Angular speed should be avoided when assessing the speed-curvature power law of movement

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Abstract. The speed-curvature power law is an intriguing constraint between the geometry and the kinematics of a movement trajectory. It dictates, for instance, how much slower the hand should move in curves. The empirical phenomenon has led to various interesting theoretical proposals for its origin and possible generating processes. Its formulation based on tangential speed ($V \sim C^{\beta}$) is considered equivalent to its angular speed counterpart ($A \sim C^{\beta+1}$), with the corresponding change in the exponent. In the case of drawing ellipses, these relationships are respectively known as the "one-third power law" or the "two-thirds power law", after the values of the exponents. Here we show that using angular speed instead of tangential speed tends to result in much stronger data correlations, impacting on the interpretation of the strength of relationship, the putative value of the exponents, and even the very existence of the power law. We explain how and why this is the case, using both empirical and synthetic data. We conclude that angular speed should be avoided when expressing the speed-curvature power law.

Keywords: motor control, speed-curvature, power law, geometry, kinematics.

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1. INTRODUCTION

The relationship between instantaneous speed and local curvature in human movement is a classic, fascinating, and well-studied phenomenon in the field of motor control. Its historical roots can be dated to the observations of Binet and Courtier (1893) and Jack (1895) that in 'natural handwriting' the speed of the pen tends to be lower in curved segments of the path and higher in the straight segments. Lacquaniti and colleagues (1983) proposed that the relationship can be expressed as a power law and reported its validity for a large class of hand movements. The law has since been observed also in human locomotion, speech, eye movements, and even in the trajectories of animals (Zago et al, 2016). For a recent review, see (Zago, Matić et al, 2017).

The speed-curvature power law is expressed in several ways, involving either tangential speed or angular speed as kinematic variables and curvature or radius of curvature as geometric variables. Different formulations are often explicitly considered to be *equivalent* (e.g. Lacquaniti et al. 1983; Viviani and Cenzato, 1985; Wann et al. 1988; Viviani and Schneider, 1991; Schaal and Sternad, 2001; Vieilledent et al. 2001; Ivanenko et al. 2002; Perrier and Fuchs, 2008; Maoz et al. 2009; Tesio et al. 2011; Huh and Sejnowski, 2015; Karklinsky et al. 2016; Zago, Matić et al. 2017; Rybarczyk and Carvalho 2019; Matić and Gomez-Marin 2020).

Indeed, examining the algebraic relationships shows that the exponents in different formulations of the power law are mathematically related. Namely, with angular speed A, curvature C, and a constant k (related to average speed, also called the velocity gain factor), one has the "AC power law" as:

$$A = kC^{\beta} \tag{1}$$

With tangential speed V, since A=VC, one gets the "VC power law":

$$V = kC^{\beta-1} \tag{2}$$

If instead of curvature C, one uses the radius of curvature R = C^{-1}, the exponent of the power law would simply be: $A = kR^{-\beta}$ or $V = kR^{1-\beta}$. See also Figure 1 for definitions of variables.

It turns out that, in drawing ellipses and scribbling movements, the exponent of the AC power law (equation 1) is empirically found to be β≈2/3 (Lacquaniti et al. 1983), a phenomenon known as the "2/3 power law". If one chooses to analyze the same trajectories with the VC power law (equation 2), the exponent is near -1/3, and the phenomenon is sometimes called the "1/3 power law". Regardless of the name used, most published research uses tangential speed and curvature, the VC power law, as expressed in equation (2). The use of angular speed is relatively rare, but it is frequent enough to warrant scrutiny (e.g. Lacquaniti et al. 1983, Ivanenko et al 2002; Zago, Matić et al. 2017; James et al. 2019).

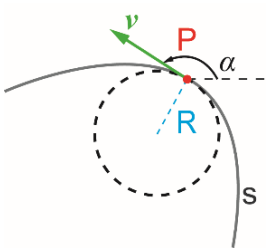

Figure 1. For a particle P moving along the path s, speed (V) is defined as the magnitude of the (tangential) velocity vector, $V = |\mathbf{v}| = \left|\frac{ds}{dt}\right|$, while angular speed (A) is the absolute rate of change of the direction of velocity, $A = \left|\frac{d\alpha}{dt}\right|$, and curvature (C) is the reciprocal of the radius of the osculating circle, or the rate of change of direction with respect to arc-length, $C = \frac{1}{R} = \left|\frac{d\alpha}{ds}\right|$. From this we see that, naturally, A=VC.

Crucially, these mathematical relationships show an equivalence of exponents between relationships, but not an equivalence of *strength* of the underlying relationship.

In the research on the power law of movement, power functions (1) and (2) are typically converted to linear functions by taking a logarithm of both sides. Next, the exponents β, the speed gain factor k and the strength of the relationship, expressed as the coefficient of determination r^2 are estimated by linear regression from models:

$$\log A \approx \log k + \beta_{AC} \log C \tag{3}$$
$$\log V \approx \log k + \beta_{VC} \log C \tag{4}$$

In linear regression, the coefficient of determination (r^2), shows how much variance in the outcome is explained by the predictors, expressing the strength of the linear relationship. When there are only two variables in a regression, the r^2 is the square of the correlation coefficient r (Pearson's or other). More generally, it is calculated as $r^2 = 1 - RSS/TSS$ where RSS is the sum of squares of residuals and TSS the total sum of squares of the outcome. If the relationship between variables has a high r^2 (near 1), one can predict with high accuracy the value of the outcome variable from the values of the predictors.

Here we look carefully at the differences between the coefficients of correlation and determination in the formulations of the power law involving different kinematic variables, tangential speed V and angular speed A. We show that there are big differences in empirical and synthetic data, explain why the differences exist, and suggest how to interpret different formulations of the power law.

2. EMPIRICAL DATA: Tracking elliptic targets at different frequencies

Methods

We analyzed pen trajectories recorded in a target tracking experiment (Matić and Gomez-Marin, *in preparation*).

Participants were seated, looking at a computer monitor, while holding an electronic pen in their dominant hand on a graphics tablet. There was a small circular target displayed on the screen. The pen determined the position of a circular cursor on the screen, and the task of the participant was to track the target with the cursor. Targets always moved in an elliptical pattern. Target trajectories varied across trials (i) in 9 different frequencies, from slow rhythms to fast, and (ii) in 3 different speed profiles: constant speed (β_{VC}=0), 'natural speed' (β_{VC}=-1/3) and 'extra slowing' (β_{VC}=-2/3). Each participant (three males) performed 27 trials of target tracking, lasting 16 seconds, presented in a random order.

The recorded trajectories were smoothed by a second-order, low-pass Butterworth filter with a cutoff at 10Hz, and differentiated to estimate speed and curvature. The power law exponents β and the coefficients of determination r^2 were estimated using linear regression. Conformity to the power law was chosen at an arbitrary point of $r^2 > 0.75$, considering that pure gaussian noise can produce a fit of up to $r^2 \approx 0.6$ (Maoz et al, 2005).

All of the data and the code in python for analysis and generating the figures are available in an online repository: https://github.com/adam-matic/AngularSpeedPowerLaw.

Results

The plot of the VC power law across frequencies (Figure 2) and the plot of the AC power law across frequencies (Figure 3) show that the values of exponents are equivalent; for each trajectory the following equality holds: $\beta_{AC} = \beta_{VC} + 1$. However, there is a dramatic difference in the coefficients of determination. For the VC power law, only the trajectories performed at fast rhythms of $f \geq 0.94$ conformed to the power law with the coefficient of determination of $r^2 \geq 0.75$. On the other hand, angular speed is strongly correlated to curvature across the entire frequency range, with only a few exceptions of the power law estimates with $r^2 < 0.75$, and even then, the lowest value is $r^2 \approx 0.5$.

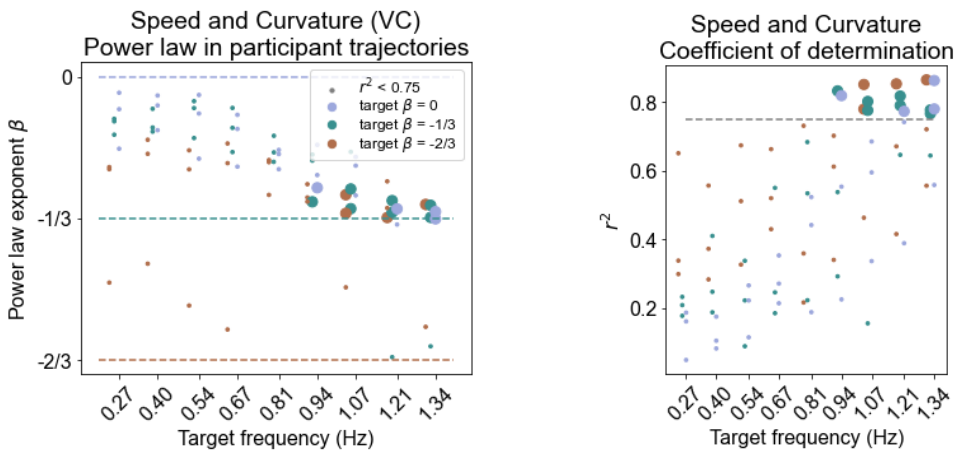

Figure 2. Using tangential speed and curvature to estimate the speed-curvature power law, only high-frequency participant trajectories (freq ≥ 0.94) conform to the power law with a coefficient of determination $r^2 \geq 0.75$. Many trajectories have a low r^2.

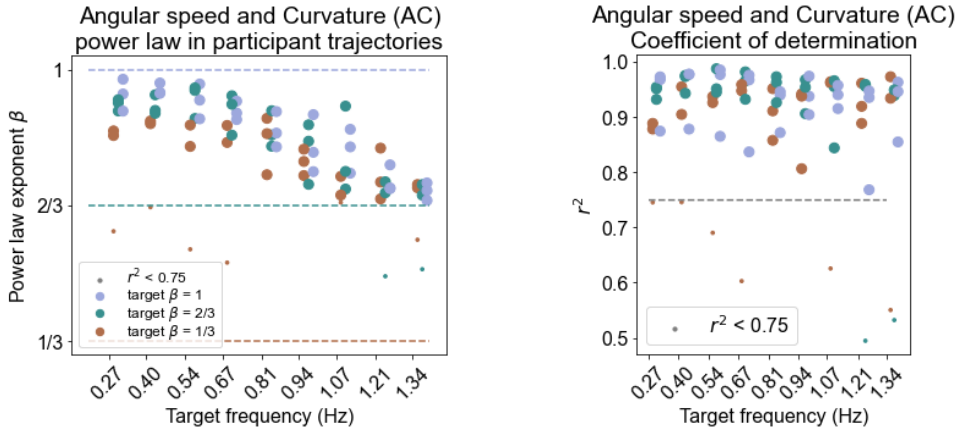

Figure 3. Using angular speed and curvature to estimate the speed-curvature power law, the r^2 is higher than 0.75 for most trials, making the AC power law apparently much stronger than the VC power law (see Figure 2).

Clearly, the VC and AC power laws are not equivalent (compare Figure 2 to Figure 3); the AC power law tends to be much stronger.

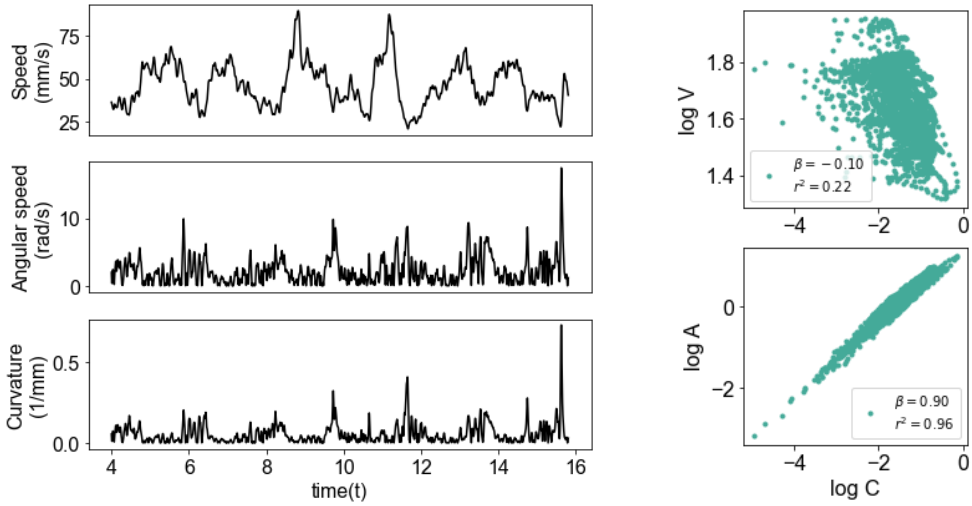

Figure 4. Example data from a single trial (target frequency = 0.27Hz), showing tangential speed, angular speed and curvatrue over time. The VC power law is fairly weak, showing that tangential speed is hardly related to curvatrue, while in this case the AC power law is very strong, angular speed being strongly correlated with curvature.

Figure 4 shows a common type of the differences in the estimated strength of the speed-curvature power law: when using angular speed, the power law is very strong while, for the same trajectory, tangential speed is very weakly related to curvature, apart from a slight tendency to slow down in more curved parts of the path.

In other words, there can be plenty of empirical cases in which the speed V of the cursor is hardly related to the curvature, while the speed of 'turning', or the rate of change of direction, is strongly related to curvature. In the following section we explore why this is the case by means of synthetic data.

3. SYNTHETIC DATA: Correlation between curvature and the two speeds

Methods

In simple linear regression, the coefficient of determination r^2 is equal to the square of Pearson's correlation coefficient r. The Pearson's correlation coefficient r is in turn defined as normalized covariance. To investigate the differences between the two power laws, here we focus only on the algebraic manipulations of correlations and covariances between the variables, and using simulations and random variables.

Elliptical trajectories were generated in the following way: first, we generated an elliptic trajectory with a given semi-minor and semi-major radius, frequency, and number of cycles. Next, we created a gaussian noise speed profile of the same length as the trajectory, smoothed it with a low-pass butterwort filter at 10Hz cutoff, and then generated a time profile for the elliptic trajectory by dividing the distances between the points with the desired speeds obtain from the random speed profile. Finally, we used a cubic spline to create constant time-sampling for the final elliptic trajectory.

Results

According to equation (4), in the VC power law one is correlating log C and log V; according to equation (3), for the AC power law one is correlating log C and log A. However, we also know that A=CV, and therefore log A = log C + log V. In considering the AC power law, we are, in effect, correlating log C with log C + log V.

Letting Cov = covariance, a = log A, c = log C, and v = log V, we can write:

Cov(c, a) = Cov(c, c+v) = Cov(c, c) + Cov(c, v) = Var c + Cov(c, v)

We can then see that Cov(c, a) will be dependent not only on the covariance of curvature and speed, but also on the variance of curvature. This means that the correlation coefficient will tend to be large when the curvature has a big variance relative to speed, regardless of the actual correlation between speed and curvature.

One can simply and effectively explore such a dependency as follows. Let us take two random normal variables x and y (Figure 5) and calculate the correlation coefficient between x and x + y, as an analogy to correlation of log C and log A. Letting Corr be equal to the Person's correlation coefficient, obviously, the correlation r = Corr(x, x+y) is *not equivalent* to the Corr(x, y), which is zero in all four cases. Instead, if we keep the standard deviation of y stable, the Corr(x, x + y) increases with the standard deviation of x.

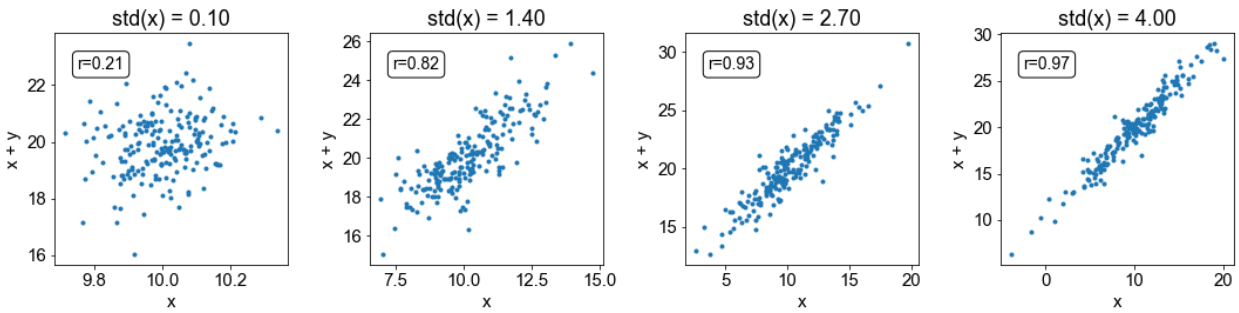

Figure 5. For two random normal variables x (mean 10, std in range [0.1, 4]) and y (mean 10, std 1), the variable x is not necessarily correlated to x+y, but it gets more strongly correlated as the variance of x increases.

The same effect of increasing the range of curvature can be illustrated with a random speed profile along a circular and along an elliptic trajectory. As shown on Figure 6A, the circular trajectory is designed to have a constant curvature profile and a random-smoothed speed profile. As expected, the correlation between logC and logV is very low, and so is the correlation between logC and logA. In the case of the ellipse (Figure 6B), the curvature profile has a bigger range, but the speed profile is identical to the circle. This single change was enough to increase the correlation between logC and logA to 1.

We further explore this effect in Figure 7. We show how the Pearson's correlation coefficient Corr(logC, logV) is near zero for all values the semi-major axis. This is expected, since the speed profile is randomized. However, for the correlation between angular speed and curvature, even a small increase in the semi-major axis, making the ellipse slightly eccentric, and making the curvature have slightly larger variance, increases the correlation Corr(logC, logA) to very high levels.

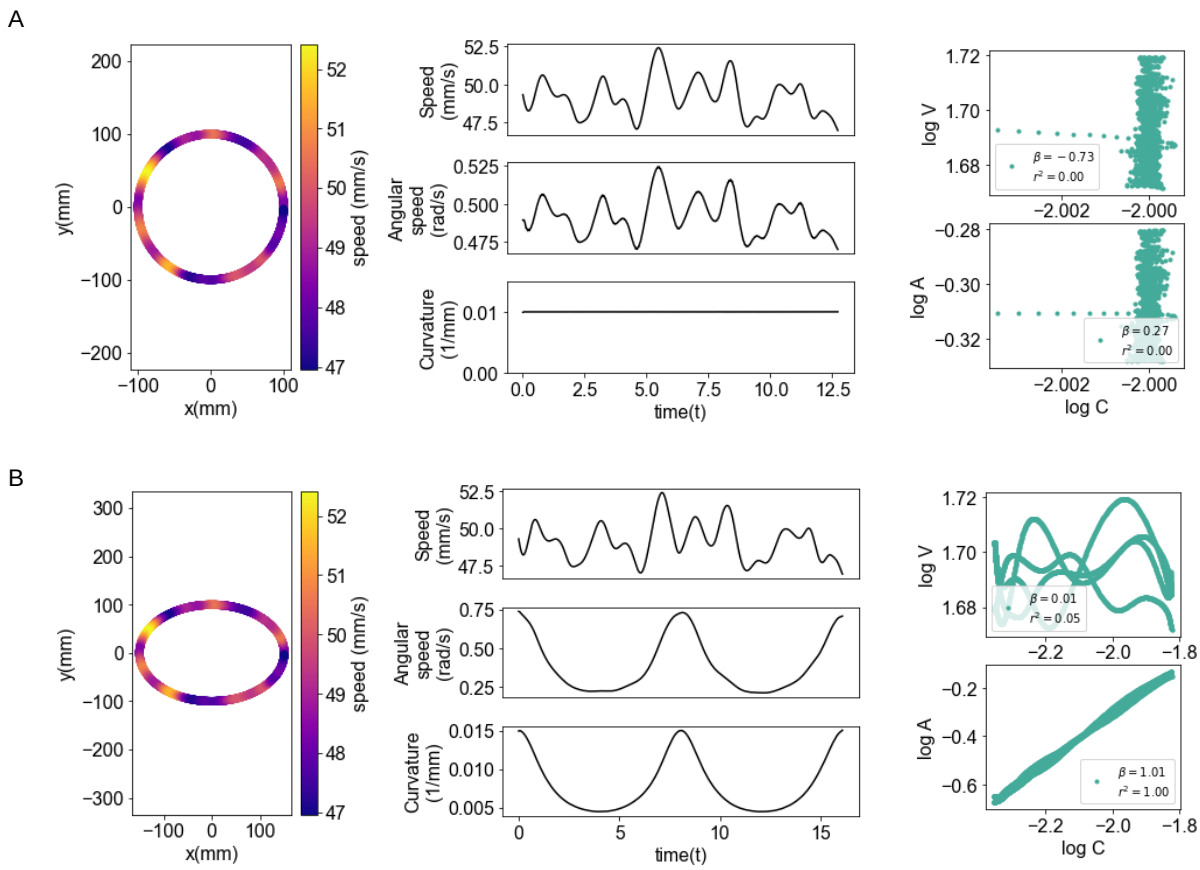

Figure 6. Correlation depends on the range of curvature in generated ellipses. **(A)** For a circular trajectory, radius of 100 mm, with a random-smoothed speed profile, both the logC-logV and logC-logA correlations are near zero. **(B)** For an elliptic trajectory, ra = 150 mm, rb =10 0mm, with the same speed profile, the logC – logV correlation is still low, while the logC-logA is very high. In this case, it is worth realizing that an exponent tending to β=1 for the AC power law already indicates, if one recalls the mathematics, that there will be no relation at all between tangential speed and curvature.

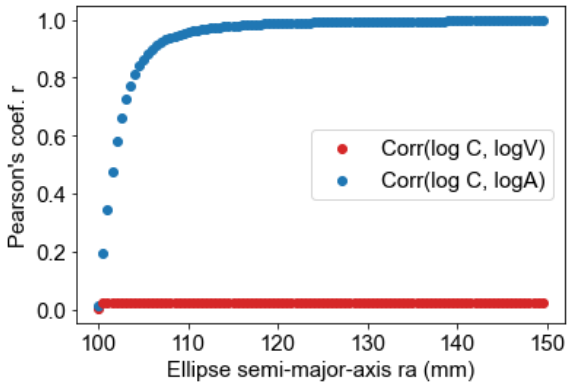

Figure 7. The correlation between logC and logA depends on the range of curvature, even in the absence of a speed-curvature power law. All ellipses had the same random-smoothed speed profile (see Figure 5), and the same semi-minor axis rb=100mm, while the curvature range was varied by increasing the semi-major axis. Note that, for instance, at 105 mm, which is 5% increase, the correlation is 0.87, and at 110 mm, which is a 10% increase, the correlation is 0.96.

4. DISCUSSION

Here we asked whether tangential speed and angular speed are equivalent in estimating the speed-curvature power law from data.

The analysis of empirical data from an experiment with human participants (Figures 2, 3 and 4) shows a huge difference between the two forms of the speed-curvature power law, generally considered to be equivalent. This difference can be explained in the following way: the VC version of the law is relating a purely kinematic variable (tangential speed) with a purely geometric variable (curvature), and is adequately reflecting the strength of the constraint in human movement. The correlation between angular speed and curvature, and the AC power law, have to be interpreted differently.

We also found that some types of movement (here fast drawing of ellipses at about 1s of cycle time or faster) show a strong speed-curvature power law, always with an exponent tending to $\beta vc \approx -1/3$. Other forms of movement don't show a strong speed-curvature power law, namely, drawing of slower ellipses (Figure 2).

We stress that this correlation is measuring a genuine phenomenon in movement, an invariance that needs to be explained by theories of human motor control. However, the correlation between angular speed and curvature will reflect the tendency that small-curvature segments have smaller angular speed; a generic fact we know a priori from A = CV.

Conceptually, one could think of the correlation between angular speed and curvature as the correlation between widths and areas of rectangles (area = width * height, or log area = log width + log height) in a set of rectangles of different sizes. In this idealized example, the correlation is not necessarily there, but in most cases, if we have a sufficient range of widths, the small-width rectangles can be associated with small areas, and large-width rectangles with large areas, regardless of their heights. The larger the range of widths, the stronger the correlation.

Although it is possible to measure (Figure 4) or construct (Figure 6) trajectories where angular speed and curvature that are hardly correlated at all, in most movement trajectories a great deal of the correlation between angular speed and curvature arises from the fact that the rate of change of

direction increases in direct proportion to curvature. As we have illustrated, a high correlation between logA and logC in an elliptical trajectory can arise it complete absence of kinematic-geometric constraints (Figure 6B and Figure 7).

Ever since the pioneering works of Binet and Courtier (1883), Jack (1895), and a century latter Lacquaniti et al (1983), one goal of movement science is to understand why there is a relationship between movement speed and curvature. In such a quest, the main message of this manuscript is that when estimating the correlation between kinematics and geometry of movement, one should better use mutually independent kinematic and geometric variables, such as tangential speed and curvature, avoiding angular speed, as it can often inject spurious correlations to curvature, leading to misinterpretations of the strength of the speed-curvature power law of movement.

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Conflict of interests. The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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