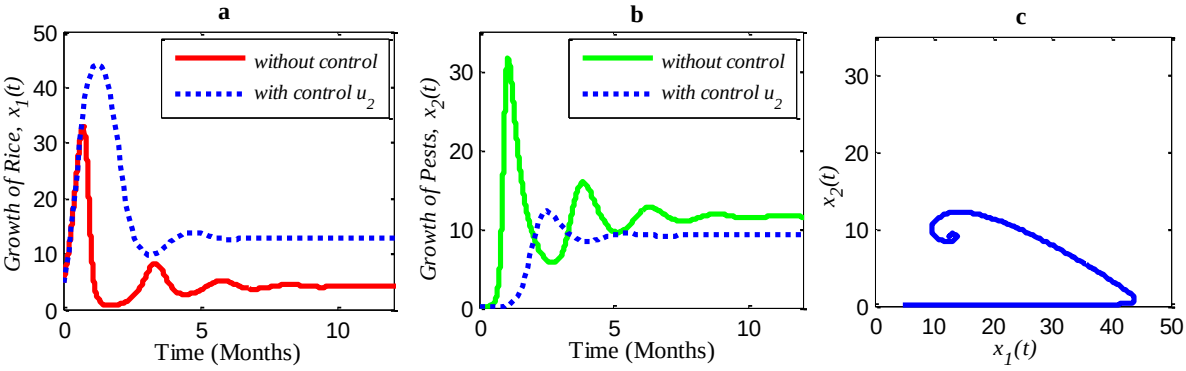

Fig. 12. Time series of (a) annual production of rice under control u_2 only, (b) pests population under control u_2 only, (c) phase portrait of rice-pests under control u_2 and the solution is $(\hat{x}_1(t), \hat{x}_2(t)) = (12.86, 9.17)$.

(iii) When the cultural methods are used continuously, and chemical controls are applied only in the emergency (Eq. (23)) the density of the pest population decreases significantly by 33.02% to 7.73% (Fig. 13 (b)). As a result, the annual losses of rice dramatically fall from about 41.05 to 27.49 MMT. After simultaneously applying controls the annual rice production increases considerably to about 365% (19.18 Mt/h) as presented in Fig. 13 (a) thereby approaching the equilibrium point $(\hat{x}_1(t), \hat{x}_2(t)) = (19.18, 7.73)$ as shown in Fig. 13 (c).

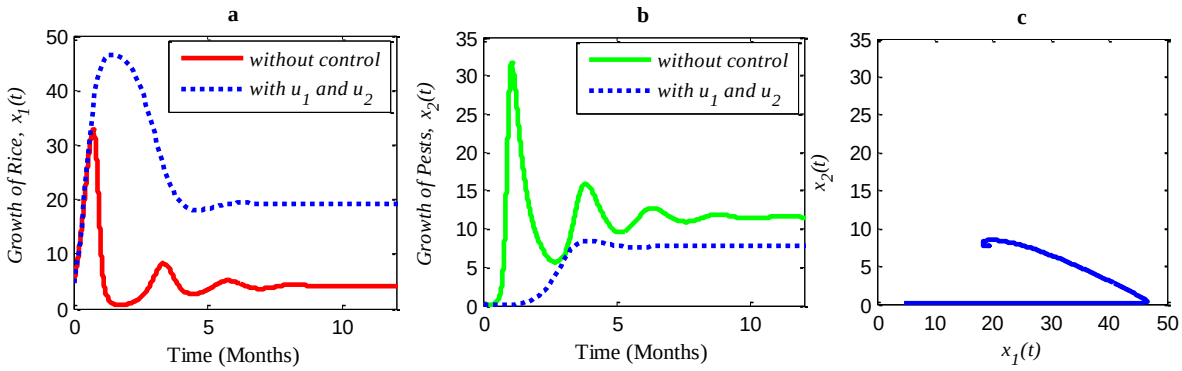

Fig. 13. Time series of (a) annual production of rice under controls u_1 and u_2, (b) pests population under controls u_1 and u_2, (c) phase portrait of rice-pests under controls u_1 and u_2, where the solution is $(\hat{x}_1(t), \hat{x}_2(t)) = (19.18, 7.73)$.

Next, we made a numerical comparison to analyse the results of all situations. The dynamic changes in the growth of the state variables in the three scenarios are shown in Fig. 14. As can be seen from Fig. 14, the annual production of rice increases significantly under both control strategies instead of just one, in contrast, the rice pests decrease dramatically. It also shows that cultural methods can control the density of a pest population, but chemical controls are comparatively more effective. From scenarios (i)-(iii), it is concluded that scenario (iii) is the best strategy to increase the annual rice production and reduce the density of rice pests.

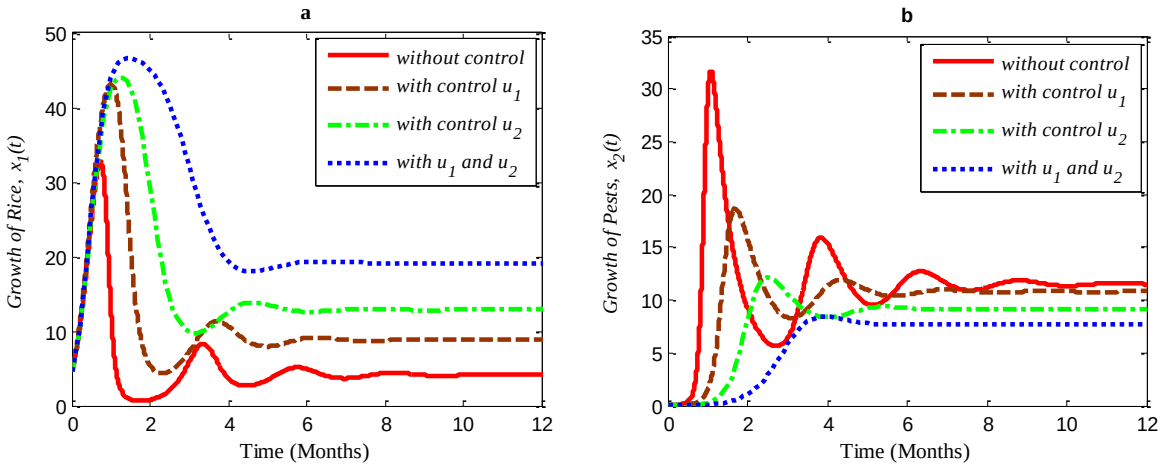

Fig. 14. A comparison between the scenarios (i)-(iii). Here, (a) represents the time series of annual production of rice, and (b) shows the time series of pest populations under three different scenarios.

5. Conclusion

Our study describes the interrelationship between yield of rice and loss through pest species and effective ways to increase the annual yield by controlling insects and improving growth conditions. First a new formulated dynamic mathematical model of a rice-pest system and how the pests harm the annual production of rice is set out. The annual production of rice decreases with the increasing number of rice pests and vice versa. The rice-pests system is examined through studying transcritical bifurcations (Fig. 4 and 5), which indicate stability for a number of pests within the acceptable thresholds and instability for above or below the acceptable thresholds.

An optimal control (minimization) problem is setup, considering cultural and chemical control as two control variables to minimize the density of rice pests and the annual losses of rice simultaneously. We conducted a positivity and a stability analysis at each equilibrium point to achieve model vigour and obtained necessary conditions for optimality of the problem to determine optimal control variables. The change in the annual production and paddy-loss due to implementing either one or both controls has been compared in detail with the annual production and paddy-loss without control.

Chemical controls, however, are potentially toxic unsafe with detrimental effects on the environment and nearby ecosystems. Therefore, it is recommended to apply chemical control only in emergencies. We have developed a decision model to determine when pesticides should be applied. The decision model is activated only when the level of insects exceeds the acceptable threshold and the model becomes inactive only when the level of acceptable damage threshold reaches the maximum level, which means that the model becomes inactive when environmental pollution reaches the most acceptable level. This situation continues to recur until the end of cultivation. The acceptable thresholds of pest population can vary with the pest species. In this case, the pesticide application time can change for the change in the acceptable thresholds.

Cultural methods have the potential to control rice pests in agriculture sustainably and simultaneously increase the annual production of rice as shown in Fig. 11. However, the use of chemical controls can further improve the yearly rice yield as shown in Fig. 12. Comparing Fig. 11 and Fig. 12, chemical controls show to be more effective and easier to apply than using cultural controls, contributing to higher annual yield thus contributing to local and global food security. However, when cultural methods are used widely and until the end of cultivation with chemical controls being also applied in emergency situations, the system can reduce the growth rate of rice pests by about 33.02% which reduces the annual losses about one-third, as represented in Fig. 13. We also carried a comparison among the presented three scenarios displayed in Fig. 14. The result of the comparison finds that if cultural methods can be used till the end of cultivation and chemical controls can be applied only in the emergency situations, it would be the best control strategy for increasing the annual rice production and controlling rice pests.

This study could be helpful in agriculture by enriching rice production and controlling pests naturally and artificially. However, the long-term effects of using chemicals even in smaller numbers is an outstanding

issue – chemical control can bring short-term relief but may damage the ecosystem with time. Also, the introduction of biological controls (predators and parasites) as alternative to chemical controls would make sense to be studied in this context as does the inclusion of changes of water quality because of the application of pesticides in crop fields and nearby aquatic ecosystems such as rivers or canals.

Conflict of Interest

The authors declare to have no conflict of interest.

Author contribution

Author contributions have been evaluated according to CRediT.

Sajib Mandal: Conceptualisation, Methodology, Software, Validation, Formal Analysis, Investigation, Data Curation, Writing - Original Draft, Writing – Review & Editing, Visualisation.

Sebastian Oberst: Conceptualisation, Methodology, Investigation, Validation, Writing - Review & Editing.

Md. Haider Ali Biswas: Conceptualisation, Methodology, Validation, Resources, Supervision, Project administration.

Md. Sirajul Islam: Conceptualisation, Supervision.

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