Memory as a Topological Structure on a Surface Network

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Abstract:A special charged surface network with surface spin half particles on it, that can be arranged in topologically inequivalent ways, is introduced. It is shown that action potential-like signals can be generated in the network in response to local surface deformations of a particular kind. Signals generated in this way carry details of the deformation that create them as a form of plasticity that influences the pathways they traverse leaving a topologically stable helical array of spins: a potential memory substrate. The structure is a non transient alignment of surface spins in response to the transient magnetic field generated by the moving charges present in the action potential-like voltage signals generated since particles with spin have magnetic properties. The structure has a natural excitation frequency that may play a role in memory retrieval. Signal generation and memory storage are proposed to depend on the existence of a surface spin structure. We show that such a surface network can capture the intricate topological features of any connectome in the brain. In addition biophysical properties of such a network are examined in order to constrain predictions of how it may function.

keywords:surface network, solitons, spin structure,memory substrate, topology

Highlights

Global method for generating one dimensional non-dissipative voltage signal pulses, analytic expressions for them with numerical examples, memory substrate creation, memory lifetimes, surface distortion waves.

Introduction

The importance of topology for exploring global features of the brain is now beginning to be recognized in neuroscience and elsewhere. For example there is growing interest in using topological methods to identify hidden structures present in the vast amount of data available that describe the connectivity architecture of the brain's axon network. This is an important area of current research. Another area of exploration where topological and algebraic geometry methods are being used is to unravel the information contained in brain signals. A few representative references in this growing area of research are: (Bassett,2020; Curto,2017; Reimann,2017; K.Hess, 2018; M.Marcolli,2019).

Recently there has also been growing interest in using unconventional ideas to throw light on the functioning of the brain. The aim is to relate global and local features of the functioning brain. We briefly mention two general examples of this nature in which geometric and topological ideas are used. In the first it is suggested that ideas from gauge theories of physics (Sengupta,2016), may help provide a framework for relating local activity of neurons to their system level behavior. Gauge theory converts a global symmetry of a system described by a Lagrangian (O'Raifeartaigh,1997) to one that is local and in the process introduces specific types of local interactions. The underlying geometric structure of gauge theory is that of a mathematical fiber bundle space (Nash,1983). For the brain, the analogue of the Lagrangian, is assumed to be a metastable free energy with a symmetry. The hope is that introducing gauge

theory ideas in such a framework would help tackle problems previously outside the realm of computational neuroscience, such as finding links between action and perception. The scheme described is aspirational and novel but no concrete free energy expression is proposed nor is there a proposal for its symmetry. The major underlying problem addressed is to try to relate the functioning of the brain to its form.

There is also the suggestion that the brain may be regarded as an infinite genus object (Tozzi,2021) where spontaneous creation of Riemann surface networks happen whenever brain signals are generated. A geometrical picture of such a process is suggested where inhibitory neurons are represented by vortex holes which are then given a topological interpretation. The aim of the approach was to understand the functioning of the brain in terms of a global balance between inhibitory(I) and excitatory (E) neurons. A simple geometrical method is proposed to determining the ratio E/I and is carried out. The major idea of the approach is to suggest that a global balance between inhibitory and excitatory neuron activity underpin the functioning of the brain.

In an entirely different area of research, quantum computing, topological ideas are also gaining importance. For example in one influential approach, a topological genus g Riemann surfaces with surface spins has become an area of intense research activity. In these works the aim is to find a way to store and manipulate quantum information that preserves the quantum nature of the system all the time. Quantum systems with large number of atoms or a large number of spins can be made to be in a certain quantum state but

manipulations of the state, necessary to carry out computing, can spoil the quantum nature of the information being processed. There is a need to produce long term stable quantum states. Recent work on a network of spins on a genus g Riemann surface attempts to address this problem by exploiting the inherent stability of topological properties (Hamma,2008). Thus on a genus g Riemann surface two classes of quantum spin configurations are considered, those that form closed loops and those that are not so aligned. Each class of configuration carries an energy label. It is shown that if the strength of an external magnetic field that interacts only with the non aligned spins crosses a critical value, the stable topologically protected lowest energy configuration of the system consists of coherent quantum spins loops rather than of individual spins or open string configurations. This is an example of a topological phase transition. Methods of creating such configurations and their properties are being studied.

In the spirit of searching for global ways of understanding the functioning of the brain, we suggest a scheme in which the form of a special surface network system, represented by its connectivity architecture, can be used to generate action potential-like excitations, by input surface deformations. The underlying idea is to show how the form of a system, the surface network, determines its function as a signal generating device. The scheme proposed differs significantly from existing methods of signal generation, (Scott,2002; Heimburg,2005; Shrivastava,2020) in two important ways: signals are generated by global means where the topology of the surface network and the presence of

surface spin half particles play an essential role in generating action potential-like soliton signals. Signals are produced in response to input signals that are local surface deformations, defined by signal specific parameters, that reduce the topology of signal generating subsystem to that of a sphere. We show that the signals produced by topolgy changes carry away with them the topolgy change structure as well as the signal specific surface deformation information that create them. The soliton pulses, carry charge and as they move produce a transient magnetic field that acts on the surface electron' spin and alignes them in topologically stable static signal specific magnetic structure that can be excited by a signal specfic frequency. The structure, a memory substrate is created in the pathways traverse by the soliton pulses. These remarks suggest that memories, stored in static structures, may be retrieved by oscillating electromagnetic signals either external or internally generated that have frequencies that overlap with memory label frequencies by a resonance excitation mechanism.

We explain three important features of the scheme: the reason for choosing a special surface for the network, the reason for choosing special type of input signals and how the surface network produces one dimensional action potential-like voltage pulses that are soliton solutions of a special non-linear differential equation called the non-linear Schroedinger equation.

We now prove that the special surface network we choose exactly captures any hypothetical connectome of the brain. It thus captures a important global

feature of a brain. Suppose we are given a hypothetical connectome of a specific brain. If this connectome is covered by a smooth surface then a topological theorem, the theorem on classifying surfaces (Munkres,2014) tells us that no matter how complex and intricate are the unknown connection details of the underlying connectome the surrounding surface is topologically equivalent to an orderly array of donut surfaces. Thus the theorem tells us that (Fig 1, Fig 2A, Fig 3) all represent a Riemann surface. The unknown details of the connectome are reflected by one unknown number of the surface, namely, the number of donut surfaces required to represent it. We will call this mathematical surface an "ubersurface" as it can viewed as a mathematical surface that covers the brain's assembly of three dimensional individual linked neurons. If we suppose the ubersurface is smooth then it becomes a well studied mathematical surface known as a Riemann surface. We will explain what we mean by a smooth surface shortly. The number of donuts g present in the ubersurface is called the genus of the Riemann surface. Thus a Riemann surface which looks like a collection of thin tube like surfaces joined together in regions to form a compact overall structure exactly capture the topology of a given connectome. It captures the form of the brain.

There are, however, many smooth non equivalent mathematical Riemann surfaces with spin structure that can capture the brain's intricate connectivity architecture. However within this broad class, it can be proved (Arbarrello,2001) that there is a special subclass of surfaces can generate the wide variety of brain-like signals we want provided this special class of surfaces are

charged and have a spin structure. A spin structure means that on the surface there are spin half particles (electrons) that can be arranged in many topologically inequivalent ways. Let us explain this idea by an example. Consider an arrangement of spins in the form of a chain. We view each spin half electron as a magnet and visualized an array of individual spin units as arrows, that represent the north pole to south pole direction of the magnet, linked together, with all the arrows aligned, that encircles a tube of the surface. The spins are linked by spin-spin magnetic forces. Such an array cannot be changed to one in which the arrow directions are reversed in a continuous way. The two arrays represent two possible spin structures. A given spin structure is topologically stable and cannot be changed by smooth deformations.

The next step is to introduce input signals. An input signal on a surface will be taken to be a local set of surface deformations that are consistent with the mathematical features of the ubersurface. This is physically reasonable as our system is charged surfaces so that surface deformations will lead to local voltage changes. The response of the surface to these deformations is the signal. Surprisingly these are, as we will show, one dimensional soliton solutions of the non-linear Schroedinger equation. Let us explain how such a precise result follows from the qualitative geometrical picture described.

To do this we need two mathematical results. The first tells us that a geometrical charged Riemann surface with spin structure can also be described by an algebraic function called the Riemann theta function where all its variables are constructed from the coordinate variables of the Riemann surface

while the spin structure of the Riemann surface is represented in the Riemann theta function by a set of variables called characteristics that can only take two values, namely $(0, \frac{1}{2})$. The charged nature of the Riemann surface makes the theta function a voltage function. The second is that a Riemann theta function associated with a Riemann surface must satisfy an identity called the Fay trisecant identity (Mumford,1987). It should be noted that the Fay identity holds only if the system has a spin structure.

We now have the concepts needed to explain how signals, as soliton solutions of the non-linear Schroedinger differential equation, can be generated by input surface deformations. The essential idea is to exploit the link between geometry and algebra provided by the Fay identity. This remarkable connection was used by Mumford (Mumford,1987) to find one dimensional soliton solutions of non-linear differential equations. We will discuss this point in greater detailed later. But first we depart from Mumford and show that his solutions can be interpreted as action potential like brain signals. To establish this result requires several steps.

We start by showing that these solutions can be interpreted as signals from one part of our two dimensional Riemann surface network to another. This will be true if our Riemann surface has a modular structure so that signals can be produced in subunits and is the case for our Riemann surface that represent real hyperelliptic equations(Harnack,1876). We next establish the relevance of our network for the brain. This follows from the mathematical result that our

surface exactly captures the topological connectivity properties of any hypothetical brain connectome (Munkres,2014) and hence is related to the brain in a very precise way. Finally we interpret local surface pinch deformations as input signals of the network and show that the response of the system to such input signals are soliton solutions of the nonlinear Schroedinger equation if our Riemann surface represents a real hyperelliptic equation and has a spin structure. Thus Mumford's technical method of finding one dimensional soliton solutions now becomes a method of generating one dimensional soliton in a network.

The dynamical law underpinning our Riemann surface network which leads to these results can now be stated. It is a dynamical principle of compatibility. This requires that all allowed Riemann surface local pinch deformations input signals and the response to them must be compatible with the mathematical structure of the Riemann surface of the network. From this principle, we can show, a wide class of one dimensional signals in response to local pinch deformation signals of different kinds can be generated. We can summarize the essence of the principle by saying it shows how the form of a system can be used to perform its function. The dynamical principle simply emphasizes the need to preserve the form of the system

We have outlined how to generate signals by exploiting the form of the surface network and have also described how memories in the form of deformation

parameter details can be stored non-locally in a magnetic structure with a frequency label, forming a memory substrate, and how they may be retrieved by a resonance mechanism.

The next step is to to introduce and define the mathematical concepts we need to establish the ideas described. After that we will be able to properly define our ubersurface and pinch deformations, state the dynamical laws of compatibility, and show how pinch deformations can produce one dimensional voltage pulse solutions of the one dimensional non-linear Schroedinger equation. We start by explaining the mathematical ideas required.

Summary of Mathematical Background Ideas

We have already proved that a ubersurface can be chosen to provide an exact topological representation of a particular brain's connectivity architecture captured by its connectome and have said that such a smooth surface is known as a Riemann surface. Our next step is to properly define a Riemann surface. This is done by describing the surface by appropriate coordinates that captures its topological connectivity properties and then introducing mathematical objects that properly describe the smoothness properties of the surface. A useful starting point is a mathematical result that tells us that all Riemann surfaces can be viewed as geometrical representations of polynomial equations (Teleman,2003) of two complex variables, $P(z, w) = 0$, where,

$$P(z, w) = a_0(z) + ...a_i(z)w^i + ...a_m(z)w^m = 0$$

and the coefficients $a_i(z)$ are polynomial functions of z. The degree n of the polynomial, which means the highest power of $n = l + k$ of terms $z^k w^l$ that is present in $P(z, w)$ fixes the genus g of the surface to be $g = \frac{(n-1)(n-2)}{2} - N_s$, where N_s is the number of singular points of the polynomial equation, that is points where $\frac{\partial P(z,w)}{\partial z} = 0$. Thus a Riemann surface is a geometrical representation of a polynomial equation of two complex variables with its genus fixed by the nature of the polynomial. This representation defines the notion of smoothness of the surface as well, where the smoothness of a surface means that there are mathematical objects on the surface that can be differentiated an arbitrary number of times. We then use a mathematical result (Arberello,2001) that shows that if our ubersurface represents a special polynomial equation called the real hyperelliptic equation and has a spin structure, then pinch deformations can produce a wide variety of one dimensional brain-like signals.

Thus our reason for picking our ubersurface to be a special charged Riemann surface with spin structure is because it exactly captures the topological connectivity of the brain and, by pinch deformations (to be defined shortly) on it, action potential-like signals could be generated by topological means. We thus have a surface network with the topological connectivity properties of the brain which is able to generate action potential-like voltage pulse signals by exploiting its form.

We now proceed to define the mathematical terms introduced: a genus g Riemann surface, Σ_g, its associated the Riemann theta function and show these objects can be constructed from a real hyperelliptic algebraic equation.

We also provide a mathematical representation of the deformations, called pinch deformations, of the Riemann surface that generate brain-like solutions of interest.

A Riemann surface Σ_g of genus g with spin structure can be visualized as a smooth spherical surface with g smooth charged tube-like handles (Fig 1) that has electrons on its surface. Bending, stretching or squeezing the sphere or its handles do not change the nature of the Riemann surface as long as these operations are smooth. Consider now a ubersurface that covers the brain's assembly of individual neurons, as described earlier. This surface will have multiple closed loops surrounding voids (Fig 2). The number of these voids give the genus of the ubersurface. For reasons stated earlier such a ubersurface that captures the connectivity architecture of the brain can be represented by a Riemann surface of high unknown genus value g that can change.

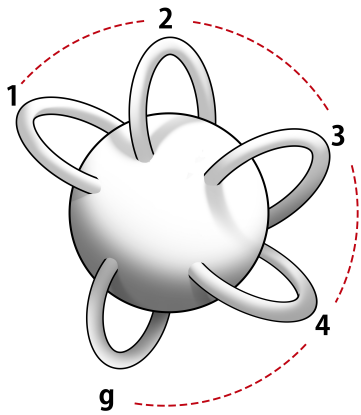

Fig. 1 Genus g surface

Thus the topological connectivity properties of the Riemann surface are displayed in (Fig 2).

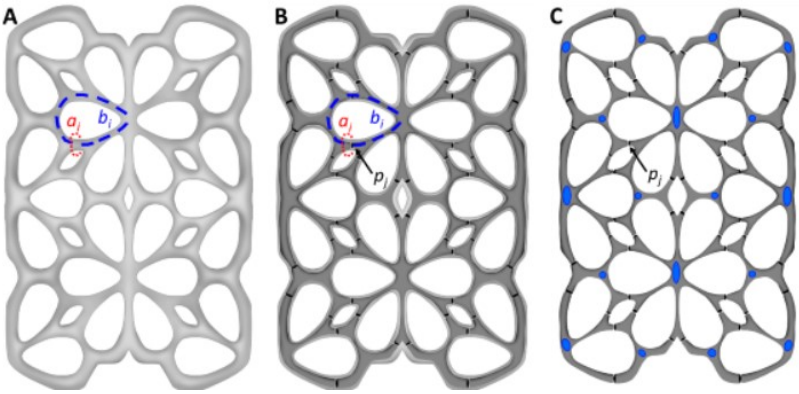

Fig. 2 Riemann surface with coordinate label,Pinched in black, Resultant g=0 surface

Riemann Surface, Σ_g: Representation and Properties

We next explain how the topological connectivity and smoothness properties can be represented. The topological connectivity properties of Σ_g are described by using topological coordinates that are a set of $2g$ closed non-contractible loops $(a_1, a_2, ..a_g : b_1, b_2, ..b_g)$ as displayed in Fig 3. The loops can be selected in different ways, a particular choice of loops is called a marking. The loops chosen have the property that two loops in the set $A = (a_1, a_2, ..a_g)$ or $B = (b_1, b_2, ..b_g)$ do not intersect while a loop a_j from A only intersects with the loop b_j from B. We write these geometric properties as $(a_i, a_j) = (b_i, b_j) = 0, (a_i, b_j) = \delta_{i,j}$, where $\delta_{i,j} = 1$ when $i = j$ and is zero when $i \neq j$ (Fig 3).

The smoothness properties of a Riemann surface Σ_g, with surface points represented by a complex number u, are represented by a set of g linearly independent holomorphic (smooth) differential one forms $\omega_i(u)du, i = 1, 2, ...g$

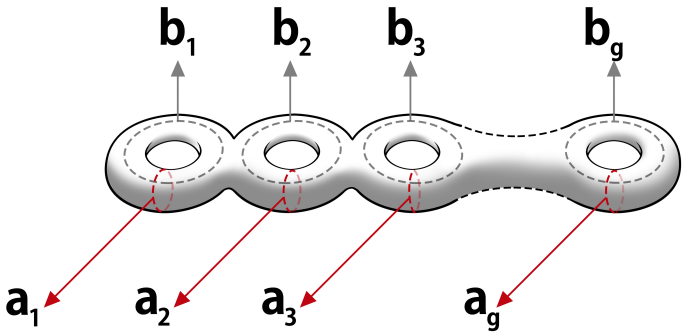

Fig. 3 Marked Riemann surface with topological coordinate loops

on Σ_g (Mumford,1987). Since Riemann showed that on a Riemann surface exactly g linearly independent holomorphic one forms exist, they form a set of basis one forms for Σ_g. This means that any holomorphic on Σ_g can be written as a linear combination of them with constant coefficients. A one form is an object that can be integrated. Thus if we write an integral $\int f(u)du$ then, $f(u)du$ is a one form. Smoothness of the one form means $\omega_i(u)$ can be differentiated an arbitrary number of times with respect to the variable u. In the calculus of complex variables this result follows if $\omega_i(u)$ does not depend on the complex conjugate of u. This result follows directly when a Riemann surface is constructed from a polynomial equation of two complex variables. We will demonstrate how this is done by constructing the Riemann surface that represents a real hyperelliptic equation.

The surface Σ_g, can also be described by an algebraic function called the Riemann theta function. The advantage of this step can be explained by a simple example. Consider the surface of a sphere. It is a geometrical object. If

we would like to study how this surface behaves under deformations it is helpful to introduce trigonometric functions. The Riemann theta function plays such a role for studying the dynamical behavior of Σ_g as we will see. Furthermore it easily incorporate the spin structure of Σ_g in terms of a set of $2g$ discrete parameters called characteristics (Mumford,1987). The other variables in the Riemann theta function are constructed from the $2g$ loops and the g smooth (holomorphic) one forms of Σ_g. Thus these variables are constructed using the smoothness and connectivity features of Σ_g. We now define the algebraic Riemann theta function.

The Algebraic Riemann Theta Function

We saw that a Riemann surface Σ_g could be defined in terms of $2g$ loops $(a_i, b_i), i = 1, 2, .g$ and g one forms $\omega_j(u)du$. From these there is a natural way to generate a set of variables (n_i, z_i, Ω_{ij}) as follows. The one forms were normalized so that an integral round a a_i loop $\int_{a_i} \omega_i(u)du = \pm 1_i$. The result represents a winding number associated with the loop a_i. The two signs correspond to the two ways a winding can happen: clockwise or anticlockwise. Thus associated with a loop a_i one can associate a winding n_i, which can take any positive or negative integer values. Next for any given point z of Σ_g it is possible to generate g complex numbers in a natural way by the map $z_i = \int_{z_0}^{z} \omega_i(u)du, i = 1, 2, ..g$ where z_0 is an arbitrary fixed point on Σ_g and we can construct a complex matrix by integrating one forms over b_i loops as $\Omega_{ij} = \int_{b_i} \omega_j(u)du, i, j = 1, 2, ..g$. Riemann proved that this matrix was symmetric and that it's imaginary part was positive. Finally an additional set of

variables (α_i, β_i) can be introduced that can only take the values $(0, \frac{1}{2})$. These discrete variables are called characteristics and they represent the spin structure of Σ_g. They allow a fractional windings when going round the a_i, b_i loops which can be shown to be related to the presence of surface particles with spin (Atiyah,1971; Mumford,1971). We can now define the Riemann theta function with characteristics (Mumford,1987):

$$\theta_{[\alpha_i,\beta_i]}(\Omega, z_i) = \sum_{n_1,..n_g=-\infty}^{+\infty} e^{\sum_{a,b} \pi i(n_a+\alpha_a)\Omega_{ab}(n_b+\alpha_b)+2i\pi(z_a+\beta_a)(n_b+\alpha_b)\delta_{a,b}}$$

where $\delta_{a,b}$ is the Kronecker delta function.

The algebraic Riemann theta function is the corner stone of all our discussions. Our one dimensional multi-soliton solutions in Σ_g are expressed in terms of them. In view of this we clarify an important conceptual point. The Riemann theta function depends on g complex numbers $z_i = \int_{z_0}^{z} \omega_i$, generated from two points (z_0, z) on Σ_g. It is thus a g dimensional space of complex numbers, where these numbers have periodicity properties, inherited from the lattice periodicity of the integrals of ω over the closed loops (a_i, b_i) of Σ_g (Mumford,1987). This g dimensional complex space, with the lattice periodicity features factored out, is called the Jacobian, $J(\Sigma_g)$ (Mumford,1987). On the other hand Σ_g is described by just one complex number that represents the points on its surface. Thus the natural question is: how can a theta function represent Σ_g, a function of just one complex variable and how can it produce a one dimensional pulse in Σ_g? There is a helpful mathematical theorem, the Jacobi inversion theorem (Guardia,2002), that answers the first question. It

tells us how these two spaces can be related. The basic idea is that for g points on $J(\Sigma_g)$ there are g related points on Σ_g. This theorem is stated in Appendix C.

The answer to the second question comes is that in the pinch deformation limit all of the g complex variables of the theta function collapse to a pair of real numbers that can be interpreted as (x, t), These numbers are multiplied by factors $W_i, V_i, i = 1, 2, ..g$ that represent the nature of the pinch deformation.

Let us give a qualitative account of this result. We start by giving a rough definition of a local pinch deformation at a point z of Σ_g. A more precise definition of the deformation is given later. Under a pinch deformation the circumference of tube at the pinch point goes to zero and there is a direct link between all g complex variables of $J(\Sigma_g)$ and one complex variable of Σ_g so that the Riemann theta function now directly becomes a function on Σ_g. Let us show how this happens in an intuitive way. Consider a pinch deformation at a point z of Σ_g and examine its effect the g variables z_i of the theta function. These variables are related by the equation $z_i = \int_p^z \omega_i(w)dw$, where z is a point on Σ_g. Now consider a circular arc joining the points z and p of Σ_g. The point p is an fixed point on this circle. In the pinch limit z approaches the p, i.e $z \to p$ as the radius of the circle with center $z = z_0$ inside the tube, shrinks to zero so that the arc length connecting z and p also become small. Thus we have

$$z_i \to \int_p^z \omega_i(w)dw$$

$$\rightarrow (z-p)\omega_i(z_0)$$

when z close to p. Since both points, z and p are close to the center of the tube, $z = 0$ we approximated $\omega_i(w)$ by its value at the center of the tube, namely, by $\omega_i(z_0)$. When this is done the g points of the theta function become, $z_i \approx \omega_i(z_0)z$. Thus in the pinch limit all the variables of the theta are all proportional to the point z which is a point on Σ_g close to the center of the tube. We have set $p = 0$. The proportionality factor $\omega_i(z_0)$, defines the nature of the pinch deformation at the point z. Thus a pinch at one point z of Σ produces g variables of the Riemann theta that are all proportional to a single point z of Σ_g which is close to the centre of the tube.

We next define a hyperelliptic equation and show how it can be used to construct a Σ_g and algebraic Riemann theta function. The hyperelliptic equation has three important features. It allows the maximal possible spin structures on its associated Riemann surface (Krazer,1903). It allows the maximum number of sub unit connected Riemann surfaces allowed (Harnack, 1876). This means it permits a large number of specialized functional subunits of the network to exist, and it can generate a vast number of diverse non-linear excitations by pinch deformations on its subunits [Arberallo,2001; Kalla,2012].

Hyperelliptic Equation, Riemann Surfaces, Theta functions

A real hyperelliptic equation for a genus g Riemann surface is an algebraic equation defined by,

$$y^2(u) = \Pi_{i=1}^{2g+2}(u - x_i)$$

$$= P(z)$$

where u, y are complex numbers but $x_i, 1 = 1, 2, ..(2g + 2)$ are either all real numbers or they occur as a pair of complex conjugate complex numbers. Let us sketch how a hyperelliptic equation describes Σ_g as well as its associated Riemann theta function.

From $P(u)$ to Σ_g

The hyperelliptic equation $y^2 = P(u)$ leads to $y = \pm\sqrt{P(u)}$. Near a zero point x_a of $P(u)$ the function becomes $P(u) = (u - x_a)K$ when all the zeros are distinct. Setting $w = (u - x_a)$ we have $P(w) = wK$. Thus a circle round $w = 0$ changes the sign of y. It is not well defined. This problem is overcome by introducing a branch cut, a line, that starts at the point $w = 0$ and the rule that when circling round the point $w = 0$ this line cannot be crossed but the variable moves on to another copy of the complex plane. By this means the function y can be made a well defined function of the variable z near all of its zeros. Thus the zeros of $P(u)$ determine (Belokolos,1994) the branch point locations of y that define the cuts necessary to make $y(u)$ well defined. These cuts define the topological coordinates, of the Riemann surface associated with the hyperelliptic equation, namely its homology cycles, $a_i, b_i, i = 1, 2, ..g$ as suitable circuits round the branch points. In particular the a_i cycles, that represent loops round the circumference of tubes of Σ_g, are constructed as circuits round a pair of neighboring zeros of $P(x)$ linked together by a branch cut. We next write down the rule for constructing g smooth one-forms from

$P(u)$. We have,

$$\omega_i(u)du = \frac{u^{i-1}du}{\sqrt{P(u)}},\ i = 1, 2, ...g.$$

Now the complex variable u represents a point on the Riemann surface of genus g constructed. We will replace u by z to represent such points. Thus we have sketched how the topological cycles (a_i, b_i) and the smooth one forms ω_i that define a Riemann surface can be constructed starting from the hyperelliptic equation. It is possible to check that the set of one-forms defined from the hyperelliptic equation are smooth functions, that is they can be differentiated an arbitrary number of times.

From $P(z)$ to the Riemann theta function

We next show how the algebraic Riemann theta function associated with $P(z)$ is constructed. Since we have already sketched how to construct the cycles (a_i, b_i) and the one forms ω_i this is straightforward. We simply have to determine the Riemann theta function variables $(\Omega_{ij}, z_i,)$ and z_i. We have, $\Omega_{ij} = \int_{b_i} \omega_j(z)dz$, and $z_i = \int_0^z \omega_i(w)dw$. The characteristics of the theta function are not fixed by $P(z)$ but are additional structural information about Σ_g that has to be added to the Riemann surface generated by the hyperelliptic equation (Krazer,1901,Kallel,2010).

We are now ready to define our surface network in terms of two postulates and define what is meant by a pinch deformation.

The Surface Network

The network is defined by the following two postulates:

Postulate 1: The surface network is represented by a charged Riemann surface Σ_g with spin structure of genus g associated with a hyperelliptic equation, where g represents the number of Riemann surface loops.

Postulate 2: The surface Σ_g is dynamic and evolving. The response of the surface to local pinch deformations, initiated by mechanical, electrical or chemical means, is determined by requiring them to be compatible with the Riemann surface structure of Σ_g. The responses are the signals of the network. A pinch deformations reduce the circumference of a handle of Σ_g at a point to zero.

We briefly comment on what we mean by compatibility. It simply means that all the mathematical objects introduced are globally defined on the Riemann surface and continue to respect the mathematical features of Riemann surfaces under deformations. The input signals of the network are local topology changing pinch deformations that reduce the topology of a subunit to that of a sphere and in this process can produce a wide variety of excitations, including one dimensional soliton pulses as well as transient localized excitations. These results hold only if the surface contains electrons. Intuitively the excitations produced carry away the topology of the subunit with them.

Pinch deformations

We have already given an informal description of a pinch deformation. Now we give a more precise description of it. Recall a pinch deformation reduces the circumference of a loop handle of the Riemann surface at a point to zero. We define such a pinch on a genus g Riemann surface using its g smooth one forms. There are a number of reasons for this. Firstly, such an approach makes a pinch deformation compatible with the Riemann surface structure of our network

since the one forms are all globally defined on the Riemann surface. Secondly, since a one form can be integrated it can naturally be used in an integral along an arc round a central point of a Riemann surface tube that can be shrunk to size zero and hence can be used to describe a pinch. Finally pinch deformations introduced in this way introduce natural deformation parameters.

One forms

We now define a pinch deformation with greater precision. In our applications we will use one forms $\omega_j(z)$ on a Riemann surface that represents a real hyperelliptic equation. We showed that they are given by $\omega_j(u) = \frac{z^{j-1}du}{\sqrt{P(u)}}$ where $P(u) = \Pi_{i=1}^{(2g+2)}(u - x_i)$. Thus these one form depends on the $(2g + 2)$ roots of $P(u)$. We also saw that the a_i cycles of this Riemann surface have circumferences related to the difference between adjacent roots, $(x_{i+1} - x_i)$ of $P(z)$. Hence a pinch corresponds to two roots of the hyperelliptic equation approaching one another as then the circumference of the corresponding loop $a_i \to 0$. The pinch point is taken to be inside the interval. There are g cycles that can be pinched. Thus in applications these details are important. Next we discuss pinch deformations in a general way. There are g functions $\omega_j(u)du$ whose variables can be deformed differently at a given surface point p. This point has two coordinates $p_a, a = 1, 2$ so that we can write a deformation as $\omega_j(p_a + \delta_{a,j}(p))$. Finally we can write $p = F(z_1, ..z_g) = F(\vec{z})$ as a function of g variables of $J(\Sigma_g)$. This is an inverse of the map $p \to (z_1, ..z_g)$. This function is not known. Putting all these pieces together we can write the pinch deformed

one form $\omega_j(p)$, near a pinch point p as,

$$\begin{aligned}
\omega_j(p) &= (V_{a,j}(p) + W_{a,j}(p)k_a(p) + U_{a,j}\frac{k_a(p)^2}{2} + ..)dk_a(p) \\
D_aF(\vec{z}) &= \sum_{j=1}^{g} \partial_{z_j} F(\vec{z}) V_{a,j} \\
\Delta_a F(\vec{z}) &= \sum_{j=1}^{g} \partial_{z_j} F(\vec{z}) W_{a,j}
\end{aligned}$$

The set of points $j = 1, 2, ..g$ represent pinch locations while $k_a(p), a = 1, 2$ can be identified with a point on Σ_g. While the functions $V_{a,j}(p), W_{a,j}(p)$ are the signal specific pinch distortion functions on Σ_g that represent the distortion code of the signal generated. Details of how this procedure is used to prove the Fay trisecant identity and find soliton solutions are given in Kalla (Kalla,2012).

Results

The underlying ideas for generating one dimensional non-dissipative multi-soliton excitations by pinch deformations have already been explained. Let us now write down the solutions, pointing out that they carry pinch deformation information and then show how this information can be transferred and stored non-locally in the form of a topologically stable helical spin-aligned magnetic trace along paths traversed by the signal. It is substrate structure for memory. The substrate being the pathways traversed by the signal.

Generating Solitons

Pinch distortions have three effects: they increase local voltage values (since the surface is charged), they reduce the original genus g of the system to zero

as shown pictorially in Fig 4 and they force the response signal to be one dimensional.

Compatibility Principle and Soliton Signal generation

We would now like to determine the response of the surface to a pinch deformation by using the compatibility principle. The basic idea has already been stated. Namely, that the Riemann theta function describes a Riemann surface only if it satisfies the non-linear Fay trisecant identity given in Appendix A (Arbarello,2001; Raina,1988) and that the Fay identity, deformed by the pinch deformation, becomes a non-linear differential equation. For the deformations we consider, the differential equation is the non-linear Schroedinger equation. Thus the compatibility postulate requires that the Riemann theta function, with its variables distorted by pinch deformations, must satisfy this equation. Thus the response of the surface to a pinch deformation is a theta function solution of the non-linear Schroedinger differential equation.

There are a wide variety brain signal-like soliton solutions that are solutions of the non-linear Schroedinger equation. Thus there are bright soliton trains that are brain spike train-like and dark soliton solutions that are brain burst signal-like. Non-propagating local transient excitations are also possible solutions of the non-linear Schroedinger equation. The one dimensionality of the solution comes, as we saw, from the fact that a pinch deformation forces surface points to the center of the tube handle pinched. However the solutions do not describe the motion of the pinch point but have a profile determined by the nature of the pinch and the global properties of the Riemann surface that

is used to generate it. Here the idea is that a subunit of the Riemann surface generates signals that then propagate to other units connected to it by axons. Such a scheme is possible since a Riemann surface representing a hyperelliptic equation has a modular structure (Harnack,1876).

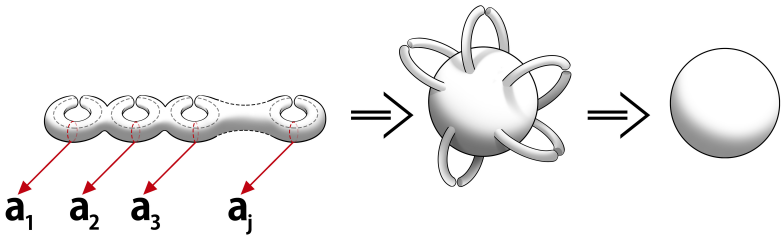

Fig. 4 Degenerate Riemann Surface

Analytic expression for multi-soliton voltage pulses

We now write down multi-soliton pulses solutions (Mumford,1987; Lakshmanan,1995) (Appendix A and B). They can be numerically evaluated (Fig 5). An example of such a solution found by Kalla (Kalla,2012) is:

$$\psi_j(x,t) = A_j \frac{\theta(\vec{Z} - \vec{d} + \vec{r}(j))}{\theta(\vec{Z} - \vec{d})} e^{-i(E_j x - F_j t)}$$

where $r(j) = \int_{a(n+1)}^{a_j} \omega_j(z) dz$.

$$\begin{aligned}
\vec{Z} &= i\vec{V}_{a_{n+1}} x + i\vec{W}_{a_{n+1}} t \\
K_1(a,b) &= \frac{1}{2} \frac{\Delta_a \theta[\delta](0)}{D_a \theta[\delta](0)} + D_a \ln \theta[\delta](\int_a^b) \\
K_2(a,b) &= -\Delta_a \ln \theta(\int_a^b) - D - a \ln(\theta(\int_a^b)\theta(0)) - (D_a \ln \theta(\int_a^b) - K_1(a,b))^2 \\
E_j &= K_1(a_{n+1}, a_j)
\end{aligned}$$

$$
\begin{aligned}
F_j &= K_2(a_{n+1}, a_j) - 2\sum_{k=1}^{n} q_1(a_{n+1}, a_k) \\
q_1(a,b) &= D_a D_b \ln \theta[\delta](\int_a^b) \\
q_2(a,b) &= \frac{D_a\theta[\delta](0) D_b\theta[\delta](0)}{\theta[\delta](\int_a^b)^2}
\end{aligned}
$$

where $\vec{d}, A_j$ are constants. Each pulse solution contains the same pinch deformation parameters through the variable $\vec{Z}$.

It should be noted that n pinches are involved in the solution $\psi_j(x,t)$. Thus a large multi-soliton train is generated by contribution from a large number of pinch deformations. Such a train reflects a global event. We also note that the term $S_j = e^{-i(E_j x - F_j t)}$ where E_j, F_j are constants, present in the solution is a coordinate artifact. This can be seen by multiplying the non-linear Schroedinger equations by S_j^{-1} and defining new variables t', x' by the equations:

$$
\begin{aligned}
\frac{\partial}{\partial t'} &= S_j^{-1}\frac{\partial}{\partial t} S_j \\
\frac{\partial}{\partial x'} &= S_j^{-1}\frac{\partial}{\partial x} S_j
\end{aligned}
$$

Thus $\psi_j(x', t')$ is a solution of the non-linear equation without the factor S_j but now the x', t' variables are j dependent which makes the physical interpretation of a train of solutions is not clear. The soliton trains given by Kalla (Kalla,2012) also do not have the same amplitudes or propagation speeds. These results may be artifacts of Kalla's solutions. We hope to address this issue in future work by directly using biologically relevant boundary conditions to generate solutions. Here we are recording the fact that a wide variety of solutions of

the non-linear Schroedinger differential equation exist and that a class of such solutions has been found by Kalla which have brain signal like features that are distorted by oscillatory factors.

Digression on the units and dimensions

An important technical feature of the model is that all the variables, functions and equations used are dimensionless. Thus a direct comparison of theoretical results with observations is not possible. This is also an advantage, because it allows a great deal of flexibility regarding the nature of the system that is being modeled and allows us to interpret the solutions found in terms of voltage pulses as we now show. We have already offered general reasons why the multi-soliton solutions may be interpreted as voltage pulses. Now we use the flexibility of the mathematical equations to show that the solutions can be interpreted as voltage pulses. Consider the following non-linear Schroedinger with dimensional parameters,

$$i\frac{\partial}{\partial t}\psi = -D\nabla^2\psi + g\psi^2\psi$$

where t is in time units, x is in length units, D is a dimensional diffusion constant, ψ is in volts and g is a dimensional constant which is a measure of non-linearity. We can rewrite this equation using dimensionless variables, we have,

$$t \rightarrow \tau = \frac{t}{t_0}$$
$$x \rightarrow \frac{x}{x_0}$$

$$D \to D_0 = \frac{x_0^2 D}{t_0}$$
$$g \to g_0 = \frac{g V_0^2}{t_0}$$

Thus we need to choose three units t_0, x_0, V_0 to compare theoretical predictions with observations. Once chosen, the response solutions generated in the model represent one-dimensional non-dissipative pulses of voltage, and the input signals can also be interpreted as surface deformations that induce voltage changes. Appropriate units of the model can be fixed from observations. Examples of multi-solitons belonging to two different classes, namely light and dark solitons, are numerically determined.

Let us now briefly comment on the way the distortion memory structures appear in our numerical evaluation of multi-soliton solution.

Examples of Multi-soliton solutions

We considered four examples. We note that the oscillatory terms in Kalla's solutions can be eliminated by a coordinate change. In view of this we evaluate ψ_j^2 and numerically checked that it is a soliton for the dark solution and that its inverse is a bright soliton. Two represent bright soliton trains with $g = 2, 3$ and two represent dark solitons that are excitations on top of an elevated background potential (Fig 5). We first found the dark solutions numerically by choosing parameter values and then inverted it and numerically checked that it represented a corresponding bright soliton train solutions. Direct evaluation of these two cases is possible and will be done elsewhere. Here we are simply

giving a qualitative feel regarding the nature of soliton solutions. Let us briefly comment on bright and dark solitons.

Solutions of the non-linear Schroedinger equation represent responses, to local pinch deformations of the Riemann surface network. The equation has a non-linear term of the form $\sum_{k=1}^{g} s_k \psi_k(x,t)^2 \psi_j(x,t)$ where $s_k = \pm 1$, for a soltion $\psi_j(x,t)$. Different choices for the sign factor s_k lead to different types of solutions. For dark solutions, of the simplest type, $s_k = +1$ for all k values, while for bright soliton trains $s_k = -1$ for all k values. Other sign selections produce voltage pulse excitations of different shapes and are topologically distinct.

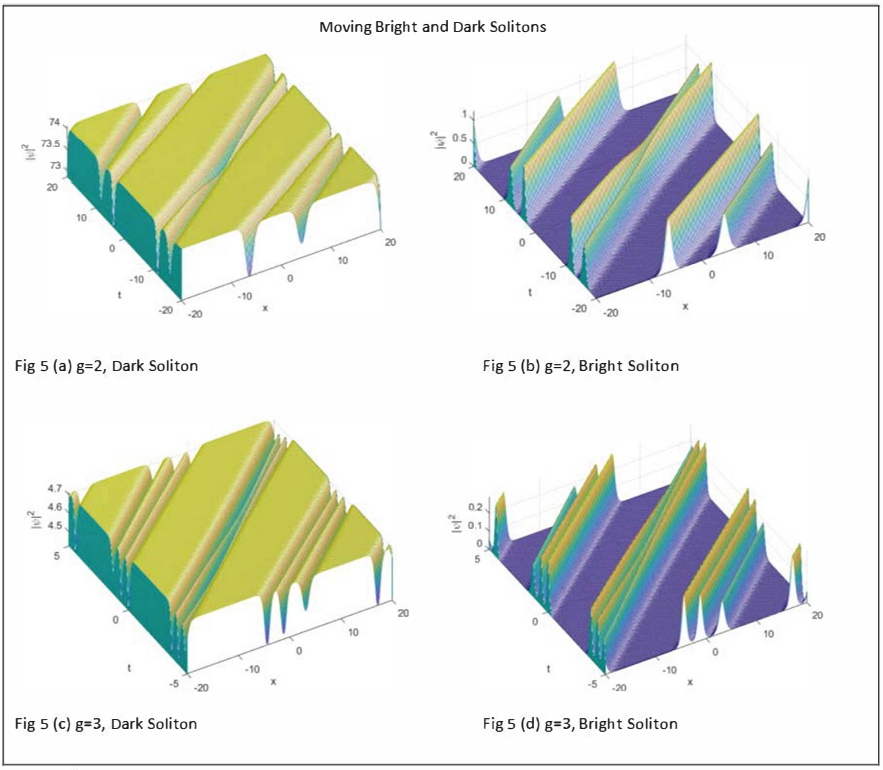

Fig. 5 Moving Dark and Bright Soliton Trains for g = 2, 3.

The numerical solution studied fixed the index j of the solutions displayed to $j = 2, j = 3$ to numerically determine two and three dark soliton solutions. The propagation speed of the solitons and their amplitudes found numerically can be adjusted to fit observations by a suitable choice of scale parameters that were discussed. In the case $j = 2$ there are eight and for $j = 3$ twelve distortion parameters. These parameters are assembled together by characteristics, α_i that identify the brain network responsible for the excitation. All characteristics are set to be equal to $\frac{1}{2}$ this means that in the soliton limit all networks paths of the unit producing the signal are used. We note the vector $\vec{Z}$ fixes the rate at which the spikes arrive. The gap between solitos also contains pinch deformation information. Let us write a voltage solution $\psi_j(x,t)$ as $\psi_j(A_j x - B_j + C_j)$ The parameter A_j, B_j, C_j contain pinch deformation information. The A_j, B_j parameters are in the variable Z while the parameter C_j comes from the diagonal part $\Omega_{j,j}$ of the period matrix. This is because in the pinch limit the loop a_j goes to zero. This means that the loop variable b_j has zero radius. Evaluating this limit gives C_j. Multi-soliton solutions are displayed in Appendix B.

Summarizing: the shape of each multi-soliton pulse contains regarding input distortions signal parameters at multiple pinch locations of the surface network. This substructure is carried directly by the solutions and also indirectly in the appropriate connectivity matrix elements Ω_{ij} of the network responsible for the signal.

We have already discussed in qualitative terms how multi-soliton trains with pinch distortion information generate helical transient magnetic fields that acts on the surface of the network tube that are present, aligning them to form a topologically stable static copy of the pinch deformation information: a static magnetic memory trace. We now proceed to determine the form of this static structure and establish that it has natural signal dependent excitation frequency using which the static memory may be retrieved.

Helical spin magnetic memory structure formation

Let us now determine the nature of memory substrate. The voltage profile, $\psi(z,t)$ of moving soliton trains along the center line of a tube of the network, was theoretically determined earlier. We label points of this center line by the variable z. By using Poisson's equation (Jackson,1999) of electromagnetic theory, we can directly find the corresponding moving charge profile density $\rho(z,t)$. We have the equation,

$$\frac{d^2\psi_j(z,t)}{dz^2} = 2\rho(z,t) \approx \frac{Q_j v_j}{\Delta_j}$$

where we have approximated the pulse by replacing it by a moving charge density, given by $\frac{Q_j}{\Delta_j}$ where Δ_j is the width of the jth pulse and v_j is the pulse speed. We note that we found that the jth voltage pulse depends on (z,t) variables in the following way, $\psi(x,t) = \psi_j(A_j z - B_j t + C_j)$ where A_j, B_j, C_j are pinch deformation parameters. We use a slightly different notation to describe the pinch deformation parameters. We next note that the mathematical form of ψ_j is that of an exponential function. Consequently $\psi_j = \frac{\rho_j}{A_j^2}$ and $\rho_j = \frac{B_j}{A_j^3}\frac{Q_j}{\Delta_j}$

so that $v_j = \frac{B_j}{A_j^3 \Delta_j}$ is completely determined by the distortion parameters. The speed of the soliton is determined by deformation parameters and hence it is signal specific.

Thus to calculate the magnetic field generated by a moving soliton voltage pulse we replace the voltage by a specific charge distribution that generates it. This step has an interesting implication. It suggests that the neurotransmitters released at a synapse also have a charge distribution that reflects the potential of the soliton pulse. In this way a mathematical explanation of the transition of voltage pulses to a collections of neurotransmitters can be understood. The neurotransmitter molecules carry the message of the pulse as it as a different representation of the same voltage profile.

A charge moving with speed v_j along an axis, creates a moving circular transient helical magnetic field along the axis of motion (Jackson,1999): a transient helical magnetic field. The link between the voltage pulse ψ_j and its charge density ρ_j described suggests that measuring the magnetic field generated by a soliton pulse can be determine its form. This has been done by (Schoss,2016).

To simplify our analysis we replaced ρ_j by $\frac{Q_j}{\Delta_j}$ where Δ_j is an effective width of the pulse. For N moving soliton pulses the magnetic field created by them $\vec{B}$is given by $\vec{B} = \sum_{j=1}^{N} \vec{B_j}$. If the jth soliton moves with speed v_j along the z-axis ,the magnetic field is a helical field circling round a tube of radius r, with magnitude $B_j = \frac{Q_j v_j}{cr^2}$, where c is the speed of light in the medium.(Jackson,1999). The transient helical magnetic field can be written

down in terms of its coordinates (B_x, B_y, z) as $B\cos(\frac{2\pi z_j}{L}), B\sin(\frac{2\pi z_j}{L}, z_j = v_j t$, where L is a loop length travesred by the pulse. We can also write this as $(B_x + iB - y, z)$

This field that moves along a helical path and interacts with spin magnets in its path and aligns them to produce a static helical structure that has a natural excitation frequency. We now determine this natural excitation frequency and then determine the static helical structure produced by the transient magnetic field of a soliton.

Helical Spin-Magnet Structure and Excitation Frequency

The transient magnetic field moving on the surface in a helical path, acts on the surface electron spin magnets that it encounters and aligns them in its direction. This process needs to be described using quantum mechanics where the spin-magnetic property of an electron is represented by a two component vector and its interaction energy with the transient moving magnetic field is represented by the operator Hamiltonian $H = \mu_e \vec{B}.\vec{s} = \mu \vec{B}\vec{\sigma}$, where $\vec{\sigma}$ are Pauli spin matrices used to describe the spin magnet nature of an electron, in quantum theory. $\mu_e = 2\mu$ where μ is the Bohr magneton (Landau,1977). In quantum mechanics the operator H controls the time evolution of electron spin. The Pauli spin are matrices that act on electron spins, represented as a two component vector, and change them. The two spin components represent spin up and spin down configurations with numerical value $\pm\frac{1}{2}$. We take the spin up vector to be in the direction of the magnetic field, tangential to the surface of the network, where the spins are located.

The interaction energy term H, in quantum theory, has a natural excitation frequency ω_j given by the formula $\omega_j = \frac{H}{\bar{h}} = \frac{\mu B}{2\bar{h}}$ where $\bar{h}$ is the Planck constant. We can estimate this energy. We replace $\vec{B}.\vec{\sigma} \approx$ by $B = \frac{Q_j v_j}{cr^2}$ and evaluate B. To do this we write the charge carried by the soliton pulse as $Q_j = N_j e$, with e the charge of an electron, its speed $v_j \approx 10^3 cm/s, r \approx 10^{-4} cm$,(Hursh,1939) set $c \approx 10^9 cm/s, , e \approx 10^{-10}, \mu \approx 10^{-19}, \bar{h} \approx 10^{-27}$ in CGS units(NBS data base), then the value of this excitation frequency is given by $\omega_j \approx N_j$. Thus for a pulse carrying N_j units of electron charge, in the range $1 \leq N_j \leq 10^2$, will lead to the excitation frequencies ω_j of the static memory trace to be in the range $1 \leq \omega_j \leq 100$ Hz that overlap with observed EEG waveform frequencies.

The helical pattern $S_j(z)$ generated by the jth soliton pulse can be determined by using the quantum evolution equation, $S_j(t) = e^{-i\frac{Ht}{\bar{h}}} S_j(0)$. The evolution operator is defined as $U(t) = e^{-i\frac{Ht}{\bar{h}}}$. It tells us how a given spin vector $S_j(0)$ evolves in time. In our case the electrons are at fixed locations along the path of the transient magnetic field and interact with it as it passes through the spin location and aligns it. Thus our interest is not in the evolution of a single spin vector but in the way spins at different locations, interacting with the transient magnetic field, get aligned in the direction of the field. This can be done by replacing t by $t = \frac{z}{v_j}$, where z is a point in the pathway of the transient magnetic field. This gives us an evolution operator $U(z)$. At this

stage we use the following identity:

$$e^{-i(\frac{\mu}{v_j\hbar})\vec{\hat{n}}.\vec{\sigma}Bt} = \cos[\frac{\mu}{v_j\hbar}\vec{\hat{n}}Bt] - i\vec{\hat{n}}.\vec{\sigma}\sin[\frac{\mu}{v_j\hbar}\vec{\hat{n}}Bt]$$

The result follows from the properties of the Pauli matrices, namely $\vec{\sigma}.\vec{\hat{n}}^2 = 1$. We can thus write the operator $U(z)$ as

$$U_j(z) = e^{-i(\frac{\mu}{v_j})\vec{B}.\vec{\sigma}z} = (\cos[(\frac{\mu}{\hbar v_j})Bz] - i\vec{\sigma}.\vec{\hat{n}}\sin[(\frac{\mu}{\hbar v_j})Bz], 0 \leq z \leq L$$

we have set $\vec{B} = \vec{\hat{n}}B$ where $\hat{n}$ is a unit vector in the direction o $\vec{B}$. We now set $\vec{\sigma}.\vec{n} = \frac{1}{2}$ when it acts on spins and alignes them. The first term acting on a spin agian gives a value $\frac{1}{2}$. Thus our static helical spin-magnet distribution,$S_j(z)$ is,

$$S_j(z) = (\cos[(\frac{\mu}{\hbar v_j})Bz] - i\vec{\sigma}.\vec{\hat{n}}\sin[(\frac{\mu}{\hbar v_j})Bz], , 0 \leq z \leq L$$

It is a helical chain of electron spin magnets. The natural frequency of this structure, as we stated, is given by, $\omega_j = \frac{H}{\hbar}$. For g pulses we get $S(z) = \sum_{j=1}^{g} S_j(z)$, a collection of static helical patterns. These patterns can intersect. Our calculation is a quantum calculation. This is appropriate for electrons. For a pulse carrying N_j units of electron charge, in the range $10 \leq N_j \leq 10^3$, the corresponding excitation frequency of the structure will be in the range $1 \leq \omega_j \leq 100$ Hz, which overlap with EEG waveform frequencies.

We will estimate the value of N_j when we next discuss the thermal stabilty of memory structures.

We have thus shown that memories are encoded as static structures along helical pathways on the surface. These different static structures can intersect but each structure associated with a specific memory has its own specific excitation frequency. Thus from a collection of such memories a specific memory can be retrieved by exploiting its specific frequency label by a resonance mechanism. This is a testable prediction of the scheme.

A picture of the helical field of the transient surface magnetic producing a helical spin structure is shown in Fig 6. The helical fields parameters used are all theoretical input values (Table 1). The exact form of the helical curve is not easy to determine. What we show is an idealized sketch using a simple representation of a helical curve, written as, $B(z)(r\cos\omega t, r\sin\omega t, r\omega t)$ which is plotted for two values of r that represent the inner, $R(2)$, and outer radii, $R(1)$, of the membrane tube. The magnetic field $B(z,t)$ has a complex shape and is related to the soliton voltage pulse in a way that was determined. The corresponding helical memory substrate is a alignment of spin magnets, that is a faithful copy of the transient field structure and has its excitation frequency, as pointed out.

The memory storing structure suggested, predicts that an external magnetic field would disrupt memory but not modify memory. The external field would act not on individual spin magnet pairs but on the entire spin memory structure leading to local smooth deformations of the structure that does not change its topology.

Table 1 Magnetic Helix.

Magnetic Field	Radius(1)	Radius(2)	Frequency
1	2	1.8	1

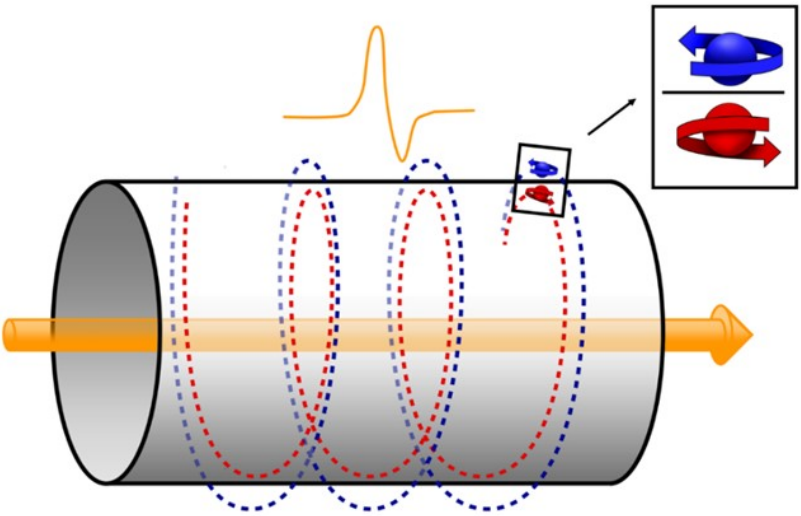

Fig. 6 Helical Transient Magnetic Structure

Biophysical Discussions

We have shown how action potential-like signals that carry a deformation code can be generated from the topological connectivity architecture of the brain as represented in a two dimensional charged surface network. The surface of the network is required to have electrons and the electron spin magnets form linking patterns that are topological in nature. In such a network local pinch deformations that carry deformation parameters, change the topology of a subunit to that of a sphere and in this process create soliton or multi-soliton or other excitations. We pointed out how the deformation code carried by the signals can be transferred to the pathways traversed by the signal using the presence of that spin structure and have suggested that memory retrieval

might be achieved by a process of resonance excitation where the presence of a frequency label attached to a memory structure is exploited.

We now assume that these signals can be identified with brain signals and that the memory storage mechanism suggested is relevant for memory storage in the brain. We would like to explore consequences of such an identification by making a number of numerical estimates. We showed that memories stored have a natural excitation frequency, that multi-soliton pulses are required to create thermally stable memories and that even though the memory traces are topologically stable they decay.

We start by posing and answering two questions: Is there any biological evidence that supports the assumption that pinch deformations and topology changes occur when signals are globally generated and can one provide numerical estimates for consequences of the scheme and show that they are realistic biological scales?

The first question is answered by experimental results of Faisal (Faisal,2005) and the other of Ling (Ling, 2020). In the work of Faisal it was shown that brain action potentials were generated if axons were pinched beyond a certain threshold value, while in the electron microscope work of Ling it was found that when a neuron produced an action potential its shape became sphere like. Both these results are consistent with the global pinch method of generating brain signals. The result of Faisal directly confirms that pinch deformations cause action potentials while the result of Ling support the picture that when a signal is generated topological spheres are created.

Further evidence of pinching and swelling during action potential generation is found in Costa (Costa,2018).

To tackle the second question we estimate the parameter values and the physical and biological characteristics of our mathematical results using the electrical and geometric properties of the brain axon membrane. In doing this we identify the handles of the Riemann surface as axons, their region of joining with the soma and the pinching points as synapse locations. Our major hypothesis is that memory is stored as a helical magnetic structure, a memory substrate, on axons created by brain signals traversing through them. We have theoretically determined the nature of static memory structures and shown that different memory structures carry different excitation frequencies that can overlap with the frequencies of EEG waveforms.

We next study stability conditions necessary to create memory and show that the a certain threshold value for charge transfer by a soliton train is required. We then show that even topologically stable memories decay due to thermal diffusion and from this estimate memory lifetimes. These are predictions.

Memory Stability

For helical magnetic memory traces to be created the binding energy of spins to the magnetic field generated by the soliton pulse should be strong enough to withstand thermal fluctuations at body temperature T and once formed the binding energy of a topological collection of spins should again be strong enough to withstand disruption due to body temperature thermal fluctuations.

Let us examine if these two conditions can be satisfied. The binding energy of an aligned spin to the magnetic field generated by a single soliton generated soliton pulse that carries N units of electric charge is given by $U = N\mu_B \vec{B}.\vec{\sigma}$, where μ_B is the Bohr magnetron, and $\vec{\sigma}$ are Pauli spin matrices. Using CGS units and physics numbers taken from the National Institute of Standards data base. Putting in numbers we get $U = N(10^{-15}$ergs

where N is the number of charges present in a single soliton pulse. We use a value for $c \approx 10^9$ cm/s in the medium,take $r \approx \times 10^{-4}$ (Hursh,1939). Now the thermal energy $U_T \approx kT = 10^{-14} ergs$ at room temperature, where k is the Boltzmann constant. Thus we need $N > 10$ for a single soliton pulse generated magnetic spin binding to be thermally stable.

We next estimate N. The area swept out in one millisecond by the brain voltage pulse is $2\pi\ \Delta vt$ If we take a action potential to travel with $v \approx 10^3$ cm/s and duration 10^{-3} cm/s, and suppose $10^{-10} \leq \Delta \leq 10^{-8}$ where Δ is the cross section radius of the one dimensional pulse, then by using the charge density numbers for the gaint squid (Segal,1968) we get $1 \leq N \leq 10^2$ (Segal,1968). Thus our estimate for N is consistant with its value required for thermal stability of forming memory structures and for the static memory structures to have excitation frequencies that overlap with EEG waveform frequencies. we have related surface charge densities in our estimate. This is motivated by the nature of pinch deformations where surface deformations play a key role.

Our crude estimate suggests that the binding energy of aligned spins per electron is stable under thermal fluctuations. Furthermore, once a memory

structure is formed its topological long range order introduces an additional topological stability to the structure. We next estimate the spin binding energy. The binding energy of a coherent N_0 spin system is, $B_s = N_0 \frac{\mu_B^2}{r^3} \approx N_0 \times 10^{-17} > kT \approx 10^{-14}$, setting $r \approx 10^{-7} cm, \mu \approx 10^{-19}$. Thus the structure is stable for $N_0 > 10^4$. Notice if we set $r \approx 10^{-8}$ then the binding energy would be stable for $N_0 > 10$. The key feature is the topological nature of the aligned spin system that makes it behave as a coherent unit. We speculate that this is the case for the memory trace and suggest that such a structure might play a role in storing short term or even long term memory.

It should also be noted that the estimate given implies that not all soliton pulses will produce stable spin magnetic structures as not all them carry the charge required to produce magnetic fields of the required strength. The magnetic storage of memory in pathways implies that they should be affected by magnetic fields and that these memories can be retrieved by a resonance mechanism using oscillating magnetic fields of low frequency.

Forgetting: Estimate of the Lifetime of Memories

We next estimate the lifetime of memory structures created in the surface network model. The inevitable decay of memories is due to thermal diffusion. The idea is that thermal diffusion inevitably lead to the movement of the particles that carry spin. This movement increases the distance between elements of the magnetic memory trace and eventually lead to their degradation. The random thermal fluctuations lead to the Brownian movement of particles and

is represented by the Einstein relation (Einstein,1905):

$$
\begin{aligned}
(\Delta x)^2 &= 2Dt \\
&= \mu k_B T
\end{aligned}
$$

where t is time, k_B is the Boltzmann constant, D the diffusion coefficient,μ the mobility coefficient and T the brain temperature. The collection of helically organized spins forms a topological configuration and hence its response to external disturbances reflects this collective property. For non-durable memory the fluctuating energy due to positional changes of the spins must be greater than the magnetic binding B_M that they have. Now B_M has a length dependence of $\frac{1}{l^2}$ which will also be increased by the fluctuation, that is, $l^2 \rightarrow l + \Delta^2 \rightarrow l^2 + \mu k_B T$. Putting these pieces together we get,

$$
\begin{aligned}
\frac{1}{2} k_s (\Delta x)^2 &> B_M \\
t(k_s k_B T) &> B_M = C_M \frac{1}{(l^2 + (\Delta x)^2)} \\
B_M &= C_M \frac{1}{(l^2 + T\mu k_B)} \\
t &\approx \tfrac{1}{T^2}
\end{aligned}
$$

Thus this estimate suggests that lifetime of the non-durable memory depends sensitively on temperature. It should be noted that this Brownian diffusion, with appropriate values for D and μ, is valid for long-term and short term memory and suggests that the Brownian diffusion process will inevitably degrade

memory, unless they are periodically consolidated. If we require working memory to have $t \approx 10$ sec and long term memory $t \approx 10^6$ sec (one month), it suggests B_L for long term memory should be $B_L \approx 10^5 B_M$.

Using these ideas an estimate for the number of pulses required to establish a long term memory can be made. Assuming each voltage potential pulse has duration $\approx 10^{-3}$ sec (Hursh,1939) and that the value of B_L is related to total charge transferred by pulses for a month long term memory to be created would require a pulse stream duration of $\approx 10^2$ sec.

We next estimate the electrical properties of the surface network.

Comments on Electrical Features of Surface Network.

We now show, that if we assume that our surface tubes have the known electrical and geometrical properties of the axon membrane we can theoretically estimate the value for the tube's voltage potential. This voltage value will thus estimate the expected resting potential of the system. The calculation uses dynamical laws from quantum electrodynamics. We then show that by introducing additional biological information regarding the presence of fluids inside the surface tubes, a formula linking the voltage pulse propagation in a tube to a surface distortion wave can be found. We start by estimating the equilibrium potential value in a surface tube of the network.

Estimate of Surface Resting Potential Value.

The usual way to estimate the Potential Resting value in biology is by using the generalized Nernst equation based on equilibrium thermodynamics (Sterratt,2014). There the specific equilibrium distribution of sodium and potassium

ions was required to model a membrane. Here our aim is different. We want to estimate the equilibrium value between strips of the surface separated by the diameter r of the our surface tube. It is an estimate which only uses the electrical properties and the dynamic laws of quantum electrodynamics (QED) of the tube surface. It was shown by Feinberg (Feinberg, 1970) using QED, that there is an attractive Casimir-Poldar potential $V_{CP}(r)$, between charged dynamic dipoles layers separated by a distance r. The attractive potential is balanced by the elastic properties of the tube and the fluid inside it. We use the observed value r for an axon tube for our surface tube in order to get our result. We have (Feinberg,1970),

$$V_{CP}(r) = -\frac{V_0}{r^7}$$

which is an attractive potential energy between dynamic neutral molecules. It is the resting potential. Our problem is to estimate the value of V_0, fix the value of r to be the tube diameter r to find the value of V_{CP}. We estimate a value of ≈ 70 mV. Dimensional reasoning is used to estimate V_0.

We use the following notation. Let d be the membrane thickness (Regan,2019), r the diameter of the tube, e the charge of an electron, r_b the hydrogen atom Bohr radius, and N the effective number of molecules in the segment of the membrane chosen. These variables, other than N, are taken to have the following values (National Institute of Standards data base):

$$\frac{e^2}{r_b} = 10 \text{ eV}$$

$$r_b = 10^{-7} \text{ cm}$$

$$r = 10^{-4} \text{ cm}$$

$$d = 10^{-6}$$

We next write V_0 as

$$\frac{V_0}{r^7} = N^2 \frac{e^2}{r_b} \frac{r_b}{r} \frac{d^6}{r^6}$$

N is determined by setting $V_{CP}(r) = eV_R$, where V_R is the resting potential. Introducing the Bohr radius for hydrogen atom allowed us to set an energy scale $\frac{e^2}{r_b} \approx 10$ eV. Thus we have

$$\begin{aligned} V_{CP}(r) &= N^2 \times 10 \times 10^{-12} \times \frac{10^{-7}}{10^{-4}} \times (\frac{10^{-6}}{10^{-4}})^6 \text{ ergs} \\ &= 100 \times 10^{-3} \times 10^{-12} \text{ ergs} \\ N^2 &= 10^{13} \end{aligned}$$

Thus we get $N \approx 3 \times 10^6$, which gives an estimate of the number of dipoles in a segment area of the surface. Taking each molecule to have diameter 3×10^{-8} cm the segment length is $\approx 10^{-1}$ cm. Thus the number is roughly the number of lipid dipole molecules per unit area in units of microns (Tansey,2019).

Next we show that by using the Casimir Poldar potential and adding in the biological fact that there is a complex fluid inside the surface network, it follows that multi-soliton voltage pulses $A(x,t)$ must be accompanied by surface distortion waves $\Delta(x,t)$ and a formula relating the two is established.

A Formula for the Surface Distortion Wave

The formula follows from two inputs: a mathematical link between geometric distortions of our surface tube and voltage changes and the appropriateness

of using low Reynold number physics for describing the transverse motions of the tube. The first input has already been established by the Casimir Poldar potential. The second input can be easily justified. The Reynold number R of a fluid is given by $R = \frac{\rho v d}{\eta}$, where v is the transverse fluid velocity, d is the diameter of the tube, ρ the fluid density and η the dynamic viscosity coefficient of the fluid. In the transverse direction $d \approx 10^{-4}$cm and v is small so that R is small. In a low Reynolds number environment there are no time delays and with no inertial effects (Purcell,1977) so that a voltage change directly translates to a deformations in the transverse direction to the propagating voltage pulse (Fig 7).

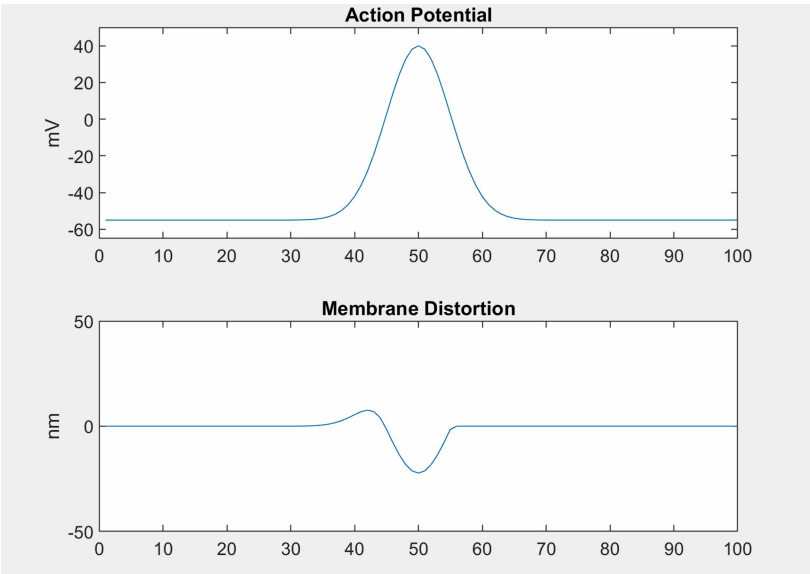

Fig. 7 Predicted Distortion Wave for Rat's Hippocampus Axon radius

Thus the membrane distortion, $r \to r_E + \Delta(x,t)$ where r_E is the equilibrium axon diameter, is proportional to the soliton pulse potential, and we have,

$$\Delta(x,t) = \Delta_T \frac{A(x,t)}{V_T}$$

The proportionality factor is taken to be $\Delta_T \approx r_E \frac{V_E - V_T}{7V_E}$, where V_E is the resting potential, the membrane distortion when $A(x,t) = V_T$, where V_T is the value of the threshold potential, required for a soliton pulse $A(x,t)$ generation. We can only follow the evolution of $\Delta(x,t)$ when it is small compared to r_E. This condition holds for the cycle of values for $A(x,t)$ that starts and ends at $A(x,t) = V_T$.

If we use our theoretical formula to a biological system we see that the predictions made do give the right range for the distortion wave amplitudes and their speeds. We use $r_E \approx 10^{-4}$ cm, $V_T \approx -65, V_E \approx -80$ mV and the range $-80 < A(x,t) < +20$ mV for the rat's hippocampus axon radius (El Hady,2019) and action potential, our formula predicts that the distortion wave will have $\Delta_T \approx 36, \Delta_M \approx -12$nm. The associated distortion wave is plotted in Fig 7. In the plot we have identified our soliton pulse voltage with the observed action potential pulse voltage recorded. The picture of the distortion wave plotted (Fig 7), uses the parameter values given in El Hady (Table 2). The threshold potential value of -65 mV is an estimate.

Table 2 Distortion Wave Parameters.

Resting Voltage	**Threshold Voltage**	**Reference**
-80	-65	26

A number of different approaches for studying membrane distortion waves have been proposed including electrostriction (El Hady,2019) converse flexoelectricity (Chen, 2019; Petrov,2002) or solitary waves (Anderson,2009; Mussel,2019).

Conclusions

The key result of this paper is that it is possible to construct a system, in our case a two dimensional network, whose form decides its function. This was done by introducing a dynamical principle that required that all input deformations and the response of the surface to them must respect the mathematical form of the surface introduced. All our results follow from this principle. We highlight four significant results established

The first is, we proved (Munkres,2014) that there exist smooth surface networks, a Riemann surfaces, that can exactly capture the topological connectivity properties of any hypothetical brain connectome. Thus the connectivity topological form of any brain, its form, can be represented by a mathematical surface, a Riemann surfaces of unknown but large genus g, where a Riemann surface is a clever geometrical representation, discovered by Riemann, of any polynomial equations in two complex variables (Teleman,2003).

The second result is that all input signals can be represented by local pinch deformations that reduce the topology of a subunit to that of a sphere and in the process generate a wide variety of excitations, including one dimensional soliton pulses and local transient excitations. All of these excitations were solutions of one dimensional nonlinear Schroedinger equation. For this result

to hold it was essential that the surface had electrons. The dynamical principle that requires the mathematical form of surface fixed the structure of allowed local pinch deformations input signals but allowed them to have variable, signal specific parameters. The principle also required the response of the network to these input signals, as discussed, be soltions of the nonlinear Schroedinger equation.

The third result is that soliton pulses generated carry away with them the topology of the subunit that produce them as well as the signal specific pinch deformation parameters. There are two topological numbers that describe the topological information carried away by the solitons. The first comes from the networks connectivity of a subunit, its genus k. This number is the number of solitons produced by the subunit. The second topological number comes from the fact that surface electrons are magnetic and these magnets can link together to form different topological structures which can be described by an integer W, defined in terms of the characteristics of the theta function as $W = \sum_{i=1}^{k} \alpha_i \beta_i$ where $(\alpha_i, \beta_i), i = 1, 2, ..k$ for a subunit of genus k pinched to produce a signal. The solitons carry this number as a phase $e^{i\pi W}$ (Appendix B, Eq1). The topological number W reflects the pathways involved in the creation of the signal and the characteristics used to define W describe the presence of electron spin.

The fourth result is a suggestion of a method to store information in the pathways traversed by soliton pulses. Since the soliton pulses carry charge they produce magnetic fields. This transient magnetic field of moving solitons

acts on the spin-magnet of electrons to produce a static magnetic copy of the signal specific pinch deformation carried by the solitons along the pathways traversed by them. We showed that these static helical structures have natural excitation frequencies in the range $(1 - 100)$Hz.

Whether memory storage is in spin structures or not will ultimately have to be resolved by experiments. However we note that helical arrays of spin have been found in condensed matter physics (Uchida,2006). In our approach the soliton signals carry information directly. In a widely used approaches (Bialek,1991) it is suggest that information in a train of acction potentials is hidden in the frequency of action potential production and/or in the distribution of a train of action potentials.This hypothesis is consistent with our explicit form of information storage in deformation parameters, namely by noting that the postion x and time variables t of a soliton have the structure: $(x, t) = (A_j x - B_j t + C_j)$ where (A_j, B_j) are directly related pinch deformations while C_j is indirectly related to pinch deformations: it depends on $\Omega_{jj} = \int_{b_j} \omega_j$ and the a_j radius of the b_j loop shrinks to zero in a way determined by the distortion parmaters. The parameter A_j determines the rate of pulse production, while C_j gives the gap between pulses.

The mathematical scheme described produces non dissipative signals and is thus very different from dissipative models of signal generation such as the Hodgkin-Huxley model (Scott,2002). It also requires the network to have a number of loops.

It is well known that one dimensional soliton and multi-soliton solutions can be obtained in the real line by a variety of ways Newell(Newell, 1985) but finding them for a one dimensional network faces a number of conceptual problems. For instance the branching details of the network (Yusupov,2019)need to be specified and the boundary conditions for solitons at these junction points need to be fixed resulting in a lack of general network independent results for such systems. In these approaches the choice of the nonlinear system is usually imposed on the network. It is not a natural required feature of the network, unlike the topological surface network where the nonlinear Schroedinger equation emerges from a dynamical principle. This is a surprising result that is only valid for a very special Riemann surface network. Natural soliton signals appropriate for axon tubes exist have been found and investigated (Heimburg,2005; (Shrivastava,2014). The motivation was to see if it was possible to have non dissipative ways of transmitting brain signals. In these approaches the signals were sound like elastic wave solitons that were natural for an axon membrane as they were generated by pinch deformations using the special thermodynamic and elastic properties of fluid of the axon biomembrane. Later it was shown that such signals also induce voltage pulse action potential like signals (Mussel,2019). But these soliton signals do not carry information and do not require loop topology about the pinch deformations that create them, the nature of pinch deformations required to generate solitons is not clear and extending these solutions to a network or finding producing multi-solitons is problematic. The speed of the solitons are not theoretically determined.

Let us briefly comment on the existing mechanism suggested for storing memory. One review (Queenan,2017) discusses the difficulty of understanding long term memory storage in the brain by examining different mechanistic hypothesis that have been proposed, with various degrees of supportive evidence. The most widely accepted suggestion is that memory is stored by changing the magnitude of synaptic weights (Kandel,2019; Ortegao de San Luis, 2022). However, recent studies indicate that this form of plasticity may be important for memory encoding and retrieval, but is dispensable for long term information storage itself (Tonegawa,2015). Another hypothesis posits that memory is stored in an intracellular level by chemical means (Glanzman,2018; Gershmann,2021). However this view faces an immediate problem, as most mammalian proteins exist for a few days while synaptic proteins likely to be relevant, can manage to live maybe up to a week (Tsien,2013). There are few plausible candidate proteins currently available that can play such a role, and there is no clear mechanism for quickly storing information in such long lived objects such as polynucleotide chains (but see Akhlaghpour, 2022). A viable system exists that plasticity of synaptic wiring between cells may alter the connectome to store a stable memory as stable microanatomical engrams (Ryan, 2021). Though plausible this hypothesis provides only a substrate for engram formation but no clear insight into information coding. In view of this impasse, alternative unconventional but testable suggestions such as the one suggested in this paper may be appropriate.

Summarizing one can say that all the results of the surface network come from its topology, smoothness and its spin structure using the dynamical principle that required all surface deformations and the networks response to them respect the mathematical form of the network surface. This principle relates the form of the system to its function and establishes a direct mathematical link between the trinity: signals-information-memory. Information was coded in the static arrangement of spin-magnets structure is, as explained. The process of signal generation reduces the topology of a subunit to that of a transient topological sphere in excited electrical states. This is a prediction of the scheme that should have observable consequences which we plan to investigate.

Declarations

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2. The authors declare no competing financial interests.
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4. Code supplied by C.Klein and J.Frauendiener.
5. Contribution of Authors: Siddhartha Sen developed the model and wrote the paper, David Muldowney carried out the numerical calculations, to produce the soliton and distortion wave figures, Tomas Ryan and Maurizio Pezzoli contributed to the writing and suggested key neuroscience ideas. Maurizio Pezzolli produced all the remaining figures and Tomas Ryan edited the text.

Supplementary Material

Appendix A: The Degenerate Fay Trisecant Identity

The soliton solutions were found by exploiting a link between the Riemann surface and the Riemann theta function provided by the Fay identity. In the logic chart these links are shown. In it the Fay trisecant identity is also displayed

using the notation of (Raina,1998). Here we write down the pinch degenerate limit of the Fay identity due to (Kalla,2012):

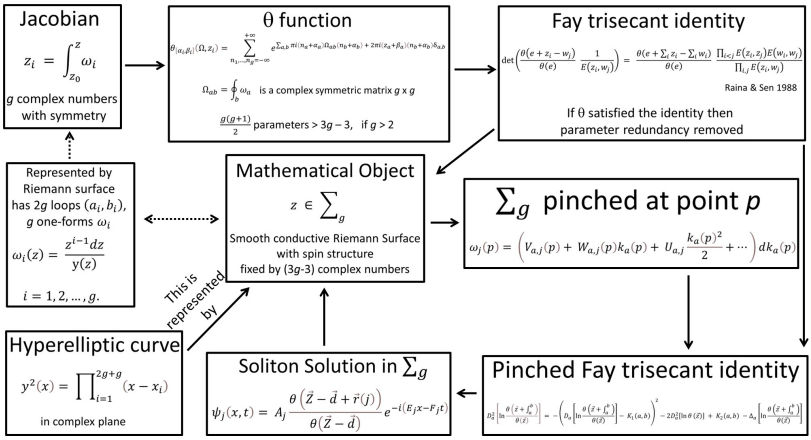

Fig. 8 Chart of Logical links.

We define the functions we need to write down the degenerate form of the Fay identity. Let a, b be distinct points on Σ_g. Fix local parameters k_a, k_b in a neighborhood of these points. Let $\delta = (\alpha_i, \beta_i), i = 1, 2, ..g$ be a non-singular odd characteristic. Then for any $\vec{z} = (z_1, ..z_g)$, Fay's identity is $A = C + D + E$ where

$$
\begin{aligned}
A &= D_a^2[\ln\frac{\theta(\vec{z}+\int_a^b)}{\theta(\vec{z})}] \\
C &= -(D_a[\ln\frac{\theta(\vec{z}+\int_a^b)}{\theta(\vec{z})}] - K_1(a,b))^2 \\
D &= -2D_a^2[\ln\theta(\vec{z})] + K_2(a,b) \\
E &= -\Delta_a[\ln\frac{\theta(\vec{z}+\int_a^b)}{\theta(\vec{z})}]
\end{aligned}
$$

where $\int_a^b = \int_a^b \vec{\omega}$ and the scalars $K_1(a,b), K_2(a,b)$ are given by

$$K_1(a,b) = \frac{\Delta_a \theta[\delta](0)}{2D_a\theta[\delta](0)} + D_a \ln\theta[\delta](\int_a^b)$$
$$K_2(a,b) = -\Delta_a \ln\theta(\int_a^b) - D_a^2 \ln(\theta(\int_a^b)\theta(0)) - (D_a \ln\theta(\int_a^b) - K_1(a,b))^2$$

where D_a is the operator of directional derivate along the vector $\vec{V} = (V_{a,1}, V_{a,2}..., V_{a,g})$ while Δ_a is the operator of directional derivative along the vector $\vec{W} = (W_{a,1}, W_{a,2}..., W_{a,g})$ where these vectors describe the pinch distortion at a point of Σ_g. Thus Fays identity is written down for a specific pinch deformation.

Next we recall that the parameters of the pinch deformation at a point p of Σ_g are given by the vectors $(\vec{V}, \vec{W})$, as follows,

$$\omega_j(p) = (V_{a,j} + W_{a,j}k_a(p) + U_{a,j}\frac{k_a(p)^2}{2} + ..)dk_a(p)$$
$$D_aF(\vec{z}) = \sum_{j=1}^{g} \partial_{z_j} F(\vec{z}) V_{a,j}$$
$$\Delta_a F(\vec{z}) = \sum_{j=1}^{g} \partial_{z_j} F(\vec{z}) W_{a,j}$$

where the set of points $a_j, j = 1, 2, ..(g+1)$ represent pinch locations. They are special points of Σ_g related to the location of the zeros of the hyperelliptic equation. The representation of the one form $\omega_j(p) = V_{a,j} + W_{a,j}k_a(p) + ...)dk_a(p)$ parametrize the signal specific deformations while the range of the label j captures the number of pinch points that contribute to the soliton excitation. Using this notation multiple component (spike trains) solutions

of the non-linear Schroedinger equation, found by Kalla (Kalla,2012) can be written down. We are interested in the soliton limit of this solution.

Appendix B: Soliton Limit

Soliton solutions emerge in the limit when a genus g surface degenerates to one of genus zero. The technical way of doing this is discussed in Kalla (Kalla,2012). As a genus zero surface is conformally equivalent to Riemann sphere there is a map $w(p) = w$ which maps a point p on the genus zero surface to a point w on the Riemann sphere. In the genus zero limit we have the following result,

$$\Omega_{kk} = \ln \epsilon + O(1)$$

$$\Omega_{ik} = \ln[\frac{(w_i - w_k)(s_i - s_k)}{w_i - s_k)(s_i - w_k)}]$$

The diagonal period matrix elements expression from $\Omega_{kk} = \int_{u_i}^{v_i} \Omega_{v_k - u_k} + O(\epsilon)$ where $\Omega_{v_k - u_k} = (\frac{1}{w - w_{v_i}} - \frac{1}{w - w_{u_i}}$ and u_i, v_i are poles that appear when double points that result when the genus g surface degenerates to one of genus zero are desigularized. The meromorphic differentials that then appear on the topological sphere are entirely defined by their behavior near their singularities. In this pinching limit meromorphic functions also appear associated with the variables z_i regarded as functions of a point z of the Riemann surface. The reason for the appearance of singularities can be easiliy understood. A pinch deformation corresponds to the shrinking of a_i loops of a Riemann surface. These loops correspond to circuits round neighboring zeros of the hyperelliptic equation $y^2 = \Pi_{i=1}^{(2g+2)}(z - x_i)$ that define branch points of the equation in the complex plane. Thus a pinch deformations correspond to two zeros coming $(x_i, x_{i+1}$

together. When this happens the associated one forms $\omega_j(z) = \frac{z^j dz}{\sqrt{P(z)}}$ become singular and so do the function $z_i = \int_0^z \omega_j(z)dz$ since $P(z)$ now becomes $P(z) \to \Pi_{i=1}^{(1=g}(z - w_i)$. One technical point is that the expression for $\omega_i(z)dz$ written down in terms of the hyperelliptic function needs to be normalized before it can be used in the theta function.

The general structure of such a meromorphic function $f(w)$ is, now,

$$f(w) = \alpha \Pi_{i=1}^{(n+1)} \frac{w - w_{a_i}}{w - w_{b_i}}$$

with a local parameter $k(w_a) = f(w) - f(w_a)$, where we also require that $f(w_{u_k}) = f(w_{v_k}$. This requiement comes from a global constraint that the distortion parameters have to satisfy. With the help of these variables we can explictly identify the memory parameters. They are the distortion parameters and the location of poles of one forms and the location of poles and zeros of the meromorphic functions. These parameters thus describe the nature of distortions and they also carry information about the network by carrying information about the location of pinch points and the deformed period matrix.

For our case $w_a = w_{a_{n+1}}, w_b = w_{a_j}$. We have

$$\begin{aligned}
V_{a,k} &= \frac{1}{k_a'(w_a)}[\frac{1}{w_a - w_b} - \frac{1}{w_a - w_{u_k}}] \\
W_{a,k} &= A_{a,k} + B_{a,k} \\
A_{a,k} &= \frac{1}{k_a'(w_a)^2}[-\frac{1}{(w_a - w_{v_k})^2 + \frac{1}{(w_a - w_{u_k}})^2} \\
B_{a,k} &= -\frac{k_a''(w_a)}{k_a'(w_a)^2} V_{a,k}
\end{aligned}$$

where $V_{a,k}, W_{a,k}$ are pinch deformation functions. Then the theta function becomes

$$e^{[\sum_{i,k=1}^{n+1}(i\pi\Omega)_{ik}\alpha_i\alpha_k+\sum_{k=1}^{n+1}2i\pi\alpha_k z_k]} = det(T)$$

$$(T)_{ik} = \delta_{ik} + \frac{w_i - s_i}{w_k - s_k}e^{z_i+z_k}$$

where each characteristic, $\alpha_i, i = 1, 2, ..(n+1)$ can only be either $0, \frac{1}{2}$ and $z_k^j = Z_k - d_k + \beta r_{j,k}, Z_k = iV_{a_{n+1},k}x + iW_{a_{n+1},k}t$ The other variables $r_{j,k} = \ln\frac{[(w_{b_j}-w_{v_k})(w_{a_j}-w_{u_k})]}{w_b-w_{u_k})(w_a-w_{v_k})]}$ and β is a constant that can take values $(-1, 0+1)$ while the parameters d_k are constants.

In terms of these variables the dark soliton train , $\psi_j(x,t), 1 \leq j \leq n$ solution (Kalla,2012) of the non-linear Schroedinger equation can be written as,

$$\psi_j(x,t) = A_j e^{i\pi W}\frac{det(T_{j,1})}{det(T_{j,0})}e^{-i(E_j x - F_j t)} \tag{1}$$

where A_j, E_j, F_j are known constants (Kalla,2012) and $W = \sum_{j=1}^{n}\alpha_j.\beta_j$, the spin structure phase. The soliton variables depend on the parameters,$(w_i - s_i)$ and the variables z_i that explicitly depend on the deformation parameters $V_{a,k}, W_{a,k}$, while the set of non zero characteristics carry information about the network responsible for generating the signal. The deformed period matrix also appears in the solutions.

We illustrated the approach by numerically evaluating four multi-solitons. In all the cases we use the fact that the oscillating factor is a coordinate artifact, as we showed and plot the absolute value of the non-oscillating part. We plan to

Table 3 g=3 Dark Soliton Parameters

Synapse	Code	Network	T or E
Synapse(1)	$(0, -2.5873, 0, 11.5975)$	$(1, 1, 1)$	T
Synapse(2)	$(0, -3.09581, 0, 14.706)$	$(1, 1, 1)$	T
Synapse(3)	$(0, -3.2384, 0, 15.6638)$	$(1, 1, 1)$	T

present solutions without oscillation terms in a future work. Here we find first dark solutions for $g = 2, g = 3$ and then invert it to get a bright multi-solitons (Figure 7). The soliton nature of both solutions is shown numerically.

Dark solitons are special high energy excitations that can come from a small cluster of pinches in a background of high excitation. Their shape is thus an elevated potential hump on top of which there are oscillations.

In the numerical evaluations of the two and three spike multi-soliton solution for dark and bright multi-solitons are considered and it is found that not all input signal parameters lead to excitations. This is an intriguing result. The solution's pinch deformation parameters fix the rate at which the soliton spikes appear as well as the gaps between them through the parameters $\vec{Z}$ and $\vec{r_j}$ of the solutions while the parameters $T_{j,\beta}$ fixes the spike shapes. We write $V = (a_j, b_j) = a_j + ib_j, W = (c, d) = c + id, i = +\sqrt{-1}, j$ and $(V_j, W_j) = (a_j, b_j, c_j, d_j), j = 1, 2, 3$ The pinching loops, for the, $g = 3$, dark solitons, are real points $(a_1 = -3.999, a_2 = -3.95, a_3 = -3.92, a_4 = -3.91)$, they are the zeros of the hyperelliptic equation. The Riemann surface is constructed using this information and choosing branch point locations in the complex plane located to be: $(-4, -3, -2, -2 + \epsilon, 0, 0 + \epsilon, 2, 2 + \epsilon), \epsilon = 10^{-10}$. The input data is displayed in tabular form in Table 3. For the $g = 2$ case the branch points are $(-2 - 1, 0 + \epsilon, 2 + \epsilon)$, the pinching loops are

$(a_1 = -1.9, a_2 = -1.11, a_3 = -1.8)$ and input distortion data is displayed below,

Table 4 g=2 Dark Soliton Parameters

Synapse	Code	Network	T or E
Synapse(1)	$(0, -1.9142, 0, 2.5916)$	$(1, 1)$	T
Synapse(2)	$(0, -2.2790, 0, 3.6734)$	$(1, 1)$	T

Appendix C: The Jacobi Inversion Theorem

The equivalence of Σ_g and $J(\Sigma_g)$ is at the level of certain abelian groups called divisors. For Σ_g a divisor D is the formal sum of selected points on Σ_g mutiplied by integers. These groups establish a relationship between the number of zeros of functions on Σ_g to the corresponding number of zeros of the Riemann theta function on $J(\Sigma_g)$. We have,

$$D = \sum_p n_p p$$

The degree of a divisor is defined as $\deg D = \sum_p n_p$. For a meromorphic function $f : \Sigma_g \to C$ a natural notion of a divisor (f) can be introduced. We write

$$(f) = \sum_p (\mathrm{ord}_p(f))_p$$

This is an enrichment of Σ_g as further structures namely special types of functions are introduced on it. A meromorphic function has simple zeros and poles. The order at a simple pole is minus one while at simple zero it is plus one. A general result is that the sum of the orders of zero and poles of a meromorphic function on Σ_g is zero. This translates to the statement that $\deg(f) = 0$. We next introduce the Abel-Jacobi map that maps points p on

Σ_g to points on $J(\Sigma_g)$. Remember a point in $J(\Sigma_g)$ is defined by g complex numbers. We have,

$$\mu : \Sigma_g \rightarrow J(\Sigma_g)$$
$$\mu = (\int_{p_0}^{p} \omega_i, i = 1, 2, ..g)$$

The point p_0 is a fixed reference point on Σ_g with respect to which other points are linked by a path. It is possible to show that the linking path chosen does not matter. The map is well defined.

This map extends to divisors. For points p in Σ_g we have,

$$\mu(\sum_p n_p p) = \sum_p \mu(p)$$

We can now state the inversion Jacobi theorem.

Jacobi Inversion Theorem

For every point in $J(\Sigma_g)$ is the image under μ of a degree zero divisor of the form $D = \sum_{i=1}^{g}(p_i - p_0)$ on Σ_g

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