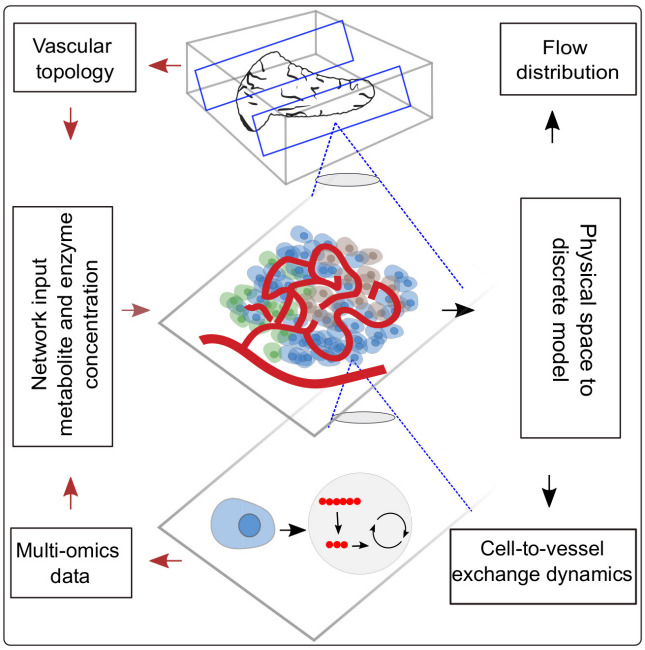
A graph-based framework for multi-scale modeling of physiological transport

Deepa Maheshvare M.[1], **Soumyendu Raha**[1], **and Debnath Pal**[1]

[1]Department of Computational and Data Sciences, Indian Institute of Science, Bengaluru 560012, India

Graphical abstract



Trillions of chemical reactions occur in the human body every second, where the generated products are not only consumed locally but also transported to various locations in a systematic manner to sustain homeostasis. Current solutions to model these biological phenomena are restricted in computability and scalability due to the use of continuum approaches where it is practically impossible to encapsulate the complexity of the physiological processes occurring at diverse scales. Here we present a discrete modeling framework defined on an interacting graph that offers the flexibility to model multiscale systems by translating the physical space into a metamodel. We discretize the graph-based metamodel into functional units composed of well-mixed volumes with vascular and cellular subdomains; the operators defined over these volumes define the transport dynamics. We predict glucose drift governed by advective-dispersive transport in the vascular subdomains of an islet vasculature and cross-validate the flow and concentration fields with finite-element based COMSOL simulations. Vascular and cellular subdomains are coupled to model the nutrient exchange occurring in response to the gradient arising out of reaction and perfusion dynamics. The application of our framework for modeling biologically relevant test systems shows how our approach can assimilate both multi-omics data from *in vitro* - *in vivo* studies and vascular topology from imaging studies for examining the structure-function relationship of complex vasculatures. The framework can advance simulation of whole-body networks at user-defined levels and is expected to find major use in personalized medicine and drug discovery.

biophysical interactions | discrete network model | functional networks | hierarchical modeling | microvasculature and microenvironment | solute transport | spatio-temporal dynamics | systems biology.

Correspondence: ***dpal@iisc.ac.in***

Significance

Within a tissue, the short-range and long-range communications that sustain metabolism is mediated by the vessels perfusing through the cells. When blood flows through the vasculature, nutrient exchange occurs from cell-to-vessel in accordance with the concentration gradients arising out of cellular and vascular dynamics. Therefore, the topology of blood flow and the anatomical arrangement of cells act as primary determinants influencing the functional response of a tissue. We introduce a graph-based model for understanding the multi-scale connectivity observed across functional networks in a tissue. Our approach offers the flexibility to couple different scales by defining each component to the desired scale so as to balance the need for completeness without compromising on necessary details to answer the questions of interest.

THE physiological system is a complex network in which each organ forms a subsystem and different subsystems interact to maintain overall homeostasis of the body. Within each subsystem functional networks exists at different levels of complexity. Metabolic and signaling networks within a cell, cell-to-cell communication networks in the extravascular region of tissue, cell-to-vessel communication networks, and the vascular network which couples the local dynamics to the global dynamics determine the functional behavior of all tissues. Bottom-up and top-down modeling approaches emulate the cellular dynamics and organ-level physiology. The ability to simultaneously capture the local and global dynamics by hierarchically bridging the communication networks existing across diverse scales is the key challenge in the holistic modeling of physiology.

Microscale cellular models use a bottom-up approach in which multi-omics data assimilated from high-throughput sources are employed to formulate and validate parameter-intensive kinetic models. These models capture the depen-

dency of intracellular dynamics on metabolic steady states and flux distributions (1–3). Owing to cellular heterogeneity and the existence of intercellular communication, autocrine and paracrine signaling (4–6), the response elicited by a single cell cannot be scaled to a cell community. Cell population models, therefore, employ discrete modeling approaches for examining cell-to-cell interactions such as intra- and inter-islet synchronization established by gap junctional coupling (7, 8).

Macroscopic organ scale compartment models (9) employ top-down modeling approach for predicting the bulk flow and elimination kinetics of biomolecules. These organ scale models rely on single-tube, parallel-tube or tank-in-series approximations for idealizing distribution volume of blood into compartments (10). For improving the mechanistic understanding of tissue-vessel interaction, multiphase porous media-based models representing the tissue volume as intravascular and multi-region extravascular compartments (e.g. capillary-interstitial-parenchymal exchange unit) (11, 12) emerged. However, these frameworks do not offer the possibility to fuse macro- and micro- scale models. Consequently, the effect of network architecture on microperfusion patterns (13) and its influence on the nutrient exchange cannot be investigated by these compartment models.

To overcome the limitations of compartment-based models, continuum approaches have been put forth for understanding the implications of morphological changes on the functional response of an organ. Conventional finite-difference, finite volume or finite element based formulations operate on a continuous domain and the equations are discretized and weak solutions are obtained. In the extravascular domain, continuum approaches are helpful in estimating the collective response of a tissue mass where the bulk of the tissue is smeared and treated as a homogeneous domain. Although homogenization simplifies the complexity of the computational domain, the approach is limited in its ability to probe aspects such as the influence of heterogeneous arrangement of cells on nutrient release patterns. In the intravascular domain, continuum approaches are suitable for analysing the effects of dilation of blood vessels, deformations that occur as a result of fluid-structure interactions. Advancements in imaging studies and the availability of microvascular datasets offer the possibility to model large scale networks. Discretizing the tortuous microvasculature vasculature for 3D modeling of the advection-dispersion physics gives rise to extensive computational overhead while employing continuum approaches. This has led to the development of graph- and hybrid- based approaches in which the vasculature infiltrating the tissue volume is represented as one dimensional network of pipes for modeling the flow and delivery of resources over networks (14–19). In summary, while efforts have been made to resolve: (i) spatial heterogeneity at subcellular scale (20), (ii) short-range communication in the microenvironment of cell communities, (iii) metabolic zonation in single sinusoid models (21), explicit models of long-range communications mediated by the vascular system remain underdeveloped at both intra-organ and inter-organ scale.

Towards this end we need a scalable hierarchical framework that allows us to bridge diverse scales for modeling production, consumption and distribution of biochemicals in a tissue microenvironment. We introduce a discrete modeling framework for simulating gradient-driven advection-dispersion-reaction physics of multispecies transport. Graph-theoretic approaches that have been proven successful in examining flow of information through large scale real-world networks is applied. We resort to discrete-vector calculus and use the operators defined on a finite-graph to spatially discretize and formulate the transport dynamics in the vascular domain as a "tank-in-series" model. Further, the computational domain for establishing the vessel-to-cell exchange and cellular dynamics within the cell are set up by combining ideas from other multiscale and Krogh cylinder models (21, 22). Dynamics of nutrient exchange from blood vessel to the layer of cells that lie in close proximity to the vessel surface is modelled; cellular reactions are explicitly modelled by representing cells as discrete volume nodes. Differential equations defining the interactions over nodal volumes embedded in the graph are solved by translating the physical domain into a metamodel in which the biophysical attributes are subsumed. This framework is suitable for the following key applications: (i) reduce the computational cost involved in the spatial discretization of large tissue volumes (Sec. A); our discrete approach is geared towards obtaining fast solution by reducing the system dimension and the metamodel is scalable into any domain; (ii) probe the effect of flow topology on scalar transport and the sensitivity of concentration dynamics to network parameters and variations in physiological setpoints. (Sec. B, D); (iii) assimilate multi-omics data from *in vitro* and *in vivo* studies and vascular topology from imaging studies (Sec. C) for examining the influence of structural changes on the functional response of a tissue.

The outline of this study is as follows. The procedure followed in translating the capillary vasculature into a weighted graph and the preliminary assumptions considered for setting up the computational domain are discussed in Section A and B. The governing equations of the flow distribution and the mathematical formulation of the discrete model of advection-diffusion-reaction physics are presented (Sec. C, D, E). We use our framework to model two physiologically relevant test systems: (i) advection-dispersion dynamics of glucose transport in the microvasculature, (ii) advection-dispersion-reaction dynamics of glucose-lactate exchange in the functionally coupled tissue-vascular domains. By applying our method, we predict glucose drift in the islet vasculature and cross-validate the flow and concentration fields of the multi-physics simulation with COMSOL simulations (Sec. A). We establish the cell-vessel link and predict the spatio-temporal evolution of glucose-lactate exchange in the extravascular and intravascular domains (Sec. C). We test the model behaviour for various flow topologies (Sec. B), different pressure drops and glucose doses (Sec. D).

I. Methodology

For setting up the discrete-modeling framework to study the multiphysics coupling in multiscale systems, we start by introducing the steps involved in constructing the computational domain which is a metamodel of the physical space, dissection of the metamodel into subdomains which form the functional units and formulation of the mathematical operators. The three main steps involved in our workflow are illustrated in Figure 1. *Create skeleton*: The topological organization and biophysical characteristics of the blood vessels are extracted by skeletonizing the reconstructed vasculature (Sec. A). *Solve flow distribution in the network*: The physical space is translated into a weighted graph representing a hydraulic circuit. The pressure and flow fields are computed over the network by establishing the relationship between the node and edge entities of the graph using Hagen-Poiseuille equation (Sec. C). *Solve advection-dispersion-reaction dynamics*: The metamodel is subdivided into functional units composed of cell and vessel subdomains. We combine multiple scales by coupling the uptake and release flux of cell domain ω^t with the carrier-mediated exchange flux occurring at cell-vessel interface. The metabolic reactions occurring at the cellular scale are modelled by biochemical rate laws. And the mass transport in the capillary domain Ω^{bv} is described by coupling the cell-to-vessel influx or outflux with the gradient-driven advection-dispersion transport in the capillary domain Ω^{bv} (Sec. D, E).

A. Construction of capillary networks

The topology of the blood vessels that exchange nutrients with the surrounding tissue is represented by one dimensional network of pipes. The biophysical attributes such as diameter and length of each segment, the blood vessel fragment connecting two branching points, are quantified as edge weights ($\mathbb{W}$) of the graph network ($\mathcal{G}$). We investigate four network configurations in this work. The 3D vasculature of pancreatic islet displayed in Fig. 2a and both the configurations of tumor network shown in Fig. 2b,c were reconstructed from binary images of segmented vessels examined in Chen et al. (23). We follow the workflow detailed below for generating a weighted graph from the binary images: (i) The multipage .tiff image was rendered into a 3D volume using the ray casting technique available in *3D Slicer* (25) by specifying the spacing of the image stack in the xyz directions. The dimensions of the input stack and domain size of the reconstructed volume are given in Table I. (ii) The largest connected region of the segmented volume was filtered using the *Island* effect available in *Slicer* and the 3D object was exported in a stereolithography file (.stl) for skeletonization in the *Vascular Modelling Toolkit library (VMTK)* (26). (iii) *vmtksurfaceclipper* was employed to open the surface at the network inlet and *vmtknetworkextraction* algorithm was utilized to skeletonize the geometry. This yielded a network with nodes (vertices) ($\mathbb{V}$) positioned at the N-furcation points or terminal ends, and edges ($\mathbb{E}$) formed by the vessel segment linking two nodes. (iv) Segment length and diameter were computed by tracing the shortest path between two nodes and extracting the maximum inscribed sphere radius (thickness), respectively. The 2D structure of the mesentery vasculature displayed in Fig. 2d was generated by parsing the diameter, length of vessels, and topology information available in *Amira* mesh file provided in Esposito et al (24). Statistics of diameter and length distributions are shown in Fig. 2e,f. We represent the skeletonized geometry of the capillaries as a weighted graph, $\mathcal{G}(\mathbb{V} \in \mathbb{R}^m, \mathbb{E} \in \mathbb{R}^n, \mathbb{W} \in \mathbb{R}^n)$ for investigating the advection-diffusion-reaction dynamics. The cardinality of all the properties and the operators defined on $\mathcal{G}$ is summarized in supplementary S1 Table.

TABLE I
Specifications of the computational domain

Tissue (unit)	Islet	Tumor Design 1 & 2	Mesentery
Image dimensions			
x (pixels)	366	529	
y (pixels)	366	529	-
z (pixels)	253	348	
Image spacing(μm)	0.6	2.31	-
Domain size			
x (μm)	219.6	1221.9	
y (μm)	219.6	1221.9	-
z (μm)	151.8	803.8	
# Vessel segments	52	63	489
v_{in} (μm/s)	160^a	400^b	400^b
Q_{in}/Q_{out}(nl/min)	3.76	23.8	40
P_{in} (mmHg)	60	60	-
P_{out} (mmHg)	-	-	0

- number; v_{in} - inlet velocity; Q_{in} - inflow rate; Q_{out} - outflow rate; P_{in} - inlet pressure; P_{out} - outlet pressure; a - (27); b - (28);

B. Preliminary assumptions

The time-dependent uptake and release of biochemicals by a tissue is determined by the gradient-driven transport across the capillary vessels which facilitate the transvascular exchange of solute with the tissue interstitium. Subsequently, the carrier-mediated sites present on the plasma membrane of cells aid the uptake of nutrient resources from the interstitial space for metabolism. Here we introduce a graph-based mathematical framework for capturing blood-tissue exchange. The following simplifying assumptions are made in our model similar to the assumptions considered in the other multiscale studies. (i) Assuming the scalar concentration in the interstitial fluid attains rapid-equilibrium with the concentration in the blood (12), the interstitial compartment is not modelled. (ii) The endothelial layer of the capillary surface is lined with metabolite transporters which promote facilitated diffusion of biochemicals across the capillary wall, similar approach has been presented in Heaton et al. (29). We consider this as a reasonable assumption since a large fraction of the endocrine cells of the pancreas lie in close contact with the surface of the capillaries (30). (iii) Due to the deficiency of lymphatic drainage in the islets of Langerhans (31), fluid exchange with lymph vessels is not modelled (32). Based on these considerations we subdivide each blood vessel branch and the layer of tissue surrounding the capil-

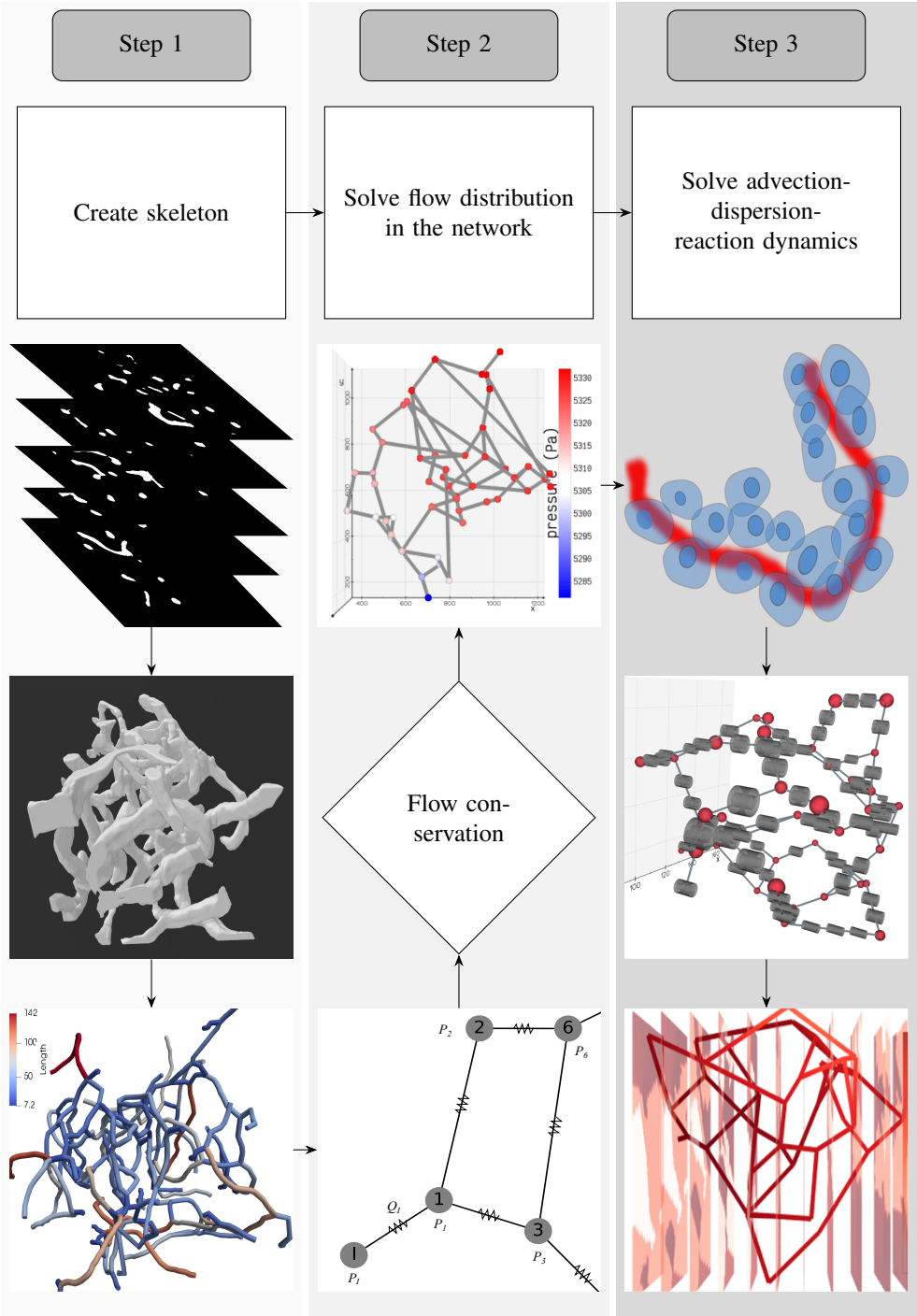


Fig. 1. The workflow involved in setting up the system for simulating the spatio-temporal evolution of biochemical species is illustrated: *1. Create skeleton*: top, z-stack of microscopic images; middle, three dimensional volume of the reconstructed vascular geometry; bottom, the length of blood vessel branches are color-coded. *2. Solve flow distribution in the network*: bottom, the vascular network represented as a hydraulic circuit; middle: from the estimated nodal pressures, flow thorough the vessel branches are computed and checked whether the mass conservation holds at the junctions; top: 3D visualization of pressure gradient observed across the vasculature. *3. Solve advection-dispersion-reaction dynamics*: top: creation of the computational domain which includes blood vessel and the layer of the cells lining the vessel surface; middle: functional unit of the computational domain consisting of the vascular and cell sub-domains; bottom: concentration gradient observed across the vasculature after solving the advection-dispersion-reaction dynamics.

lary into discrete functional units, diagrammed in Fig. 3d. The molar transport occurring through the blood vessel compartment of these functional units is modelled by one dimensional advection-dispersion equation (33) that accounts for convection flux, the axial and radial diffusive flux of the solute molecules. The transcapillary exchange flux and the cellular processes occurring in the tissue compartment of each functional unit are modelled by rate expressions that capture the kinetics of metabolite specific transporters and enzyme catalysed reactions, respectively. The governing equations that model the inter-compartment dynamics of these well-mixed volumes embedded in the finite connected network

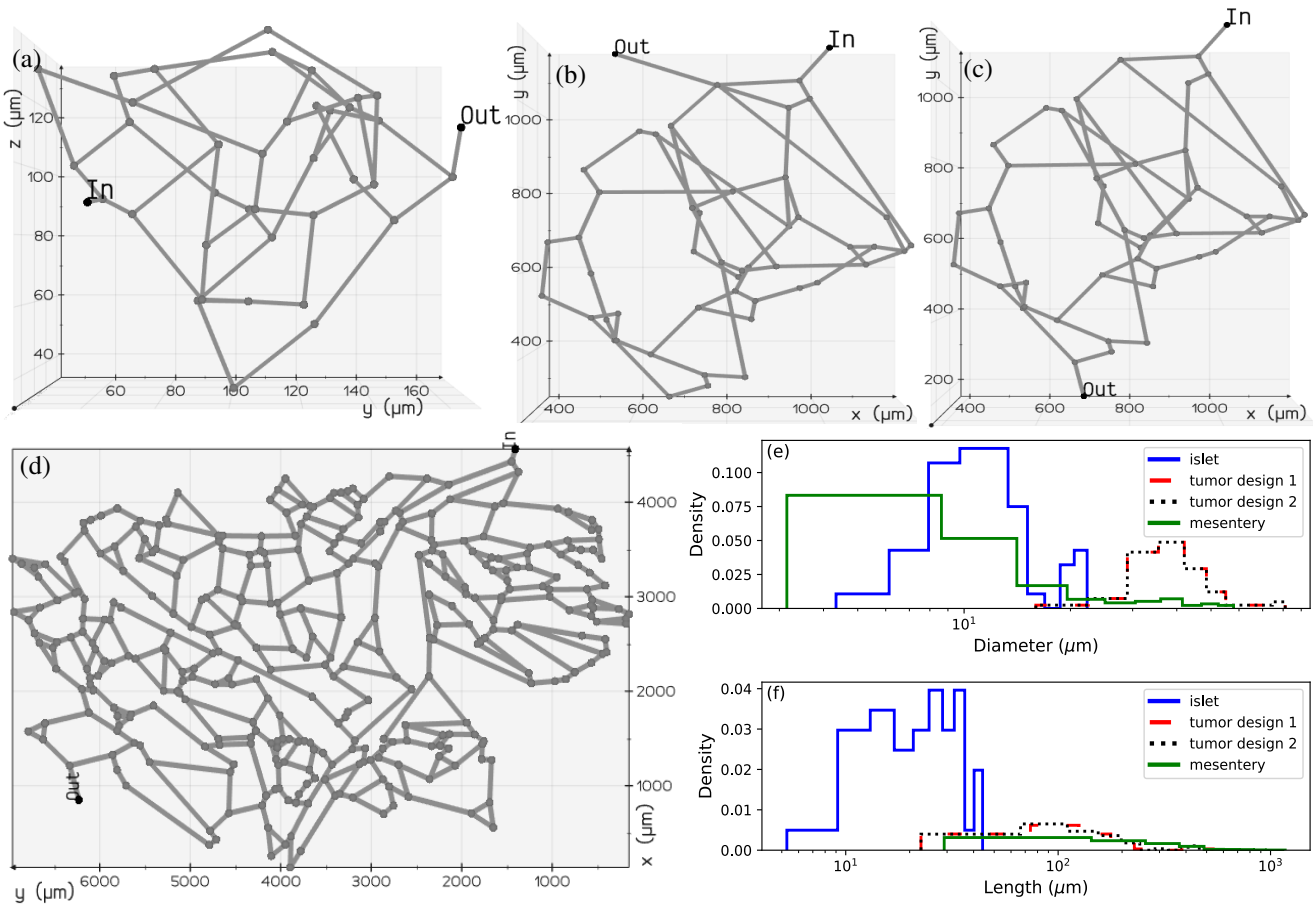


Fig. 2. Microvascular network configurations: (a) pancreatic islet, (b) tumor design 1, (c) tumor design 2, and (d) mesentery generated after skeletonization of 3D volumes reconstructed from the image geometries studied in (23) and (24). Diameter (e) and length (f) distributions of the vessel segments present in the four geometries.

representation of the capillary bed is detailed in Section D and E.

C. Mathematical formulation of flow distribution in the network

To simulate the spatio-temporal evolution of chemical species, we first solve for the flow field in the vascular branches using the approach generally applied in studies that focus on modeling flow distribution in branching networks (19, 24, 34, 35).

Domain: To perform fluid flow simulations, the oriented graph depicted in Fig. 3a was used for setting up the linear system of equations.

Equations: The distribution of flow in a microvascular network is determined by the pressure gradient and the resistance offered to flow. We assume cylindrical blood vessel segments and apply Hagen-Poiseuille equation (Eq. 2) for modeling the conductance of an edge as a function of viscosity μ, radius r_{bv}, length L.

$$Q_{ij} = \frac{1}{R_{ij}}(p_i - p_j) \tag{1}$$

Here p_i and p_j are the pressures at tail i and head j nodes of the oriented edge e_{ij}, G_{ij} is the conductance associated with e_{ij}.

$$Q_{ij} = G_{ij}(p_i - p_j) = \frac{\pi r_{bv}^4}{8\mu L}\Delta p \tag{2}$$

The vector $Q \in \mathbb{R}^n$ of volumetric flow rates in n edges is defined in Eq. 3. The negative of the oriented incidence matrix $M(\mathcal{G}) \in \mathbb{R}^{n\times m}$, denoted as M henceforth, is the gradient operator that acts on the vector $P \in \mathbb{R}^m$ of nodal pressures to result in the vector $\Delta P \in \mathbb{R}^n$ of pressure gradients. We obtain Q by premultiplying ΔP with the diagonal conductance-matrix $G \in \mathbb{R}^{n\times n}$, this scales the pressure gradient across each edge by the corresponding edge conductance.

$$Q = -GMP \tag{3}$$

In our model, we consider blood as an incompressible fluid and determine the unknown nodal pressures by imposing mass conservation at all nodes. Consequently, the net flow at any given node i is zero (Eq. 4). Here, Q_{ij} is positive when flow enters node i $\left(Q_{i\leftarrow j}\right)$, negative when flow leaves node i $\left(Q_{i\rightarrow j}\right)$, and $\mathcal{A}(i)$ denotes the set of nodes that are adjacent to i.

$$\sum_{j\in\mathcal{A}(i)} Q_{ij} = 0 \tag{4}$$

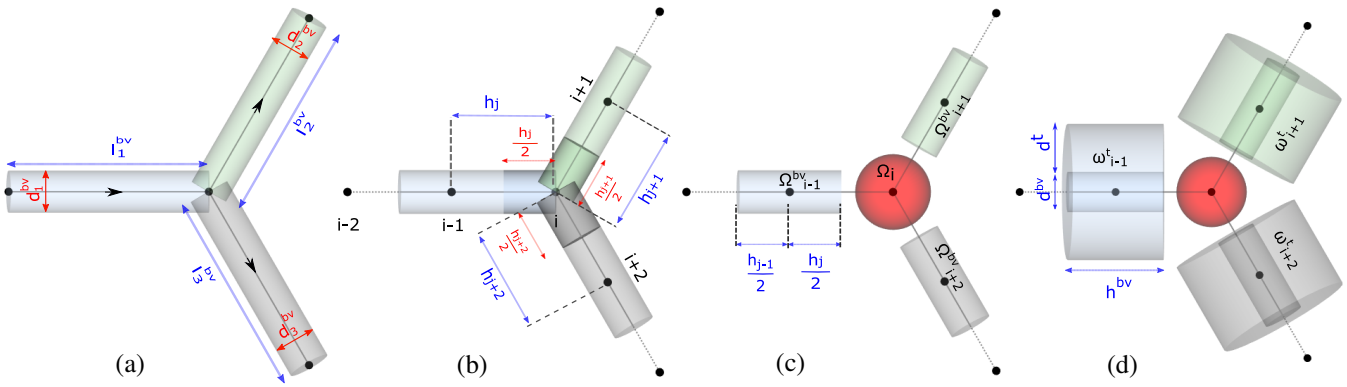


Fig. 3. Schematic representation of functional units in the model. Computational domain for studying (a) flow distribution, (b) and (c) advective-dispersive transport of biochemicals in the discrete volumes of blood vessel Ω^{bv}, (d) advective-dispersive-reactive transport of biochemicals in the discrete volumes of blood vessel Ω^{bv} and tissue domains ω^t.

Over m nodes in the network, the vector M^TQ ($M^T \in \mathbb{R}^{m\times n}$ is divergence operator which is given by the transpose of M) defines the flow conservation at all nodes excluding the terminal nodes where the flow boundary conditions are specified. The non-zero entries of vector $\tilde{q} \in \mathbb{R}^m$ (Eq. 5) contains the values of inflow or outflow rates at the boundary nodes (19). In addition to the flow rate, we specify one value of known pressure at the inlet p_{in} or outlet p_{out}. The values of these boundary conditions were specified based on experimental measurements of blood flow velocities reported in Table I.

$$\tilde{q}_i = \begin{cases} -Q_{in}, & \text{if } i = \text{inlet node} \\ Q_{out}, & \text{if } i = \text{outlet node} \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

$$-M^TGMP + \tilde{q} = 0 \tag{6}$$

Finally, the system of linear equations in Eq. 6 is solved for the unknown nodal pressures by finding the pseudoinverse. From the estimated nodal pressures, the centerline velocity u_{ij} and the volumetric flow rates are computed using Eq. 2. The linear analysis of the flow distribution presented here is applicable for laminar flow, $Re < 1$ in all segments, and this can be extended further to study the nonlinear rheology of blood using the procedure illustrated in Pries et al. (36).

D. Advection-dispersion of chemical species in blood vessel

Domain:

Discretization: The axial lines (edges) of the pipe network in Fig. 3a was spatially discretized to set up the computational domain and study the transport of biochemical species in the microvasculature. For discretizing the edges into 1D elements, in Gmsh (37), the vasculature was represented as a geometry with point and line entities. The length of the mesh elements, denoted by h in Fig. 3b, were constrained by specifying the maximum and minimum characteristic lengths i.e. $h \in [c_l + \delta, c_l - \delta]$. We derive the characteristic length c_l based on the average diameter of a biological cell, approximately 11.5 μm, calculated from the volume ranges reported in pancreas (38, 39) and we consider a deviation δ of 2.5 μm from c_l.

Volume elements: Each node in the discretized domain forms the center of the volume surrounding it. For instance, the volume of ith node V_i^{bv} (Equation 7) located at the bifurcation point depicted in Fig. 3b is the sum of half-cylinder volumes formed between nodes i-j where j $\in \mathcal{A}(i)$ (40). Here, $\mathcal{A}(i)$ is the set {i-1, i+1, i+2} of nodes adjacent to i, A_{ij} and L_{ij} are the cross-sectional area and length of the cylindrical volume between nodes i-j, respectively. At the junction nodes, we merge the half cylinder volumes of the adjoining edges and equate the sum to a spherical volume (Fig. 3c). Therefore, each branch in the network is dissected into cylindrical elements and the branches are assembled together by the spherical elements at the junctions. Since the rate of momentum transport is three orders of magnitude greater than the rate of mass transport, the accumulation term at any N-furcation junction is zero while solving for flow field and non-zero while solving the mass transport problem. As an example, for glucose species, the ratio of momentum to mass transport defined by Schmidt number ν/D is around 4000.

$$V_i^{bv} = \frac{1}{2}\sum_{j\in\mathcal{A}(i)} A_{ij}L_{ij} \tag{7}$$

Equations: The continuous formulation of the advection-dispersion physics is given by the following partial differential equation which describes the solute mole balance.

$$A\frac{\partial C^{bv}}{\partial t} = \nabla.\big(-QC^{bv} + DA\nabla C^{bv}\big) \tag{8}$$

Here, A denotes the area of flow, D is the dispersion coefficient of species, denotes the flow field, and C^{bv} is the scalar concentration of a chemical species. To shift from the continuous to discrete counterpart, first we assign the scalar concentration field C_0 to all vertices, and the flow field and dispersion coefficients form the edge weights. The net change in molar concentration of a species C_i^{bv} in the control volume V_i^{bv} at the ith node of the blood vessel depends on the contributions from advective J_i^A and dispersive fluxes J_i^D

presented in equation 9. The *influx* and *outflux* of J_i^A are the summation of molar fluxes through the edges feeding into i ($j \to i$) and leaving i ($i \to l$), respectively. And the molar flux through an edge is the flow-weighted concentration of the compartment from which the oriented edge originates (41, 42). The third term contributes to the dispersive transport. Further, $\mathcal{A}(i)$ is the set that contains nodes that are adjacent to node i, $\mathcal{A}_+(i)$ is the set of in-neighbor nodes of i, $\mathcal{A}_-(i)$ is the set of out-neighbor nodes of i, D_{ik}, A_{ik} and l_{ik} are dispersion coefficient, cross-sectional area and length of edge e_{ik}, respectively.

$$V_i^{bv}\frac{dC_i^{bv}}{dt} = \underbrace{\underbrace{\sum_{j\in\mathcal{A}_+(i)} C_j^{bv}Q_{j\to i}^{in}}_{\textit{influx}} - \underbrace{\sum_{l\in\mathcal{A}_-(i)} C_i^{bv}Q_{l\to i}^{out}}_{\textit{outflux}}}_{\text{advection flux } J_i^A} + \underbrace{\sum_{k\in\mathcal{A}(i)} \underbrace{\frac{D_{ik}A_{ik}}{l_{ik}}}_{D_{ik}^Q}(C_k^{bv}-C_i^{bv})}_{\text{dispersion flux } J_i^D} \tag{9}$$

$$D_{ik} = \tilde{D}_{ik} + \frac{u_{ik}^2 r_{ik}^2}{48\tilde{D}_{ik}} \tag{10}$$

The dispersion coefficient is determined from the Aris-Taylor's relation in Eq. 10. Here, $\tilde{D}_{ik}$, u_{ik} and r_{ik} are the diffusion coefficient of a species, centerline velocity and radius of e_{ik}, respectively.

$$V^{bv}\frac{dC^{bv}}{dt} = M^T Q M^o C^{bv} - M^T D^Q M C^{bv} \tag{11}$$

For setting up the discrete advection-diffusion equation on the network, we express the first and second order partial derivatives in Eq. 8 in terms of weighted advection Laplacian proposed by Rak et al. (43) and weighted diffusion Laplacian, respectively. This yields a system of ordinary differential equations shown in Equation 11, 12. In Eq. 11, $D^Q \in \mathbb{R}^{n\times n}$ is the diagonal matrix with diagonal entries the volumetric dispersion coefficient DA/l of each edge specified in Eq. 9, $Q \in \mathbb{R}^{n\times n}$ is the diagonal matrix with diagonal entries the volumetric flow rate of each edge, $M^o \in \mathbb{R}^{n\times m}$ is the modified incidence matrix (19, 43), $M^T D^Q M$ is the weighted dispersion Laplacian matrix $L^D(\mathcal{G}) \in \mathbb{R}^{m\times m}$ and $M^T Q M^o$ is the weighted advection Laplacian matrix $L^Q(\mathcal{G}) \in \mathbb{R}^{m\times m}$ (Eq. 12).

$$V^{bv}\frac{dC^{bv}}{dt} = L^Q C^{bv} - L^D C^{bv} \tag{12}$$

E. Advection-dispersion-reaction:

Domain: We adapt the Krogh cylinder approach presented in multiscale models (12, 21, 32) and approximate the layer of tissue surrounding the vessels as a cylindrical volume, illustrated in Fig. 3d.

Equations:

$$V_i^{bv}\frac{dC_i^{bv}}{dt} = \underbrace{\sum_{j\in\mathcal{A}(i)} C_j^{bv}Q_{ij}^{in} - \sum_{l\in\mathcal{A}(i)} C_i^{bv}Q_{il}^{out}}_{\text{advection flux } J_i^A} + \underbrace{\sum_{k\in\mathcal{A}(i)} \frac{D_{ik}A_{ik}}{l_{ik}}(C_k^{bv}-C_i^{bv})}_{\text{dispersion flux } J_i^D} - \underbrace{V_i^{cell} j_i^E}_{\text{exchange flux } J_i^E} \tag{13}$$

In the last term of Eq. 13, j_i^E (Eq. 15) is the net exchange rate that governs the bidirectional transport between tissue compartment ω_i^t and blood vessel compartment Ω_i^{bv} of the functional unit. The uptake or release flux J_i^E is computed by multiplying j_i^E with V_i^{cell} since the maximal rates are often reported in per unit volume of a biological cell. This ensures mole balance when a species moves from ω_i^t to Ω_i^{bv} which differ in volumes. Further, C_i^{cell} is the concentration of species in the tissue cell that interacts with i^{th} node of Ω^{bv}, K_m is the affinity of transporter-protein or an enzyme for a metabolite, and V_m is the maximal rate of metabolite transport. When Ω_i^{bv} is encompassed by ω_i^t, interaction between the node associated with both the compartments exists and J_i^E appears as a source or sink term in Eq. 13. At the junction nodes of blood vessel where no interaction exists J_i^E is zero.

$$V^{bv}\frac{dC^{bv}}{dt} = M^T Q M^o C^{bv} - M^T D^Q M C^{bv} - J^E \tag{14}$$

$$j_i^E = \frac{V_m}{K_m}\left(\frac{C_i^{bv} - C_i^{cell}}{1 + \frac{C_i^{cell}}{K_m} + \frac{C_i^{bv}}{K_m}}\right) \tag{15}$$

As an example, we consider a minimal model of glucose metabolism in the β-cells of islets. At basal condition, the concentration of glucose and lactate are at basal level in the bloodstream. In fed state, glucose transporters sense the high blood glucose level and export glucose from Ω^{bv} to Ω^t. The glucose-to-lactate conversion in Ω^t is presented as a lumped reaction in our model for simplicity. Ω^t acts as a source of lactate and the lactate transporter facilitates its release into the bloodstream. The direction of the exchange flux is dictated by the concentration gradient $C^{bv} - C^{cell}$ across the vessel wall.

F. Cell

The mole balance of each species in the cell domain ω^t is modelled by

$$V^{cell}\frac{dC_i^{cell}}{dt} = Nv \tag{16}$$

TABLE II
Reactions, rate laws and parameter values of exchange and cellular reaction

Flux	Reaction	Rate law
v_{TI}	$A^{bv} \rightleftharpoons A^{cell}$	$\frac{V_m^{T1}(A^{bv}-A^{cell})}{K_{m,A}^{T1}+A^{bv}+A^{cell}}$
v_E	$A^{cell} \rightarrow 2B^{cell}$	$\frac{V_m^E A^{cell}}{A^{cell}+K_{m,A}^E}$
v_{T2}	$B^{cell} \rightleftharpoons B^{bv}$	$\frac{V_m^{T2}(B^{cell}-B^{bv})}{K_{m,B}^{T2}+B^{cell}+B^{bv}}$

A: glucose, B: lactate, T1: glucose transporter *glcim*, T2: lactate transporter *lacex*, E: glucose to lactate converter *glc2lac* are the tags used in the mathematical model. Parameter values in the model: V_m^{T1} = 10; $K_{m,A}^{T1}$ = 1.0; V_m^E = 0.01; $K_{m,A}^E$ = 4.5; V_m^{T2} = 10; $K_{m,B}^{T2}$ = 0.5. All $\left(V_m \text{ in M/min and } K_m \text{ in mM} \right)$.

where, N is the stoichiometric matrix and v is the reaction flux vector. The kinetic rate expressions of the reaction and transport mechanisms, and the values of the kinetic parameters included in the model are presented in Table II.

II. Results and Discussion

The discrete model framework developed allows to bridge the cell-to-vessel exchange and explicitly model the cellular dynamics. In this section, the mathematical formulations presented in Section I are verified using several test systems and the results are compared with finite element analysis in COMSOL. The analysis includes solving two steps that are coupled: first, a flow field analysis which involves computation of pressure and velocity distribution in the blood vessel branches; second, analysis of the spatio-temporal evolution of the concentration fields.

We first present the results of pressure gradient and flow distribution observed across the islet vasculature. After validating the results of nodal pressures and edge velocities with the results from COMSOL, we proceed with the simulations of advection-dispersion dynamics of glucose species in the blood vessel. Here we compare the transient change in the blood glucose concentration obtained from our discrete model versus COMSOL simulations for islet and mesentery networks. Next, we investigate the effect of change in perfusion pattern on glucose distribution by varying the inlet and outlet locations in the tumor vasculature. Further, we examine the effect of different pressure gradients applied across the network and the effect of glucose doses supplied at the inlet on metabolite rise times observed in the islet vasculature.

All the 3D visualizations of flow and concentration fields presented in this paper are rendered using *vedo* (44).

A. Comparison of flow and concentration fields

To illustrate how the results from our discrete formulation compare with the finite element implementation available in COMSOL, we first solve the static flow problem (Eq. 6) and use the flow profile for simulating the advection-dispersion dynamics (Eq. 12) in the islet vasculature.

The pressure and velocity field computed across the islet vasculature are shown in Fig. 4a,b. We observe a net pressure drop of 34.83Pa for an inlet pressure and flow rate of 60Pa and 3.76nl/min, respectively. The velocity distribution lies in the range reported by Diez et al. (27). The conservation of flow at each node is cross-verified by computing the divergence of the flow field M^TQ and the visualization is presented in Fig. 4c.

Validation in COMSOL Multiphysics

Here we describe the simulation implementation of the coupled multiphysics problem in COMSOL to validate the flow fields and concentration fields of advection-dispersion simulations from our discrete model. *Geometry:* The generation of the geometrical model of the islet and mesentery vasculature (Fig. 2a,d) was automated using an AutoLISP script (45). The coordinates of points and the connectivity information of the lines were specified as inputs for creating the CAD geometry. The DXF file containing the geometry data was imported in COMSOL and *Normal* mesh size was used to generate the mesh elements. *Parameters:* Values of diffusion coefficient of glucose species, viscosity of blood, diameter of branches in the vasculature, inlet pressure, and outlet flow rate, specified in Table I and S2 Table , were defined as input parameters. *Computation*: Fluid flow was studied as static problem in the *Pipe Flow* module considering blood as a Newtonian fluid. This stationary problem was solved as a linear system using the *Direct* solver by specifying the pressure and flow boundary conditions. Then, one-way coupling of the flow physics was established with *Transport of Dilute Species in Pipes* module to solve for the time-dependent advection-dispersion physics of glucose species in the finite element solver. For this transient simulation, Dirichlet boundary condition was specified at the inlet and mass outflow was modelled by setting the diffusive flux to zero.

Fig. 4d,e show the nodal pressures and edge velocities computed from our model are consistent with the results from COMSOL for the islet vasculature. Time-varying concentration profile of glucose species obtained from our discrete model is compared with COMSOL simulations at nodes highlighted in the inset of Fig. 4f. After comparing the results of the islet vasculature containing 52 edges and 125 discretized elements in our discrete model, we further extend this analysis to compare the results (Fig. 4g) of the large mesentery network with 489 edges and 9033 discretized elements. Supplementary videos S1 and S2 show the comparison of evolution of concentration profiles from our model versus the results from COMSOL for the entire region of the islet and mesentery vasculature, respectively.

Scalability

The procedure involved in solving the partial differential equations can be split into two steps: assembly and solve

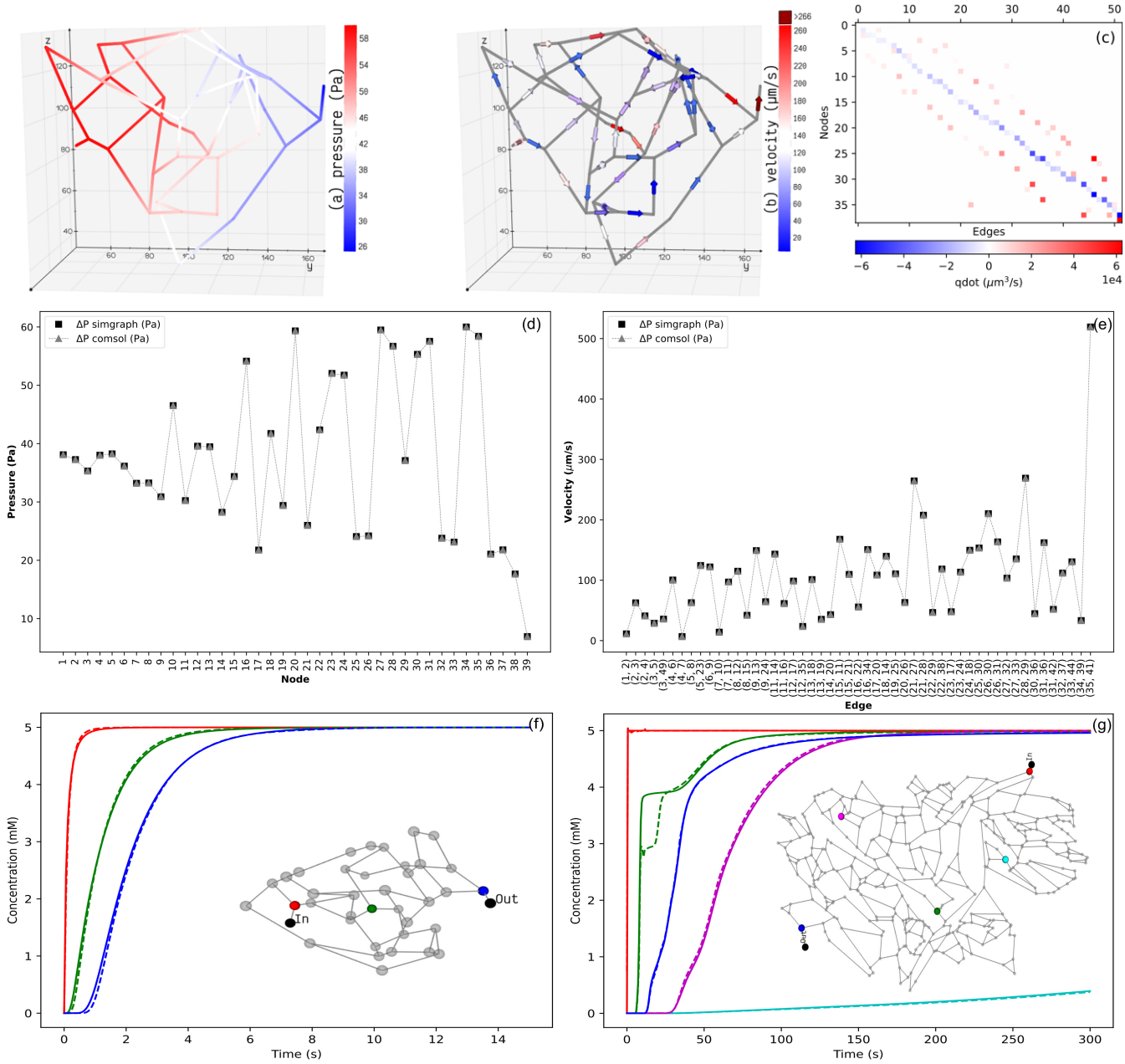


Fig. 4. Distribution of the static properties of islet vasculature. (a) Pressure gradient from inlet to outlet and (b) directional flow from inlet to outlet. (c) Flow conservation at all nodes. The summation of flows through the edges entering and leaving each node, except the boundary nodes, add to zero. **Validation of static results of the islet vasculature.** (d): Comparison of nodal pressures between the current work and COMSOL simulation. (e): Comparison of edge velocities between the current work and COMSOL simulation. **Comparison of the concentration profiles at various locations in the islet and mesentery vasculatures.** (f) Scalar concentrations observed at positions 12.07 μm (red), 54.85 μm (green), and 110.27 μm (blue) from the inlet node. (g) Scalar concentrations observed at positions 1267.78 μm (blue), 1692.3 μm (cyan), 3476.96 μm (green), 5085.33 μm (magenta), and 6319.57 μm (blue) from the inlet node. Solid and dashed lines indicate the transient change in glucose concentration from our model and COMSOL, respectively.

step. In the assembly step, discretization is done and matrices are generated for the second step which solves the system. For the islet vasculature, we report the time taken for solving the stationary fluid flow and the transient advection-dispersion dynamics for a time span of 15s. Our discrete method implementation in MATLAB takes 7s and the finite element solver in COMSOL takes 10s. For the mesentery network, running the advection-dispersion dynamics for a time span of 300s takes 267s and 168s in MATLAB and COMSOL implementations, respectively. We provide the sparse Jacobian pattern as an additional input to *ode15s* to speed up the compute time. Solving the same system in Julia using QNDF method, which is a translation of the MATLAB's ode15s, gives a speed up of 127x when compared to COMSOL. A relative error tolerance of 1e-3 is set for carrying out all simulations and an absolute error tolerance of 1e-6 is set for both ode15s and QNDF.

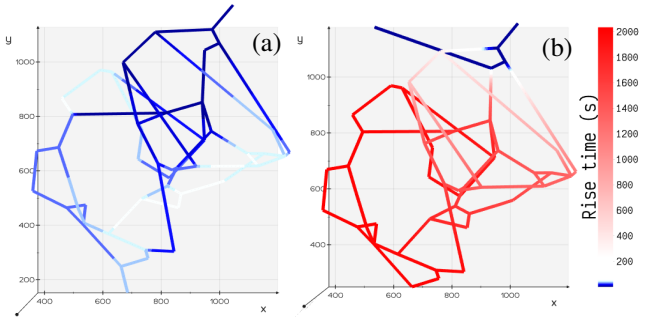


Fig. 5. Comparison of rise time t_r for two different configurations of tumor vasculature. (a) Tumor Design 2: the outlet is positioned away from the inlet and the rise time is shorter in this configuration. (b) Tumor Design 1: the outlet is positioned closer to the inlet and the rise time is longer in this configuration.

B. Influence of flow topology on scalar transport

Engineered perfusable vasculatures have been useful for investigating the structure-function relationship of complex vasculatures(46). Computational models that capture the influence of flow topology on the scalar transport will enable experimental scientists to design and test the efficacy of optimized drug delivery systems.

Motivated by the experiments carried out by Chen et al. (23) on microfluidic devices imprinted with blood vessel vasculatures, we study the effect of variation in perfusion patterns on the metabolite rise times t_r in two different geometric configurations (Fig. 2b,c) of the tumor vasculature examined in their study. Flow distribution was computed for an inlet pressure of 60Pa and fluid flow rate of 23.8nl/min. Spatial distribution of glucose species was obtained and the rise time in each volume element Ω_i^{bv} is computed by finding the time taken for the concentration to rise from 10% to 90% of the steady-state value. Fig. 5 illustrates the time taken for the distribution of glucose in *Tumor Design 1* (Fig. 5b) is much greater when compared to *Tumor Design 2* (Fig. 5a). These findings support the dye distribution patterns observed in these two configurations reported in Chen et al.. Supplementary video S3 shows the distribution of glucose computed by our discrete model for both the configurations of tumor vasculature.

C. Functional coupling of blood vessel-cell exchange

In the mathematical framework presented in Section I, it is practical to model heterogeneous cell types exhibiting heterogeneous enzyme activity at different vascular and cell densities observed in various pathophysiological conditions.

In this study, for the ease of demonstration of our method, we present the results of a minimal model of cellular glucose metabolism with uniform enzyme activity and homogeneous cell type exchanging biochemicals with bloodstream. In Fig. 6, we present the volumetric spatial analysis of the islet vasculature enveloped by a layer of homogeneous cell mass which forms the annular region of the tissue domain ω^t. The extracellular glucose and lactate concentrations in Ω^{bv} are initialized to 5mM and 1.2mM at the network inlet. The intracellular concentrations of glucose and lactate in ω^t are initialized to zero. Due to the high concentration of glucose in the bloodstream, the gradient established between Ω^{bv} and ω^t in response to the advective-dispersive transport promotes the uptake of glucose by the annular region ω^t. The cellular enzymes metabolise glucose-to-lactate and the concentration of lactate in ω^t rises above the basal value in the bloodstream. This gradient drives the export of lactate into the bloodstream until the system equilibrates.

D. Sensitivity of concentration dynamics to ΔP and glucose dose

The pressure conditions observed in a vascular tissue may vary due to several physiological factors. To study the influence of pressure gradient on the time taken to reach the steady-state concentration, we vary ΔP across the vasculature by specifying the inlet pressure and zero outlet pressure. Fig. 7a illustrates the concentration profiles generated by simulating the advection-dispersion dynamics of glucose species by varying the pressure drop from 20Pa to 200Pa. When ΔP is high, the velocity of fluid is high in each branch. Consequently, convective flow dominates over dispersion and this results in short rise times. The effect of change in pressure gradient results in change in transit time of the fluid from 90.76s at ΔP = 20Pa to 9.08s at ΔP = 200Pa. In Fig. 7b, we show the sensitivity of the net glucose uptake flux and net lactate release flux to different glucose doses set at the inlet. It is observed that with increase in glucose dose the tissue units uptake more glucose from blood and this drives the formation of lactate in the tissue subdomains. The excess lactate is then transported to the vessel subdomains until equilibrium is attained and the driving potential is zero.

In conclusion, here we have presented a mathematical framework for understanding the multiscale connectivity existing in the functional networks of tissues. The test cases presented above demonstrate how experimental data from different sources can be encoded in our framework to build explicit models of cellular and vessel-to-cell interaction dynamics to better predict pathomechanisms. At the intra-organ scale, this work can be further extended to include cell-to-cell communication networks and decipher the order of communication occurring in the microenvironments with different vasculature architectures (e.g. periphery to the center, center to periphery, one pole to other patterns in islets (47)) and cytoarchitectures (e.g. mantle-core and heterogeneous distribution of β and α cells in islets (13)). The metamodel is easily mutable; it is possible to induce different vascular phenotypes and disease states by altering vascular density and incorporating fold changes of metabolic and enzyme concentrations in the cellular units. To further research efforts involved in carrying out virtual experiments of inter-organ communication, our framework can be utilized for piecing together the top-down and bottom-up modeling approaches and accommodate each organ at the desired spatial resolution. Example, a comprehensive understanding of multiorgan disease states such as diabetes can be developed by interpreting the intra- and inter- organ interaction as communication occurring within "network of networks"; in whole body models, the inter-organ communication can be modelled by abstract-

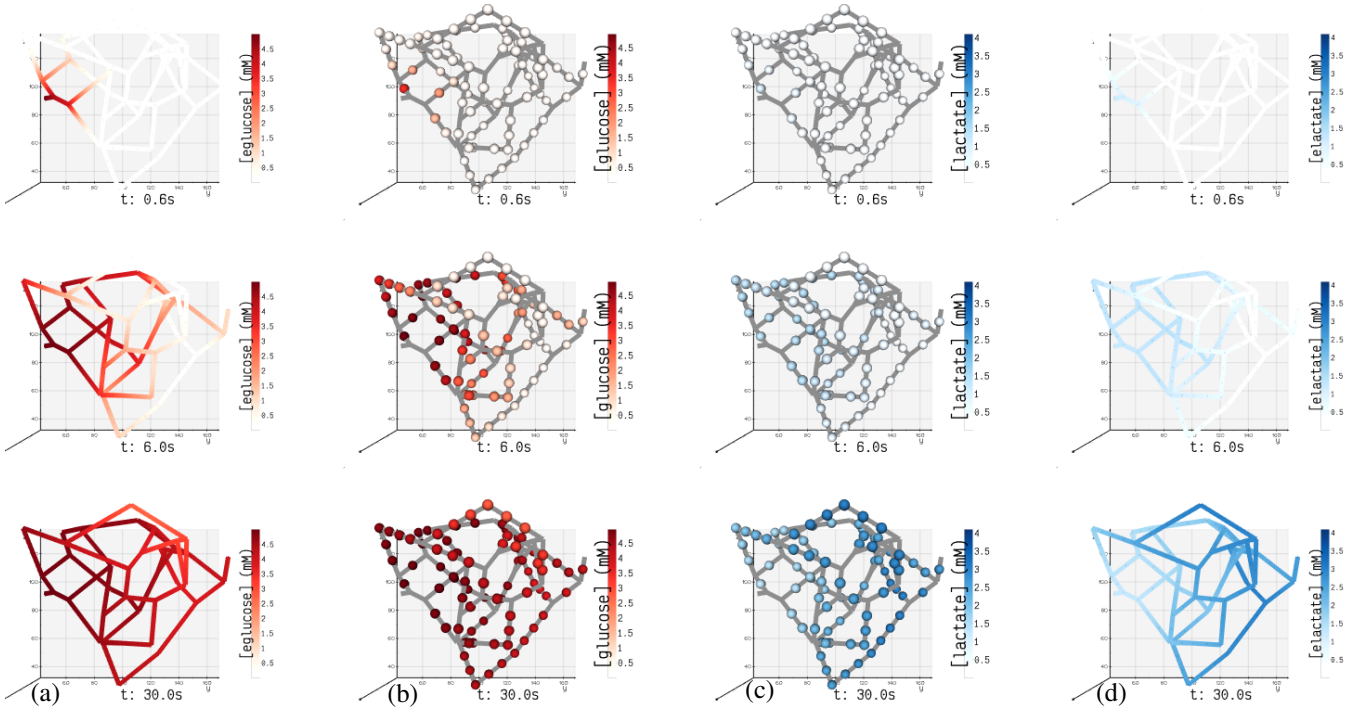


Fig. 6. Spatial concentrations of metabolites in blood vessel and tissue domains of the islet vasculature observed at three different instants in time (0.6, 6, 30 seconds). (a) [eglucose]/ (b) [elactate] and (c) [glucose]/ (d) [lactate] denote the concentration of glucose/ lactate species observed in blood vessel Ω^{bv} and tissue ω^t compartments, respectively. In panel a and d, the concentration of cylindrical and spherical volume elements embedded in the blood vessel domain is simulated and the gradient displayed along the length of the blood vessel is generated by interpolation. In panel b and c, the color-coded spheres represent the concentration evaluated in the annular region of ω^t.

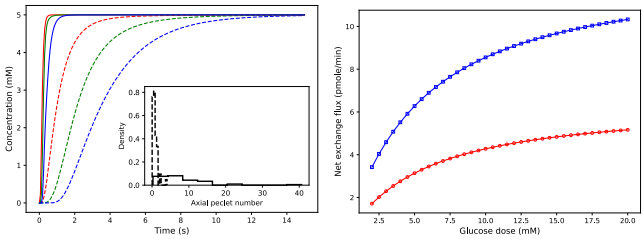


Fig. 7. Sensitivity of concentration dynamics to $\triangle$P and glucose dose. (a) Left: influence of varying pressure gradient across the network on metabolite rise times. Glucose concentration observed at positions 12.07 μm (red), 54.85 μm (green), and 110.27 μm (blue) from the inlet node. Inset displays the distribution of peclet number obtained by computing $ul/\tilde{D}$ for each branch. Dashed and solid lines indicate the dynamic change in concentration observed at 20Pa and 200Pa, respectively. (b) Right: variation in the net glucose uptake (red) and net lactate release (blue) from the cells in response to change in glucose dose is displayed.

ing organs as compartments forming nodes of the global network and the detailed local dynamics occurring in the functional networks of an organ can be modelled by including subnetworks in the global network.

DATA AND SOFTWARE AVAILABILITY

All model simulations were performed with stiff-equation solver in MATLAB, Release R2019b, The MathWorks, Inc., Natick, Massachusetts, United States. Code is available upon request.

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AUTHOR CONTRIBUTIONS

DM, SR and DP conceived and designed the study. DM implemented the code, performed the analysis and drafted the manuscript. All authors read, discussed the results, revised and approved the manuscript.

COMPETING FINANCIAL INTERESTS

The authors declare no competing interests.

SUPPLEMENTARY MATERIAL

Supplementary Material 1

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S1 Table. Notation

$\mathcal{G}$		Weighted graph
m		Number of vertices or nodes in $\mathcal{G}$
n		Number of edges in $\mathcal{G}$
$\mathbb{V}$	$\in \mathbb{R}^m$	Dimension of vertex
$\mathbb{E}$	$\in \mathbb{R}^n$	Dimension of edge
$\mathbb{W}$	$\in \mathbb{R}^n$	Dimension of edge weights
$\mathbb{W}_G$	$\in \mathbb{R}^n$	Edge weight vector with entries the conductance of each edge
$\mathbb{W}_Q$	$\in \mathbb{R}^n$	Edge weight vector with entries the volumertic flow rate along each edge
$\mathbb{W}_{D^Q}$	$\in \mathbb{R}^n$	Edge weight vector with entries the volumertic dispersion coefficient of each edge
G	$\in \mathbb{R}^{n\times n}$	Diagonal matrix with diagonal entries $\mathbb{W}_G$
Q	$\in \mathbb{R}^{n\times n}$	Diagonal matrix with diagonal entries $\mathbb{W}_Q$
D^Q	$\in \mathbb{R}^{n\times n}$	Diagonal matrix with diagonal entries $\mathbb{W}_{D^Q}$
$M\left(\mathcal{G}\right)$	$\in \mathbb{R}^{n\times m}$	Incidence matrix of $\mathcal{G}$
$M^T\left(\mathcal{G}\right)$	$\in \mathbb{R}^{m\times n}$	Transpose of incidence matrix of $\mathcal{G}$
$M^o\left(\mathcal{G}\right)$	$\in \mathbb{R}^{n\times m}$	Modified incidence matrix of $\mathcal{G}$
$L^D\left(\mathcal{G}\right)$	$\in \mathbb{R}^{m\times m}$	Weighted diffusion laplacian matrix of $\mathcal{G}$ computed as $M^T D^Q M$
$L^Q\left(\mathcal{G}\right)$	$\in \mathbb{R}^{m\times m}$	Weighted advection laplacian matrix of $\mathcal{G}$ computed as $M^T Q M^o$
$\mathcal{A}\left(i\right)$		Adjacent nodes of i
P	$\in \mathbb{R}^m$	Dimension of vector containing nodal pressures
$\tilde{q}$	$\in \mathbb{R}^m$	Dimension of vector containing boundary data
C	$\in \mathbb{R}^m$	Dimension of concentration vector
J	$\in \mathbb{R}^m$	Dimension of exchange-flux vector
V	$\in \mathbb{R}^m$	Dimension of vector containing the volume of nodes

S2 Table. Values of parameters used in transport equation

Symbol	Description	Value/expression	Units	Reference
r_{bv}	Radius of blood vessel		μm	
r_{cell}	Radius of cell		μm	
d_{cell}	Diameter of cell	$2r_{cell}$	μm	
Δx	Length of blood vessel element; distance between two consecutive nodes	d_{cell}	μm	
$r_{bv,in}$	Radius at the inlet of blood vessel		μm	
$u_{bv,in}$	Velocity at the inlet of blood vessel		$\mu m/s$	
A_{cell}	Cross-sectional area of tissue element	$\pi\left((r_{bv}+d_{cell})^2-r_{bv}^2\right)$	μm^2	
A_{bv}	Cross-sectional area of blood vessel element	πR^2	μm^2	
V_{bv}	Volume of blood vessel element	$A_{bv}\Delta x$	μm^3	
V_{cell}	Volume of tissue element	$A_{cell}\Delta x$	μm^3	
P_{in}	Inlet blood pressure		mmHg	
Q_{in}	Volumetric flow rate at the inlet of blood vessel	$u_{bv,in}\pi r_{bv,in}^2$	$\mu m^3/s$	
$\tilde{D}_A$	Diffusion coefficient of species A in blood	5.46E-4	cm^2/min	Berndt et al. (2018)
$\tilde{D}_B$	Diffusion coefficient of species B in blood	7.71E-4	cm^2/min	Berndt et al. (2018)
D_A	Dispersion coefficient of species A in blood		cm^2/min	
D_B	Dispersion coefficient of species B in blood		cm^2/min	
D_A^Q	Volumetric dispersion coefficient of species A in blood	$\frac{D_A A_{bv}}{\Delta x}$	cm^3/min	
D_B^Q	Volumetric dispersion coefficient of species B in blood	$\frac{D_B A_{bv}}{\Delta x}$	cm^3/min	
ν	Viscosity of blood	0.004	Pa.s	Berndt et al. (2018)
J_i^E	Exchange flux		mole/min	

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