Emergent complexity in slowly driven stochastic processes

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We consider the distribution of first passage time events in the presence of non-ergodic modes that drive otherwise ergodic dynamics on a potential landscape. We find that in the limit of slow and large enough fluctuations the distribution of first passage time events, $f(t)$, exhibits heavy tails dominated by a power law with exponent $f(t) \sim t^{-2}$, and corrections that depend on the strength and the nature of fluctuations. We support our theoretical findings through direct numerical simulations in illustrative examples.

Introduction.—Complex dynamics are ubiquitous in the natural world. Despite their intrinsic irregularity and unpredictability, they can nonetheless exhibit coherent and universal emergent properties. Of particular importance in the study of complex systems is the understanding of the time it takes for rare events to occur [1–3]. Notable examples include natural disasters [4] or the spreading of a virus [5]. In fact, first passage times are central to many fields within physics and beyond, with important examples stemming from chemistry, biology and finance (see, e.g., [6–12] and references therein). Biology in particular is ripe with examples where time is of the essence [13], such as fertilization [14], intracellular events [15–21], search processes [22–24], neural activity [25, 26] or population dynamics [27].

We here consider the estimation of first passage time distributions (FPTDs) from finite-time observations in an experimental context. In particular, we are interested in systems with intrinsic time scales comparable to the observation time, for which weak ergodicity breaking becomes evident [28, 29]. Such dynamics can be found for instance in glassy systems [30–33], where the time scales of equilibration are so long that one can decompose the dynamics into a stationary component and an "aging" component that breaks time-translation invariance.

Our main inspiration comes from the less traditional branch of the physics of animal behavior [34, 35]. Remarkably, recent advances in machine vision (see, e.g., [36–39]) have resulted in an explosion of high spatio-temporal resolution behavioral data. Analysis of fine-scale posture movements shows that much like the run-and-tumble behavior of bacteria [40], more complex organisms also exhibit stereotyped behaviors, albeit with a more intricate structure [41–48]. The notion of stereotypy in behavior inherently stems from the time scale separation between variations on what is defined as a behavioral state, and the transitions between behavioral states, much like a particle hopping between wells in a potential landscape. For example, while foraging for food the nematode worm *C. elegans* transitions between coarse-grained "runs" and "pirouettes", which are stereotyped sequences of finer scale movements [48, 49]. However, unlike the particle hopping among potential wells which has a characteristic exponential distribution of transition times, the time spent in a given behavior can be heavy-tailed (see, e.g. Fig. 4E of [48] or Fig. 3 of [50]). We here hypothesize that such heavy-tailed distributions reflect the slow continuous modulation of behavior on longer time scales, resulting from environmental factors or fluctuating internal states driven by neuromodulation, such as hunger, stress or arousal (see, e.g., [51–53]). Indeed, it has been shown that *C. elegans* continuously modulates its rate of reorientation events to explore larger and larger arenas in search of food [54]. In order to truly capture the multiscale nature of behavior, we therefore need to account for the fact that it can be modulated on time scales comparable to the observation time.

In this Letter, we introduce a general model of behavior in which the pose dynamics evolves in potential landscapes that fluctuate over time. We then study how these dynamics impact the estimation of the distribution of times spent in a given behavior. In the first section, we introduce our phenomenological description of the behavioral dynamics, decomposing it into ergodic dynamics on a potential landscape and the non-ergodic modulation of the landscape. We then derive a general result for the distribution of first passage times, and illustrate it through direct numerical simulations in three example systems.

Slowly driven ergodic dynamics.—Given a set of observations of animal locomotion (e.g. from video imaging), we consider that the dynamics can be decomposed into ergodic and non-ergodic components. The former are the state-space variables that mix sufficiently well and define the potential wells that correspond to the stereotyped behaviors; the latter non-ergodic components evolve on time scales comparable to the observation time and slowly modulate the potential landscape. The full dynamics is thus given by

$$\begin{cases} \dot{\vec{X}} = F(\vec{X}, \vec{\lambda}) \\ \tau_\lambda \dot{\vec{\lambda}} = G(\vec{X}, \vec{\lambda}) \end{cases} , \tag{1}$$

where $\vec{X} \in \mathbb{R}^D$ represents the ergodic components, $\vec{\lambda} \in \mathbb{R}^{D_\lambda}$ represents the non-ergodic degrees of freedoms, F and G are nonlinear, possibly noisy, functions, and τ_λ

is assumed to be of the order of the measurement time $T_{\rm exp}$, $\tau_\lambda = \mathcal{O}(T_{\rm exp})$, such that the $\vec{\lambda}$ dynamics do not mix. Given the time scale separation between the dynamics of $\vec{X}$ and $\vec{\lambda}$, we assume that the dynamics of $\vec{X}$ is well approximated by quasi-stationary Fokker-Planck dynamics $\dot{\rho} = \mathcal{L}\rho$, where $\mathcal{L}$ represents the Fokker-Planck operator. Since we are primarily interested in the long time scale behavior of the system, we consider a projection of the dynamics onto the slowest mode of $\mathcal{L}$, yielding a generalized Langevin equation [55, 56] with history-dependent friction and fluctuations. Assuming that we can sample the system on a time scale longer than the noise correlation time, we obtain an effective overdamped description :

$$\dot{\vec{X}} = F(\vec{X}, \vec{\lambda}) \Rightarrow \dot{x} = -\partial_x U(x, \lambda) + \sqrt{2T_x}\eta_x(t)\,, \quad (2)$$

where T_x captures the effective temperature, η_x is Gaussian white noise, and λ is a slow control parameter that modulates the effective potential landscape on slow time scales. Similarly, we consider that λ also obeys an effective overdamped Langevin equation,

$$\dot{\lambda} = -\tau_\lambda^{-1}\partial_\lambda V(\lambda) + \sqrt{2T_\lambda \tau_\lambda^{-1}}\eta_\lambda(t), \quad (3)$$

where V is assumed to be uncoupled from the dynamics of x for simplicity, T_λ captures the degree of fluctuations in λ and η_λ is Gaussian white noise.

First passage time distributions.—We are primarily interested in studying the time spent in a given behavioral state. Within the context of the Langevin dynamics of Eq. 2, this is given by the first passage time to reach an energy barrier x_f from the bottom of the potential x_0, defined as,

$$\tau_{x_0,x_f}(\lambda) = \inf\left\{\tau : x(t+\tau, \lambda) = x_f | x(t, \lambda) = x_0\right\}\,. \quad (4)$$

Despite the general interest in this concept, finding analytical expressions for the density of first passage time events is generally a formidable task [57]. Remarkably few closed-form expressions for the FPTD are known, with most results concerning only the mean first passage time (MFPT) which is more tractable (see, e.g., [1, 6, 9]). However, the MFPT provides only limited information, especially when multiple time scales are involved [15]. Here, we are interested in studying the behavior of the full first passage time distribution, with particular focus on its long time behavior in the presence of weakly non-ergodic dynamics, Eqs. 2 and 3.

As previously discussed, the measurement time $T_{\rm exp}$ essentially separates ergodic from non-ergodic dynamics. In addition, it also sets a lower bound on the slowest observed hopping rates $\omega_{\min} \sim T_{\rm exp}^{-1}$, such that when $\tau_\lambda = \mathcal{O}(T_{\rm exp})$ we can make an adiabatic approximation and assume that transition events occur within a nearly static potential. For a given hopping rate ω, the first passage time distribution is given by

$$f(t, \omega) = \omega e^{-\omega t}\,,$$

where $\omega(\lambda) = 1/\tau_{x_0,x_f}(\lambda)$ is the dominating slow kinetic transition rate which implicitly depends on the dynamics of λ. When we allow λ to fluctuate slowly, the distribution of first passage times $f(t)$ is given by the expectation value of $f(t, \omega)$ over the distribution of ω, $p(\omega)$, weighted by the effective number of transition observed within $T_{\rm exp}$, which is proportional to ω. Marginalizing over ω we get

$$f(t) \sim \int_{\omega_{\min}}^{\omega_{\max}} p(\omega)\omega^2 e^{-\omega t} d\omega. \quad (5)$$

While the barrier height is going to depend on the dynamics of a slow control parameter λ, the tail of the distribution is going to be dominated by instances in which the barrier height is the largest, motivating the use of Kramers approximation (see, e.g., [2]),

$$\omega(\lambda) = \omega_0 \exp\left\{-\frac{\Delta U(\lambda)}{T_x}\right\}, \quad (6)$$

where $\Delta U(\lambda) = U(x_f, \lambda) - U(x_0, \lambda)$ and ω_0 is a constant. For multiple realization of Eq. 3 with different initial conditions, the distribution of λ is given by the Boltzmann weight [58],

$$p(\lambda) \sim \exp\left\{-\frac{V(\lambda)}{2T_\lambda}\right\}. \quad (7)$$

Leveraging Eqs. S1,6,7 we can obtain an asymptotic approximation of the FPTD in the large t limit (see Supplemental Material),

$$f(t) \sim t^{-2} \exp\left\{-\frac{V(\Delta U^{-1}(T_x \log(\omega_0 t)))}{2T_\lambda}\right\}, \quad (8)$$

where $\Delta U^{-1}(\cdot)$ represents the inverse function of the potential difference defined by Eq. 6 and we have kept only the dominant order of the asymptotic approximation (see Supplemental Material). For very general conditions on $V(\lambda)$ and $U(x, \lambda)$, we thus get $f(t) \sim t^{-2}$ for $t \to \infty$ and $T_\lambda \gg 1$. In the following section we will demonstrate the validity of this result in three illustrative examples.

Illustrative examples: Slowly-driven harmonic oscillator.—Consider that x evolves in a harmonic potential, $U(x, s) = (x - sx_f)^2$, that is driven by a slow parameter s that fluctuates within $V(s) = s^2/2$, pushing $U(x, s)$ closer or further from x_f in a time scale τ_s, Fig. 1(a). The equations of motion are given by a set of Ito stochastic

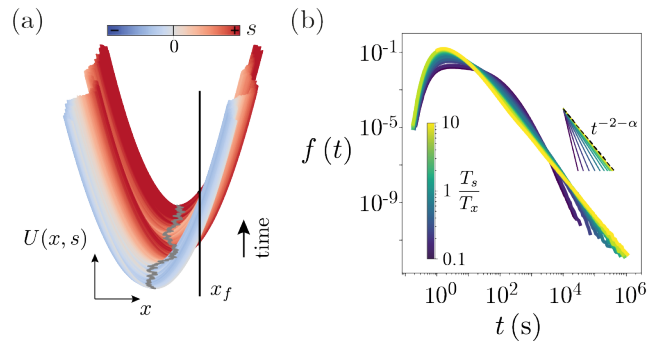

FIG. 1. Heavy-tailed first passage time distributions for a slowly-driven overdamped harmonic oscillator. (a) We simulate the dynamics of a particle in a harmonic oscillator while slowly driving the potential landscape, and estimate the distribution of times it takes to reach x_f. The gray line represents the minimum of potential, $x_0 = s$, and the color scheme different values of s. (b) FPTDs obtained from direct numerical simulations of Eq. 9 for different values of the temperature T_s that controls the level of fluctuations for the parameter driving the slow variations of the potential landscape. As predicted, the tail of the distribution behaves as a power law with an exponent $f(t) \sim t^{-2-\alpha}$, with $\alpha = \frac{T_x}{2T_s}$. The color scheme represents different ratios of temperatures, and the black dashed line the $T_s \to \infty$ limit.

differential equation, corresponding to coupled Ornstein-Uhlenbeck processes,

$$\begin{cases} dx_t = -(x_t - s_t x_f)dt + \sqrt{2T_x}dW_t \\ ds_t = -\tau_s^{-1}s_t + \sqrt{2T_s\tau_s^{-1}}dW_t \end{cases}, \tag{9}$$

where T_x and T_s captures the degree of fluctuations, dW_t is a Wiener Gaussian white noise process. We are interested in the density of first passage time events from the minimum of the potential $x_0 = s$ to $x_f = 1$, for which it is challenging to find a closed form analytical expression, even when $s(t) = s \in \mathbb{R}$ [57]. In the Supplemental Material, we derive the FPTD in Laplace space [59] and leverage it to estimate the FPTD through numerical inversion [60] for varying values of τ_s (as in Ref. [61]), see Fig. S2. We find that when s fluctuates fast enough, $\tau_s \to 0$, we can average out s and get the simpler dynamics $dx_t = -(x_t - \langle s\rangle x_f)\,dt + \sqrt{2T_x}dW_t$. In this case, the FPTD is well approximated by $f(t) \approx f(t, \langle\omega\rangle) = \langle\omega\rangle e^{-\langle\omega\rangle t}$, where $\langle\omega\rangle$ is the average hopping rate which is set by $\langle s\rangle$. Even when $\tau_s > 0$ but short, it is possible to obtain a self-consistent Markovian dynamics for $x(t)$ (see e.g., [1]). In this case, the distribution of first passage times is still dominantly exponential, but with a corrected first passage time which depends on the ratio of temperatures T_s/T_x and the slow time scale τ_s. However, as we have shown in the previous section τ_s is large enough, $\tau_s \sim T_{\text{exp}}$, the distribution of first passage times becomes heavy-tailed. In this limit, we can leverage Eq. 8 to derive an asymptotic approximation to

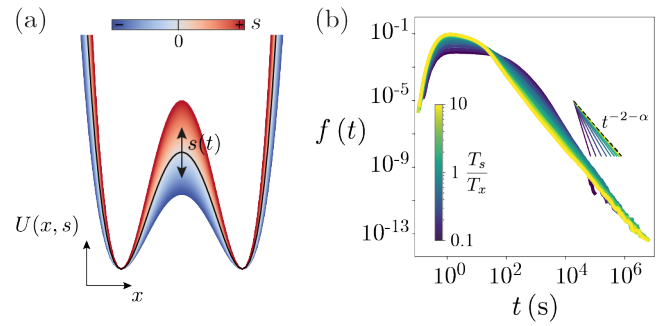

FIG. 2. Heavy-tailed first passage time distribution of a slowly-driven double-well potential. (a) Schematic of the variation in the double-well potential with s (colored from blue to red; the black line represents $s = \mu_s$). (b) FPTDs from direct numerical simulations of Eq. 11 for different values of T_s. As expected, the tail of the distribution behaves as a power law $f(t) \sim t^{-2-\alpha}$, where $\alpha = \frac{T_x}{4T_s}$ (colored line). The black dashed line represents the $T_s \to \infty$ limit.

the distribution of first passage times. The tail of the distribution will be dominated by low ω values, which correspond to $|s| >> 1$. In this limit, the barrier height primarily behaves as $\Delta U(s) = s^2/2 + \mathcal{O}(s)$. In addition, since $V(s) = s^2/2$, we see that $V(\Delta U^{-1}(x)) = x$ and Eq. 8 yields (see Supplemental Material),

$$f(t) \sim t^{-2-\frac{T_x}{2T_s}}, \tag{10}$$

which matches what we obtain from direct numerical simulations of Eq. 9, Figs. 1(b),S2,S3(a).

Illustrative examples: Slowly-driven double-well potential.—We now consider a symmetric double-well potential in which the barrier height is slowly modulated according to an Ornstein-Uhlenbeck process, Fig. 2(a),

$$\begin{cases} dx_t = -4s_t^2 x_t(x_t - 1)^2 dt + \sqrt{2T_x}dW_t \\ ds_t = -\tau_s^{-1}(s_t - \mu_s)dt + \sqrt{2T_s\tau_s^{-1}}dW_t \end{cases}, \tag{11}$$

where all the parameters are the same as in Eq. 9 with an extra μ_s that represents the expectation value of s, which we set as $\mu_s = 1$. In this case, we have a quartic potential for x, $U(x, s) = s^2(x^2 - 1)^2$, which yields $\Delta U(s) = s^2$. Since $V(s) = s^2/2$, we see that $V(\Delta U^{-1}(x)) = x/2$ and Eq. 8 yields (see Supplemental Material),

$$f(t) \sim t^{-2-\frac{T_x}{4T_s}}, \tag{12}$$

matching what we find through direct numerical simulations of Eq. 11, Figs. 2(b),S3(b)

Illustrative examples: Slowly-driven rugged parabolic potential.—Finally, we consider a rugged parabolic potential as a simple model of the rough energy landscapes found across complex systems, from glasses to proteins (see, e.g., [19, 20, 62]). We construct a rugged landscape by superimposing a sinusoidal perturbation onto a harmonic potential [63], $U(x, s) = U_0(x, s) + U_1(x)$, where

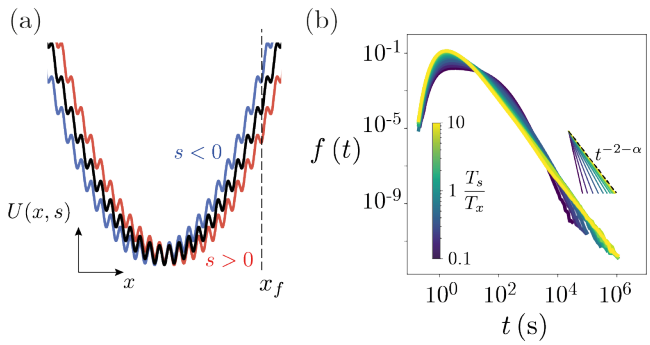

FIG. 3. Heavy-tailed first passage time distribution in slowly driven rugged parabolic potential. (a) We estimate the first passage time to reach x_f from the global minimum of a rugged parabolic potential. (b) FPTDs from direct numerical simulations of Eq. 13 for different values of T_s. As expected, the tail of the distribution behaves as a power law $f(t) \sim t^{-2-\alpha}$ (colored lines) with $\alpha = \frac{T_x}{2T_s}$. The black dashed line corresponds to the $T_s \to \infty$ limit.

$U_0(x, s) = (x - s)^2/2$ and $U_1(x) = -\cos(2\pi kx)/(k\pi)$. The corresponding dynamics are given by,

$$\begin{cases} dx_t = -\left(x_t - s_t + 2\sin(2\pi k x_t)\right) dt + \sqrt{2T_x} dW_t \\ ds_t = -\tau_s^{-1} s_t + \sqrt{2T_s \tau_s^{-1}} dW_t \end{cases}, \tag{13}$$

where k sets the number of smaller barriers between the global minimum of the potential and $x_f = 1$. We set $k = 10$ resulting in a rugged potential as illustrated in Fig. 3(a). In this case, since $U(x, s)$ is not as simple as before, it is more challenging to derive the correction terms to the power law. However, it has been shown [63] that by spatial averaging of $U_1(x) = -\cos(2\pi kx)/(k\pi)$ over one period, the resulting hopping rate is simply corrected by a constant prefactor $\omega = I_0^{-2}(k^{-1}\pi^{-1}T_x^{-1})\omega_0$, where I_0 is the modified Bessel function and ω_0 is the hopping rate in the absence of the sinusoidal perturbation (from $U_0(x, s) = (x - s)^2/2$). As such, we expect the asymptotic behavior of $f(t)$ to be the same as for the slowly driven harmonic potential, Eq. 10. Indeed, this is what we observe in Figs. 3(b),S3(c).

Discussion.—Inspired by quantitative analysis of animal behavior, we here examined how the existence of slow non-ergodic modes impacts the statistics collected experimentally, focusing on the distribution of first passage time events. Our results show the emergence of heavy-tailed distributions. In particular, we find that the distribution asymptotes to a power law with an exponent $f(t) \sim t^{-2}$ in the limit of large fluctuations, regardless of the details of the dynamics. As remarked in the Introduction, our results have important implications to a wide variety of fields, and we here discuss some of these in detail.

In the context of animal behavior, heavy-tailed first passage times with an exponent $f(t) \approx t^{-2}$ have been found extensively across multiple species, from bacteria [64], termites [65] and rats [66] to marine animals [67, 68], humans [69] and even fossil records [70]. In the context of search behaviors (e.g., when foraging for food), such observations have led researchers to hypothesize that Lévy-flights (power law distributed run lengths) are efficient search strategies and thus evolutionarily favorable [71–75]. However, we here show that such fat tails may emerge when the animal is continuously adapting its behavior (slowly modulating the potential landscape), even in the absence of external drives. We therefore predict that disrupting the internal mechanisms for slow modulation of behavior (e.g. neuromodulatory pathways) should result in distribution of first passage times that have exponential tails.

Power laws have been observed in a wide variety of systems, from solar flares [76, 77] to the brain [78] and different hypotheses have been put forward to explain their emergence (for a review see e.g. [79]). Among these, work inspired by phase transitions in statistical mechanics associates power laws to "criticality", mostly due to the fact that models inferred from the data appear to require fine-tuning of the model parameters to a special regime between two qualitatively different "phases" (see, e.g., [80]). However, as we have shown here, power laws can emerge without fine tuning and far from "criticality". Indeed, slow modes that evolve on time scales comparable to the observation time are challenging to infer from data, and can give rise to best-fit models that appear "critical". While some of the arguments we have put forward have also been proposed in other contexts [22, 81–83], we here place them into the framework of out-of-equilibrium statistical mechanics, explicitly connecting the long time scale emergent behavior with the underlying effective fluctuations. In addition, unlike other approaches [82, 84], our framework does not require explicit external drives, but simply collective modes that evolve in a weakly non-ergodic fashion.

Our starting point is an effective description of the long time scale dynamics, and further work will be required to fully bridge between microscopic dynamics, and the emergent long time behavior of the first passage time distribution that we uncovered. For example, we find that for intermediate values of $1 \ll \tau_\lambda \ll T_{\text{exp}}$ the FPTD behaves as a truncated power law with an effective exponent that is slightly smaller that -2 (see Supplemental Material), which goes beyond arguments presented here. What are the minimum τ_λ and T_λ for power laws to be measurable, and how do simple exponentials ($\tau_\lambda \ll T_{\text{exp}}$) transition to power law behavior? These are important questions if one hopes to test our predictions in an experimental context (using for example a set-up akin to the ones used to test stochastic resonance [85, 86]). Additionally, we note that when $\tau_\lambda \gg T_{\text{exp}}$, the distribution of initial conditions determines the emergent behav-

ior, see Fig. S4. Inspired by experiments in animal behavior, which are typically done with multiple animals, we here assume that the initial condition for the slow mode is sampled according to its Boltzmann distribution $\lambda(t=0) \sim e^{-\frac{V(\lambda)}{2T_\lambda}}$, reflecting the variability across individuals. In this case, the emergent behavior we have derived holds true from $\tau_\lambda \sim T_{\rm exp}$ to $\tau_\lambda \to \infty$. However, if the variability across experiments is smaller than that of the Boltzmann distribution, the $\tau_\lambda \to \infty$ limit will differ from the behavior at $\tau_\lambda \sim T_{\rm exp}$. Indeed, if the variance of the initial distribution of λ is smaller than that of the Boltzmann distribution, the temperature T_λ in our derivation should be changed to a new effective temperature $T_\lambda^0 < T_\lambda$ reflecting the lower variance of the initial conditions. Making this transformation we still get a power law distribution of first passage times, but with a modified exponent that reflects the lower variance (see Supplemental Material).

To conclude, we have considered the effect of slow non-ergodic modulations and theoretically captured their effects on the distribution of first passage times, a result that we believe is widely relevant to a range of natural systems.

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Supplemental material for: Emergent complexity in slowly driven stochastic processes

General derivation.—We here derive the expression for the first passage time distribution (FPTD) in a fluctuating potential landscape. As discussed in the main text, we consider the adiabatic limit in which the FPTD can be approximated by,

$$f(t) \sim \int_{\omega_{\min}}^{\omega_{\max}} p(\omega)\omega^2 e^{-\omega t} d\omega. \tag{S1}$$

Since $\omega = \omega_0 \exp\{-\Delta U(\lambda)/T_x\}$, we have $\lambda(\omega) = \Delta U^{-1}(-T_x \log(\omega/\omega_0))$, where ω_0 is a typical (fast) frequency of the hopping dynamics [87]. The distribution $p(\omega)$ obeys $p(\omega)d\omega = p(\lambda)d\lambda$, and it is thus given by

$$p(\omega) \sim \exp\left\{-\frac{V(\lambda(\omega))}{2T_\lambda}\right\} \frac{T_x/\omega}{\partial_\lambda \Delta U(\lambda)}.$$

Plugging this into Eq. S1, we get

$$f(t) \propto \int_{\omega_{\min}}^{\omega_{\max}} \exp\left\{-\frac{V(\lambda(\omega))}{2T_\lambda}\right\} \frac{T_x}{\partial_\lambda \Delta U(\lambda)} \omega e^{-\omega t} d\omega. \tag{S2}$$

The exponential factor $e^{-\omega t}$ restricts the contributions to $\omega \sim 1/t$, which motivates the change of variable $\omega = \frac{\theta}{t}$. The above integral is then recast in the form

$$f(t) \propto t^{-2} \int_{\theta_{\min}(t)}^{\theta_{\max}(t)} \frac{\exp\left\{-\theta - \frac{V(\lambda(\theta))}{2T_\lambda} + \log(\theta)\right\}}{\partial_\lambda \Delta U(\lambda(\theta))} d\theta, \tag{S3}$$

where $\lambda(\theta) = \Delta U^{-1}\left(-T_x \log\left(\frac{\theta}{\omega_0 t}\right)\right)$, $\theta_{\min}(t) = \omega_{\min} t$ and $\theta_{\max}(t) = \omega_{\max} t$.

To grasp the structure of the integral, it is convenient to consider first the special case where V and ΔU can be written as a power series expansion $V(\lambda) \sim a\lambda^n$ and $\Delta U(\lambda) \sim b\lambda^n$, $a, b \in \mathbb{R}$ with an equal dominant (at large values of the argument, see below) exponent n. The integral reduces then to the form

$$f(t) \propto t^{-2-\frac{aT_x}{2bT_\lambda}} \int_{\theta_{\min}}^{\theta_{\max}} \frac{\theta^{1+\frac{aT_x}{2bT_\lambda}} e^{-\theta}}{\left(-\log\left(\frac{\theta}{\omega_o t}\right)\right)^{1-\frac{1}{n}}} d\theta.$$

It remains to verify that the time dependencies at the denominator of the integrand and the limits of integration do not spoil the behavior at large times. This is verified by noting that the numerator of the integrand has the structure of an Euler-Γ function of order $2 + \frac{aT_x}{2bT_\lambda}$. The integrand has its maximum at $\theta^* = 1 + \frac{aT_x}{2bT_\lambda}$, decays over a range of values of order unity and vanishes at the origin. In that range, the argument of the power at the denominator $\log(\omega_0 t) - \log(\theta) \simeq \log(\omega_0 t)$, which yields the final scaling with subdominant logarithmic corrections

$$f(t) \sim t^{-2-\frac{aT_x}{2bT_\lambda}} \times \log(\omega_0 t)^{\frac{1}{n}-1}. \tag{S4}$$

To complete the argument, we note that the time dependency of $\theta_{\min}$ is not an issue as long as values $\theta \sim O(1)$ are in the integration range. In practice, this means that the minimum hopping rate $\omega_{\min}$ should be comparable to (or larger than) the measurement time, $\omega_{\min}^{-1} \sim \mathcal{O}(T_{\exp})$.

Before moving to the general case, two remarks are in order. First, for $\omega_0 t \gg 1$ the functions V and $\partial_\lambda \Delta U$ that appear in Eq. S3 have their argument $\lambda \gg 1$. The dominant behavior of the two functions should then be understood for large values of their arguments. Second, the denominator $\partial_\lambda \Delta U$ could *a priori* be included in the exponential at the numerator but this does not modify our conclusion. It is indeed easy to verify that the maximum θ^* and the decay range would not be not shifted at the dominant order (and this holds also for the general case considered hereafter).

We can now consider the general case with different dominant exponents $V(\lambda) \sim a\lambda^n$ and $\Delta U(\lambda) \sim b\lambda^k$, $a, b \in \mathbb{R}$. The argument of the exponential in Eq. S3

$$L(\theta) = -\theta - \frac{V(\lambda(\theta))}{2T_\lambda} + \log(\theta), \tag{S5}$$

has its maximum at θ^*, defined by the implicit equation

$$\theta^* = 1 + \frac{T_x}{2T_\lambda}\frac{\partial_\lambda V(\lambda(\theta^*))}{\partial_\lambda \Delta U(\lambda(\theta^*))} = 1 + \frac{T_x}{2T_\lambda}\frac{an}{bk}\lambda^{n-k}\,,$$

where we have used

$$\partial_\theta V(\lambda) = \partial_\lambda V(\lambda) \times \frac{d\lambda(\theta)}{d\theta}\,; \quad \frac{d\lambda(\theta)}{d\theta} = -\frac{T_x/\theta}{\partial_\lambda \Delta U(\lambda)}\,.$$

For $n < k$, the maximum $\theta^* \simeq 1$ and the integrand decays in a range of order unity. Indeed, the dominant order of the derivatives $\partial^p L$ $(p \geq 2)$ at $\theta = \theta^*$ coincide with those of $\log(\theta)$. It follows that $L(\theta) - L(\theta^*) \simeq \log(\theta/\theta^*) - (\theta - \theta^*)$. The resulting integral over θ is an Euler Γ-function of order two, which indeed forms at values $O(1)$. In that range, $\lambda \sim \left(\frac{T_x}{b}\log(\omega_0 t)\right)^{1/k}$ and the integral is then approximated by $\exp\{L(\theta^*)\}$ and the function $f(t)$ in Eq. S3 by

$$f(t) \sim t^{-2}\exp\left\{-\frac{a\left(\frac{T_x}{b}\log(\omega_0 t)\right)^{n/k}}{2T_\lambda}\right\}.$$

The factor at the denominator in Eq. S3 is $\mathcal{O}\left[\exp\left\{(1/k-1)\log[\log(\omega_0 t)]\right\}\right]$ and thus of the same order as terms that we have discarded in our approximation so we neglect it as well. Since the integral over θ forms for values $O(1)$, the constraint on the minimum hopping rate is the same as for the $n = k$ case, i.e., $\omega_{\min}^{-1} \sim \mathcal{O}(T_{\exp})$.

For $n > k$, the maximum $\theta^* \sim (\log\omega_0 t)^{n/k-1}$, which is now large. The dominant order of the derivatives $\partial^p L$ $(p \geq 2)$ at $\theta = \theta^*$ is given by $(-1)^{p-1}(p-1)!\,(\theta^*)^{-p-1}$, that is they coincide with those of $\theta^*\log(\theta)$. It follows that $L(\theta) - L(\theta^*) \simeq \theta^*\left[\log(\theta/\theta^*) - (\theta - \theta^*)/\theta^*\right]$. The resulting integral over θ is an Euler Γ-function of (large) argument $\theta^* + 1$: its value is approximated by Stirling formula, which yields $\int (\theta/\theta^*)^{\theta^*} e^{-(\theta-\theta^*)}\,d\theta \simeq \sqrt{\theta^*}$. The $\sqrt{\theta^*}$ reflects the fact that the integral forms around the maximum at θ^* of the integrand over a range $\sqrt{\theta^*}$, which implies that the approximation $-\log\left(\frac{\theta}{\omega_0 t}\right) \simeq \log(\omega_o t)$ still holds, as in the previous cases $n \leq k$. The $\sqrt{\theta^*}$, as well as the $\log(\omega_0 t)^{1/k-1}$ coming from the denominator in Eq. S3, are subdominant with respect to terms that we have neglected in the expansion of L. We therefore discard them from our final approximation for $n > k$:

$$f(t) \sim t^{-2}\exp\left\{-\frac{a\left(\frac{T_x}{b}\log(\omega_0 t)\right)^{n/k}}{2T_\lambda}\right\}.$$

Since the integral over θ forms for values $O\left((\log\omega_0 t)^{n/k-1}\right) \gg 1$, the condition $\omega_{\min}^{-1} \sim \mathcal{O}(T_{\exp})$ ensures *a fortiori* that the finite value of $\omega_{\min}$ does not affect the above result.

Discarding subdominant terms, in all three cases we thus get the general expression we present in the main text,

$$f(t) \sim t^{-2}\exp\left\{-\frac{a\left(\frac{T_x}{b}\log(\omega_0 t)\right)^{n/k}}{2T_\lambda}\right\}. \tag{S6}$$

To verify the validity of the above arguments, we show in Fig.S1 how, to dominant order, asymptotic predictions agree with a detailed numerical integration of Eq. S2 for $\Delta U(\lambda) = \lambda^k$ and $V(\lambda) = \lambda^n$.

First-passage time through an absorbing boundary in a harmonic oscillator.—We here derive the first passage distribution for the harmonic oscillator with fixed $s \in \mathbb{R}$, so for

$$dx_t = -(x_t - sx_f)dt + \sqrt{2T_x}dW_t$$

The corresponding Fokker-Planck equation is given by

$$\begin{aligned}\partial_t\rho =& \mathcal{L}\rho\,;\\ \mathcal{L} =& \partial_x((x - sx_f)\bullet) + T_x\partial_x^2(\bullet)\,.\end{aligned}$$

We are interested in the distribution of first passage times from x_0 to the energy barrier located at x_f, derived in the Supplementary Information of [88]. We here reformulate this derivation and tune it to our particular case. The survival probability can be written as

$$S(x_f, t|x_0) = \int_{-\infty}^{x_f} P_{x_f}(x, t|x_0)dx\,,$$

where $P_{x_f}(x,t|x_0)$ is the propagator from $t=0$ to t with the constraint that $x<x_f$. In other words, we have an absorbing boundary condition at x_f, $P(x_f,t|x_0)=0$. From the survival probability, the first passage density can be obtained as

$$f_{x_f}(t|x_0) = -\partial_t S_{x_f}(t|x_0).$$

The backward Kolmogorov equation for the propagator $P_{x_f}(x,t|x_0)$ is $\partial_t P_{x_f}(x,t|x_0) = \mathcal{L}^\dagger_{x_0} P_{x_f}(x,t|x_0)$, where $\mathcal{L}^\dagger_{x_0}$ is the adjoint of the generator of the stochastic process, $\mathcal{L}^\dagger_{x_0} = -(x_0 - sx_f)\partial_{x_0}\bullet + T_x\partial^2_{x_0}\bullet$. Integrating the backward Kolmogorov equation and then taking the time derivative we get

$$\begin{aligned}\partial_t S_{x_f}(t|x_0) &= \mathcal{L}^\dagger_{x_0} S_{x_f}(t|x_0)\\ \partial_t f_{x_f}(t|x_0) &= \mathcal{L}^\dagger_{x_0} f_{x_f}(t|x_0),\end{aligned}$$

which vanishes for $x_0 > x_f$, $f_{x_f}(0|x_0)=0$, $f_{x_f}(t|x_f)=\delta(t)$. To solve for f we take the Laplace transform, $Lf_{x_f}(t|x) = \hat{f}_{x_f}(p|x)$

$$\begin{aligned}\int_0^\infty e^{-pt}\partial_t f_{x_f}(t|x_0)dt =& \mathcal{L}^\dagger_{x_0}\hat{f}_{x_f}(p|x_0)\\ e^{-pt}f_{x_f}(t|x_0)|_0^\infty + p\int_0^\infty e^{-pt}f_{x_f}(t|x_0)dt =& \mathcal{L}^\dagger_{x_0}\hat{f}_{x_f}(p|x_0)\\ f_{x_t}(0|x_0) + p\hat{f}_{x_f}(p|x_0) =& \mathcal{L}^\dagger_{x_0}\hat{f}_{x_f}(p|x_0)\\ (\mathcal{L}^\dagger_{x_0} - p)\hat{f}_{x_f}(p|x_0) =& 0\end{aligned}$$

where we performed integration by parts and made use of the initial condition. The unique solution to the above problem is given by $\hat{f}_{x_f}(p|x_0) = v_p(x_0)/v_p(x_f)$, where $v_p(\bullet)$ is the unique increasing positive solution of the equation $(\mathcal{L}^\dagger_{x_0} - p)v_p = 0$ [59]. For the harmonic oscillator we get

$$-(x - sx_f)\partial_x v_p(x) + T_x\partial_x^2 v_p(x) - pv_p(x) = 0,$$

which we can solve by rewriting $v_p(x)$ as

$$v_p(x) = \exp\left\{\frac{x^2 - 2sx_f x}{4D}\right\} Z_p(x),$$

yielding,

$$T_x\partial_x^2 Z_p(x) + Z_p(x)\left[-p + \frac{1}{2} - \frac{(x-sx_f)^2}{4T_x}\right] = 0$$

Rescaling $x = \sqrt{T_x}y + sx_f$, and using the chain rule, $\partial_x^2 Z_p(x) = T_x^{-1}\partial_y^2 Z_p(y)$, we get,

$$\frac{d^2 Z_p(y)}{dy^2} + Z_p(y)\left(-\frac{y^2}{4} + \frac{1}{2} - p\right) = 0$$

which is Weber's parabolic cylinder differential equation, with solution,

$$Z_p(x) = D_{-p}\left[-\sqrt{T_x^{-1}(x-sx_f)^2}\right],$$

where D_α is the parabolic cylinder function. Thus,

$$v_p(x) = \exp\left\{\frac{x^2 - 2sx_f x}{4T_x}\right\} D_{-p}\left[-\sqrt{T_x^{-1}(x-sx_f)^2}\right] \tag{S7}$$

From this we can write an expression for the Laplace transform of the distribution of times for a particle to go from $x_0 = 0$ to $x_f = 1$,

$$\hat{f}_L(p,s) = \hat{f}_{x_f=1}(p|x_0=0) = \frac{v_p(x_0=0)}{v_p(x_f=1)}$$
$$= e^{-\frac{1-2s}{4T_x}} \frac{D_{-p}\left[-\sqrt{s^2/T_x}\right]}{D_{-p}\left[-\sqrt{(1-s)^2/T_x}\right]}, \tag{S8}$$

which we can invert numerically to evaluate the first passage time distribution (see e.g. [89]),

$$f_L(t,s) = L^{-1}\left[\hat{f}_L(p,s)\right]. \tag{S9}$$

In Fig. S2(a) we used the method of de Hoog et al. [60] to numerically invert the Laplace transform and obtain the FPTDs for each fixed s. In addition, we estimate the adiabatic approximation of the full FPTD through,

$$f_L(t) = \int p(s) f_L(t,s) ds, \tag{S10}$$

where $p(s)$ is the distribution of the values of s during first passage time events.

Simulations

Driven harmonic oscillator : We generate 1000 simulations of the Langevin dynamics given in the main text for a slowly-driven harmonic oscillator,

$$\begin{cases} dx_t = -(x_t - s_t x_f)dt + \sqrt{2T_x}dW_t \\ ds_t = -\tau_s^{-1} s_t + \sqrt{2T_s\tau_s^{-1}}dW_t \end{cases},$$

through an Euler-scheme with a sampling time of $\Delta t = 10^{-4}$ s for $T_{\text{exp}} = 10^7$ s and with initial condition $x(0) = 0$ and $s(0) \sim \mathcal{N}(0, \sqrt{T_s})$ is sampled according to the Boltzmann distribution. As for the parameter values, we take $T_x = 0.1$, $T_s = 0.1$ Figs. S2, S4, and fix $\tau_s = 10^3 \times T_{\text{exp}}$ in the main text and Fig. S3(a).

Driven double well potential: We generate 1000 simulations of Langevin dynamics given in the main text for a slowly-driven double well potential,

$$\begin{cases} dx_t = -4s_t^2 x_t(x_t-1)^2 dt + \sqrt{2T_x}dW_t \\ ds_t = -\tau_s^{-1}(s_t - \mu_s)dt + \sqrt{2T_s\tau_s^{-1}}dW_t \end{cases},$$

through an Euler-scheme with a sampling time of $\Delta t = 10^{-4}$ s for $T_{\text{exp}} = 10^7$ s, $\tau_s = 10^3 \times T_{\text{exp}}$ and with initial condition $x(0)$ which is randomly chosen as $x(0) = 1$ and $x(0) = -1$ with equal probability and $s(0) \sim \mathcal{N}(\mu_s, \sqrt{T_s})$ is sampled according to the Boltzmann distribution. As for the parameter values, we take $T_x = 0.15$.

Driven parabolic potential: We generate 1000 simulations of the Langevin dynamics given in the main text for a slowly-driven parabolic potential,

$$\begin{cases} dx_t = -\left(x_t - s_t + 2\sin(2\pi k x_t)\right)dt + \sqrt{2T_x}dW_t \\ ds_t = -\tau_s^{-1} s_t + \sqrt{2T_s\tau_s^{-1}}dW_t \end{cases},$$

through an Euler-scheme with a sampling time of $\Delta t = 10^{-4}$ s for $T_{\text{exp}} = 10^7$ s, $\tau_s = 10^3 \times T_{\text{exp}}$ and with initial condition $x(0) = 0$ and $s(0) \sim \mathcal{N}(0, \sqrt{T_s})$ is sampled according to the Boltzmann distribution. As for the parameter values, we take $T_x = 0.1$.

Numerical integration

We numerically integrate Eq. S2 with $\Delta U(\lambda) = \lambda^k$ and $V(\lambda) = \lambda^n$, through a Riemman sum using the midpoint rule from $\omega_{\min} = 5 \times 10^{-10}$ to $\omega_{\max} = 1$ with $\Delta\omega = 10^{-9}$, yielding the results of Fig. S1.

First passage time distribution estimation

From the simulations of $x(t)$, we first identify all segments, $[t_0, t_f]$, in which t_0 corresponds to the first time x returns to x_0 for the after reaching x_f, and t_f is the time first to reach x_f after t_0. We then build a normalized histogram of first passages times with logarithmic bins. We note that while for the slowly driven potential landscape the position of the minima of the potential are fixed at $x_0 = \pm 1$, for the slowly driven harmonic oscillator and the rugged parabolic potential the minima of the potential are driven by s. For practical reasons, in the numerical estimation of the FPTD we take $x_0 = \langle s \rangle = 0$, which has no practical impact due to the severe separation of time scales between the first passage time events and the relaxation to the well.

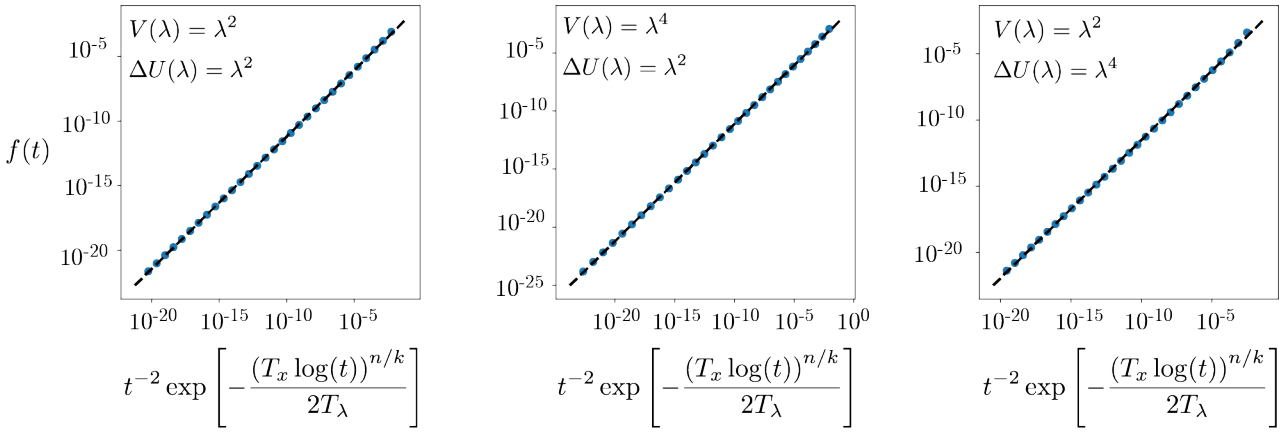

FIG. S1. Numerical integration of $f(t)$ for different choices of $V(\lambda)$ and $\Delta U(\lambda)$, compared to the asymptotic approximation of Eq. S6 (black dashed line) with $T_x = 0.1$ and $T_\lambda = 0.2$.

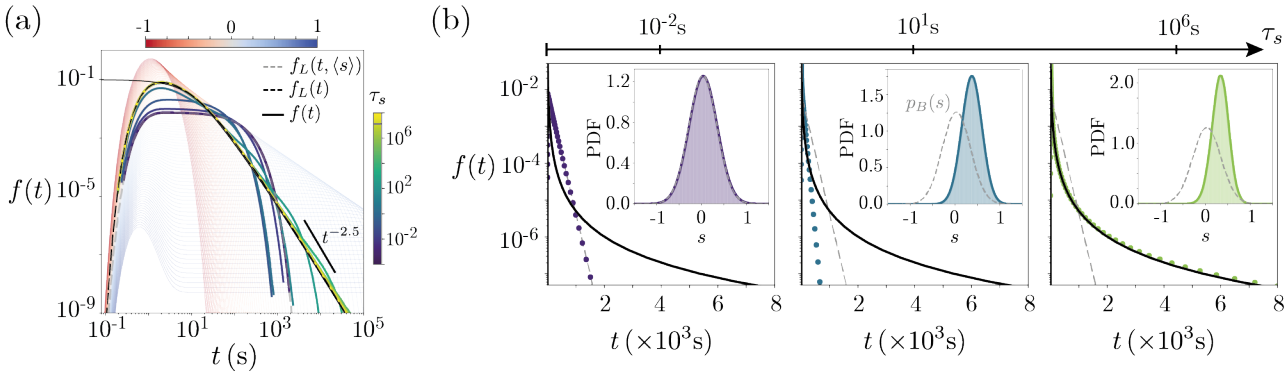

FIG. S2. Emergence of heavy-tails when the driving is sufficiently slow (a) First passage time distribution for different values of τ_s (colored from purple to yellow) show the emergence of heavy tails as $\tau_s \to \infty$. We also plot the full FPTDs $f_L(t, s)$, Eq. (S9), for different fixed values of s (color coded from red to blue). From these estimates, we highlight $f_L(t, \langle s \rangle = 0)$ (gray dashed line), which corresponds to the mean first passage time in the limit $\tau_s \to 0$ (purple line). We also plot $f_{\rm L}(t)$ (black dashed line), which corresponds to the adiabatic approximation of the FPTD obtained through a weighted average of the FPTDs obtained for fixed s, Eq. (S10); and $f(t)$, which corresponds to Eq. (10) of the main text. (b) First passage times for 3 qualitatively different regimes and the corresponding distribution of s at the first passage event (inset). When $\tau_s \to 0$ (left) the distribution of s at the first passage time events corresponds to the Boltzmann distribution of s, $p(s) \sim e^{-V(s)/T_s}$ (gray dashed line in the inset), and the distribution of first passage times $f(t)$ matches the one obtained from a fixed $s = \langle s \rangle = 0$ (dashed line); when τ_s matches the time scale of x (middle), we observe that the mean first passage time is reduced, boosted by events in which the harmonic potential moves closer to the boundary as shown in the inset; finally, when $\tau_s \gg 1$, we observe the emergence of a heavy tailed first passage time distribution (black line).

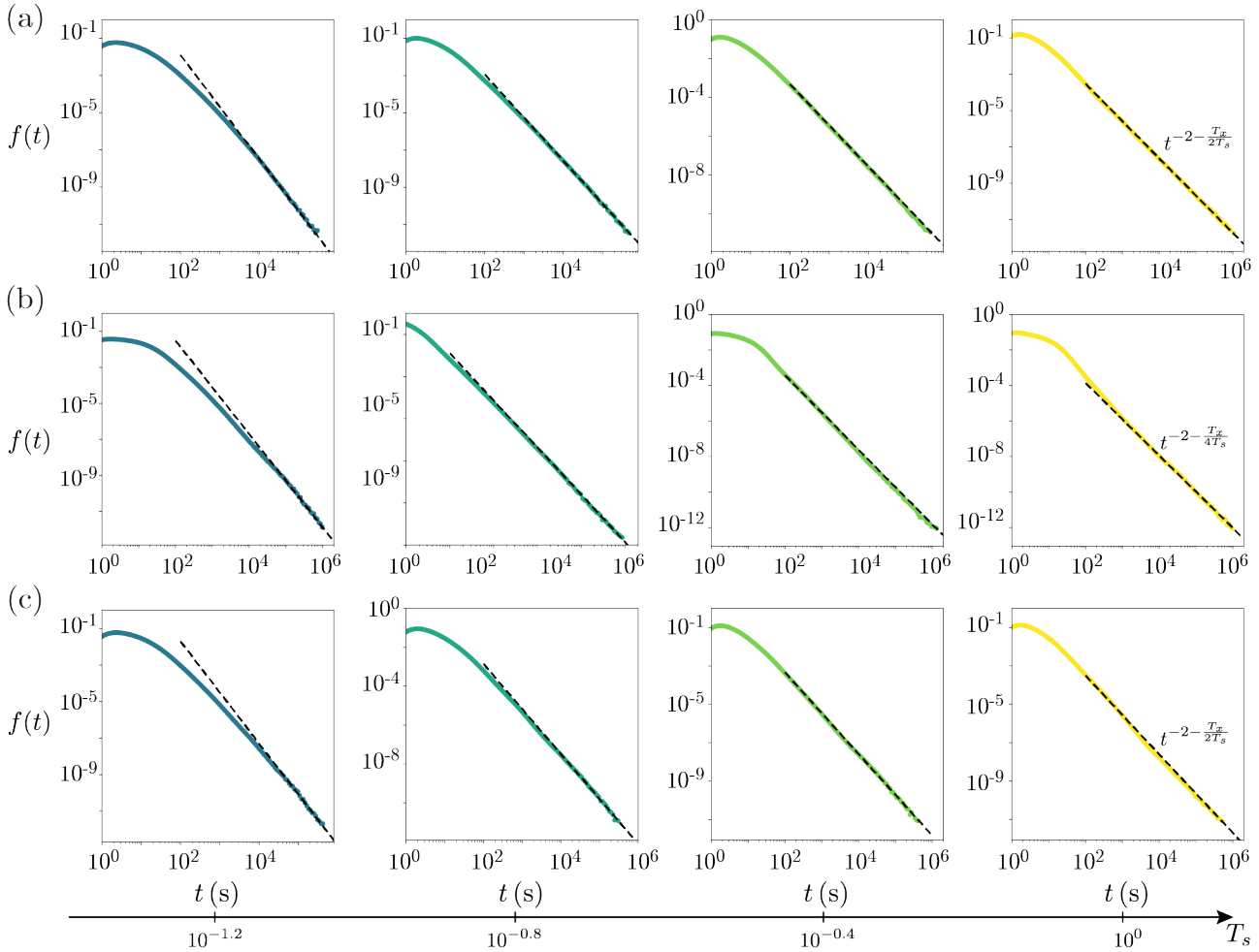

FIG. S3. Details of the accuracy of the asymptotic prediction for the behavior of the tail of $f(t)$ across temperatures T_s for the slowly driven harmonic oscillator (a), the slowly driven double well potential (b) and the slowly driven rugged parabolic potential (c).

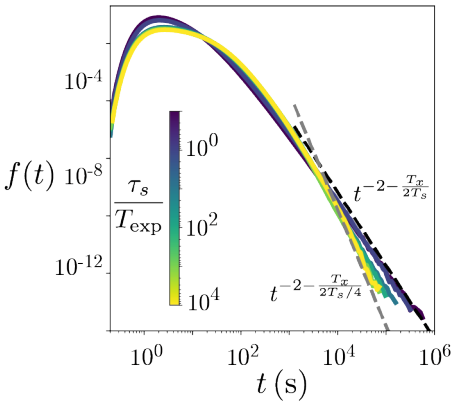

FIG. S4. First passage time distribution for different values of τ_s (colored from purple to yellow) when the initial condition is a narrower Boltzmann distribution with $T_s^0 = T_s/4$. As discussed in the main text, when $\tau_s \to \infty$ the FPTD exhibits a deeper power law exponent. When $\tau_s \sim T_{\exp}$ we recover the asymptotic behavior derived in Eq. (10) of the main text.