Kinship Demography

Inferring the expectation, variance and distribution of the number of collaterals of a dynamical process in a structured population

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Abstract

Demography influences kinship, however, as of yet, little is known of the effects of transition rates of a general, structured population, generation-overlapping, projection model on the number of kin, their structure, distribution, and how they co-vary. Thanks to a novel approach incorporating generation number into the population structure, we provide here a full description of the number of such "collaterals", in a one-sex, constant environment framework. This yields a formula for the expected number of kin (for any kin) in any class, of an individual in any class, that is much simpler than any approach published so far, easy and fast to compute and that allows future research focusing on the theory of kinship demography. This approach also leads to the first formulas for their variance-covariance and probability generating functions that yield their entire distribution.

These results will prove important for various fields studying structured population dynamics. They will allow for instance, for population and behavioral ecologists and geneticists to better understand the kinship structure of the male or female lineage of a particular population and will provide a stepping stone for a closed-form formula for the number of collaterals in two-sex populations. For epidemiologists, it will allow to analyse the distribution of "kin", that is, of individuals infected by the same source, and to compare these distributions for strains with different transition rates. For family economists, it will allow, for instance, to understand the distribution of inherited capital among members of extended families.

We further provide a formula for the unstructured number of kin and show that is symmetrical with regards to the kin relationships. The novel approach also allows to decompose the variance-covariance of the number of all kin in all classes of focal individual *ego* in a given class, according to projective stochasticity (called individual stochasticity in population ecology) and to genealogical stochasticity which characterises uncertainty about the ancestors of *ego*.

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1 Introduction

Consider a structured population ; this can be a population of wild boars structured by stage and body mass, of humans subject to an epidemics structured by stage of sickness and infectious status, of households structured by age and capital, of cancerous cells, etc. From the transition rates from all types of individuals to all types of individuals over a given time step , one can project, deterministically or stochastically, this population over time and predict its demographic (and evolutionary) fate; such analyses are the bread and butter of economics, epidemiology, population ecology, genetics, and many fields of physics. One can also retrace the ancestry of individuals; there are ways to compute the distribution of ancestors of individuals or groups of individuals of that population. Population genetics is a field where the question of the common ancestry of two sub-populations is crucial and where the time at which their coalesced can be computed (Kingman, 1982b,a; Tajima, 1983). In other words, one knows how to go both forward and backward in time, to anticipate the distribution, structure and abundances of the future descendants of the individuals in the current population and how to infer the possible distributions and structure of their ancestors in the past.

To our knowledge, however, no tool nor theory to explore directly and efficiently the distributions of the descendants of the ancestors, that is, of the kin of individuals living at the same time, has been developed until recently, despite calls by demographers (Goodman et al., 1974), mathematicians (Joffe and Waugh, 1982) and evolutionary ecologists (Pavard and Coste, 2020) for such progress.

The method most used and referred to to date is the seminal paper by Goodman et al. (1974) which provides the expected number of a few kin (sisters, aunts, mother, grand-mother, daughters and cousins) in a population structured by age, in a one-sex, deterministic context. This influential paper, one of the fifteen most significant articles published in Theoretical Population Biology (Rosenberg, 2020) provided a jumping board for the study of the interactions between demography and kinship, called Kinship Demography (Pavard and Coste, 2020). By treating each kin separately, this method, together with its matrix form Caswell (2020), makes it hard to implement, to extend to a more general framework (stochasticity, 2 sexes, etc.) and most importantly prevents any possible generalisability to any kin; a generalisability necessary to the development of Kinship Demography. Recently Coste et al. (2021) published a closed-form formula for the number of any kin, for a population with any structure. This provided insights into the bigger picture of Kinship Demography, allowing for instance to consider the global kinship structure, at a large scale, by computing the asymptotic growth rate of the number of kin at a specific kinship distance as this distance increases towards infinity. This formula however is bulky, potentially slow to compute, and proved complex to extend to stochastic and two-sex models.

Further progress has allowed to develop a new model which considers a population structured not only by the original structure of the population but also by generation. To our knowledge, such a tool has never been used yet in demography or in structured population dynamics and genetics, even though its principle - adding a trait to a model that does not affect the vital rates, and greatly simplifies the analysis at the small cost of inflating the structure, such as the addition of trait *age* to any class-structured population see 6.4 - is extremely simple. Together with the use of the genealogical Markov chains, originally developed by Demetrius (1974, 1975) and Tuljapurkar (1982b, 1993) in the framework of population entropy, and extended to Kinship Demography by Coste et al. (2021), this important step forward allows us to extend the capabilities of our approach. Before focusing, in follow-up papers, on the extension to two-sex models and to models accounting for temporal variations (i.e., environmental stochasticity), we present here a framework that accounts for individual stochasticity in the number of kin. From this framework, one can compute the distribution of the number of any kin, as well as the variances and co-variances of these numbers as well as further moments. Strikingly, this novel framework leads to a formula for the expected number of kin that is both much simpler and much faster to compute that any formula published so far. Most importantly, this formula will prove crucial for the aforementioned extensions as well as for the analysis of the sensitivity of kinship to demography. While it is presented here for the first time, this kinship formula (for the expected number of any kin) is the focus of a specific manuscript develloped in parallel, intended for a more general readership that this one, where individual stochasticity is not accounted but the illustrations extended.

2 The model

2.1 The framework

Consider a general, structured and generation-overlapping, population. We will limit ourselves to one-sex populations here and to processes that are constant in time, and relax these assumptions in follow-up articles. To be as general as possible, we consider a discrete structure and a discrete time-step (see Appendix. 6.1); the discrete structure consists of s classes. This population can be projected over time, via $\forall(i,j) \in [1,s]^2, \tilde{n}_i(t+1) = \tilde{M}_{i,j}\tilde{n}_j(t) = (\tilde{F}_{i,j} + \tilde{S}_{i,j})\tilde{n}_j(t)$, the stochastic process yielding the number of individuals in class i stemming from an individual in class in j, over the chosen time-step , that can be decomposed into a survival and a fertility process segregating surviving individuals $\tilde{S}_{i,j}\tilde{n}_j(t)$, and new individuals $\tilde{F}_{i,j}\tilde{n}_j(t)$ and where $\tilde{n}_i(t) \in \mathbb{N}$ is the (stochastic) number of individuals in class i. These stochastic processes can be grouped together in the shape of stochastic matrices $\tilde{\mathbf{M}}$, $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{S}}$, of size $s \times s$, which deterministic versions we denote $\mathbf{E}(\tilde{\mathbf{M}}) = \mathbf{M} = \mathbf{F} + \mathbf{S}$. In matrix form, the equation above becomes

$$\tilde{\boldsymbol{n}}(t+1) = \tilde{\mathbf{M}}\tilde{\boldsymbol{n}}(t) = (\tilde{\mathbf{F}} + \tilde{\mathbf{S}})\tilde{\boldsymbol{n}}(t). \tag{1}$$

Matrix $\mathbf{M}$, the deterministic projector, is the so-called population projection matrix of population ecology (Caswell, 2001), which elements, the $M_{i,j}$ are called the (expected) vital rates: $n_i(t+1) = E(\tilde{n}_i(t+1)) = M_{i,j}n_j(t) = F_{i,j}n_j(t) + S_{i,j}\tilde{n}_j(t)$ or in matrix form $\boldsymbol{n}(t+1) = \mathbf{M}\boldsymbol{n}(t) = (\mathbf{F} + \mathbf{S})\boldsymbol{n}(t) = \mathbf{M}^t\boldsymbol{n}(0)$. We can also consider the projector of a specific class of individual, and thus stochastic vector $\tilde{\mathbf{S}}_j$ describes the survival process (survival and class reached) of an individual in class j and (deterministic) vector $\mathbf{F}_j$ the expected number of offspring (in the various classes) produced by an individual in class j. We further consider, without any loss of generality, that the processes of reproduction and survival are i.i.d. (see Appendix. 6.2). Notationwise, stochastic processes are noted with a tilde, and expectations without, vectors and matrices and in bold while scalars are in plain text. All notations are summarised in Table 1.

Stochastic matrices $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{F}}$ are the building blocks for our formulas inferring the number of kin of an individual *ego* in the population. They will allow us, via their formulation in a generation structured population (sec.2.3), to go back in time towards the distribution of the ancestors of *ego*, via their genealogical Markov chains (sec.2.2), and from there to count the descendants of these ancestors, via younger, older and sisters of the same-litter than the ancestor's daughter (sec.3). In general, this formulation is not sufficient as, theoretically, the number of same-litter sisters depends both on the class of the mother and on that of the sister We provide formulas for the general case in sec.3.7, but focus mainly, in this paper, on three particular but extremely common cases, where this number only depend on the class of mother. These are (1) Bernoulli reproduction (where only one offspring maximum can be produced per timestep), (2) (independent) Poisson reproduction where each $\tilde{F}_{i,j}$ is a Poisson independent from the other and (3) One-offspring-class reproduction (where all offspring are born in the same class). We describe these processes in further details in Supplementary Material SM7.1, where we note that for the first two of these, the whole stochastic structure can be inferred from the expectations $\mathbf{F}$ and $\mathbf{S}$.

2.2 Stable State and Genealogical Markov Chains

We consider that all processes are constant over time and have been for a while, that is, we place ourselves at the demographic stable state. In that state, the population growth rate and relative abundances correspond to λ and $\boldsymbol{w}$, the maximum eigenvalue of $\mathbf{M}$ and its associated right-eigenvector (scaled to sum to 1): $\mathbf{M}\boldsymbol{w} = \lambda\boldsymbol{w}$. While $\tilde{\mathbf{M}}$ projects the population over time, its genealogical Markov chain (Demetrius, 1974, 1975; Tuljapurkar, 1982a, 1993) allows to project it backwards in time. The transition matrix of the genealogical Markov chain associated with $\mathbf{M}$ is $\mathbf{P}$, where $P_{ij} = \frac{M_{ji}\,w_i}{\lambda\,w_j}$, is the probability that the ancestor (via survival or reproduction) of an individual in class j was in class i at the previous time-step. The relationship between the Markov chain and its transition matrix is extremely simple: $P(\tilde{P}_{i,j} = 1) = P_{i,j}$, where each $\tilde{\mathbf{P}}_j = \{P_{i,j}\}_i$ is a Bernoulli scheme. Note that, in order for backwards projection to obey the same rule of left-multiplication as projection matrices, we write $\mathbf{P}$ as a left stochastic matrix ($\mathbf{P}$ is therefore the transpose of the matrix of the same name (in, e.g., Coste et al., 2021; Tuljapurkar and Orzack, 1980)). Similarly to the projection matrix, we can split the transitions of the genealogical matrix according to survival and fertility processes: $\mathbf{P} = \mathbf{P_S} + \mathbf{P_F}$, where

$$P_{\mathbf{S}}(i,j) = \frac{S_{ji}\,w_i}{\lambda\,w_j} \text{ and } P_{\mathbf{F}}(i,j) = \frac{F_{ji}\,w_i}{\lambda\,w_j}\,. \tag{2}$$

2.3 Generation structured population

The most important difference between the model we develop here and the ones developed before (including (Coste et al., 2021)) is the consideration of a different population structure that accounts for both the class and the generation number of individuals. We consider a fixed number of generations g_m and add the trait *generation number* (or simply *generation*) to the existing trait structure of the population (see Appendix 6.3). In other words, we now consider a population vector that is 2-dimensional, the first dimension corresponds to the s classes of the original structure and the second dimension to the g_m generations (that is to the g_m classes of trait *generation*). In such a model, the generation number can be relative; however, we make it absolute by fixing one end: generation 1 corresponds to the generation of the common ancestor of *ego* and its kin.
As for any multitrait model (see Appendix 6.3 on the concepts of traits in projection models and Coste et al. (2017)), the population vector needs to be vectorised, that is to be made one-dimensonal, where the number of individuals in class c_i and generation g_i are to be found in the ith entry of $\tilde{\mathbb{n}}$ where $i = c_i + s(g_i - 1)$; we will call this entry equivalently $\tilde{\mathbb{n}}_{c_i,g_i}$ or $\tilde{\mathbb{n}}_i$.

We can project $\tilde{\mathbb{n}}(t)$ forward in time via stochastic tensors $\tilde{\mathbb{M}} = \tilde{\mathbb{S}} + \tilde{\mathbb{F}}$ and backwards in time via tensors $\mathbb{P} = \mathbb{P}_\mathbb{S} + \mathbb{P}_\mathbb{F}$. As the population vector is vectorised, these tensors are simply stochastic and deterministic projection matrices of size $s.g_m \times s.g_m$. As is common with tensors, they can be described as block-matrices (see SM.7.2) and they can be easily described from the building blocks using tensor product notion (aka kronecker product): Trait *generation* does not affect the demographic processes, is constant via survival and increases by 1 via reproduction (but for individuals in generation g_m which don't reproduce, therefore $\tilde{\mathbb{S}} = \mathbf{I} \otimes \tilde{\mathbf{S}}$ and $\tilde{\mathbb{F}} = \mathbf{J_1} \otimes \tilde{\mathbf{F}}$, with $\mathbf{J_1}$ the $g_m \times g_m$ matrix with ones in the sub-diagonal. In other words, $\tilde{\mathbb{M}}$ is the empty matrix but for all diagonal blocks being $\tilde{\mathbf{S}}$ and all subdiagonal blocks being $\tilde{\mathbf{F}}$.

Such a model does not have the asymptotic properties of $\tilde{\mathbf{M}}$: the population will eventually go extinct as individuals in generation g_m do not reproduce. To understand the stable state of a generation structured model, and from it generate the transition matrices of its genealogical Markov chain, one considers $\tilde{\mathbf{M}}$ as the truncation of a model with an infinite number of generation or equivalently as of a model where the trait used is that of relative generation (see SM.7.2). This yields $\mathbb{P}_\mathbb{S} = \mathbf{I} \otimes \mathbf{P_S}$ and $\mathbb{P}_\mathbb{F} = \mathbf{J_1}^T \otimes \mathbf{F}$. In block-matrix form, we get, for $g_m = 4$, we get:

$$\mathbb{M} = \mathbb{F} + \mathbb{S} = \left[\begin{array}{c|c|c|c} \mathbf{S} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{F} & \mathbf{S} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{F} & \mathbf{S} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{F} & \mathbf{S} \end{array}\right] \text{ and } \mathbb{P} = \mathbb{P}_\mathbb{F} + \mathbb{P}_\mathbb{S} = \left[\begin{array}{c|c|c|c} \mathbf{P_S} & \mathbf{P_F} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{P_S} & \mathbf{P_F} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{P_S} & \mathbf{P_F} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{P_S} \end{array}\right] \tag{3}$$

2.4 Expectation, Distribution and Variance of the number of Kin

Kinship In that next context, we wish to build and analyse stochastic process $\tilde{\mathbb{K}}$, of same structure that $\tilde{\mathbb{M}}$ (that is, a $sg_m \times sg_m$ block matrix where the $\tilde{\mathbb{K}}_j$ are independent from each other and for each individual and time-step) that provides the number of kin of *ego*. It means that $\tilde{\mathbb{K}}_{i,j} = \tilde{\mathbb{K}}_{(c_i,g_i),(c_j,g_j)}$ is the number of kin in state i of *ego* in state j, that is the number of kin in class c_i and generation g_i of *ego* in class c_j and generation g_j; in other words it is the number of (g_i, g_j)-kin, in class c_i of *ego* in class c_j, where a (g_i, g_j)-kin is the particular kin relationship corresponding to the common ancestor $g_j - 1$ generations from *ego* and $g_i - 1$ generations from the kin. Thus, $(1,2)$-kin is the mother, $(3,1)$-kin the grand-daughters and $(3,3)$-kin the cousins. Note that this is slightly different to the characterisation the authors used in (Coste et al., 2021) where a $(g_j - 1, g_i - 1)$-kin there corresponds to a (g_i, g_j)-kin here. The realisation of $\tilde{\mathbb{K}}_{i,j}$ is independent from that of other *ego*s in states j' (even for $j = j'$), but depends on that the number of other kin $i' \neq i$, which makes it all the more important to account for all possible states and to consider the stochastic vector $\tilde{\mathbb{K}}_j$ yielding the co-varying numbers of kin descended from the g_j-ancestor of *ego* in class c_j (instead of the $\tilde{\mathbb{K}}_{i,j}$ directly) which we can do thanks to our generation structured model. For $g_j = 1$, these co-varying number of kin correspond to the number of daughters, grand-daughters, great-grand-daughters,etc. for $g_j = 2$, to the probability of the mother being alive, the number of sisters, nieces,etc, for $g_j = 3$, to the probability of the grand-mother being alive, the number of aunts, cousins, etc.

Expectation From the expectation of process $\tilde{\mathbb{K}}$, $\mathbb{K} = \mathbf{E}(\tilde{\mathbb{K}})$, we can get the expected number of (g_i, g_j)-kin in class c_i of *ego* in class c_j, directly from entry $\mathbb{K}_{(c_i,g_i),(c_j,g_j)}$. We can also expand our investigation by going beyond the mean towards the general distribution of the number of kin and its moments, of which the expectation is only a particular case. Tools stemming from probability theory and in particular branching processes, allow to investigate $\tilde{\mathbb{K}}_{(c_j,g_j)}$ via various approaches. One can compute its moments directly, or via moment generation functions, or use the related tool of probability-generating functions (*pgf*) to compute the distributions of the number of kin. In this paper, we will focus on the first moments - expectation and variance - and on the *pgf*.

Variance As for the other stochastic vectors encountered, the variance of stochastic vector $\tilde{\mathbb{K}}_{(c_j,g_j)} = \tilde{\mathbb{K}}_j$,$\mathbf{Var}(\tilde{\mathbb{K}}_j)$, is a variance-covariance matrix of size $s.g_m \times s.g_m$, which provides, on the diagonal, the variance of the number of specific each kin in state (c_i, g_i) for *ego* in state j , that is $Var(\tilde{\mathbb{K}}_{(c_i,g_i),(c_j,g_j)})$ and away from the diagonal, the covariances $Cov(\tilde{\mathbb{K}}_{(c_i,g_i),(c_j,g_j)}, \tilde{\mathbb{K}}_{(c_i',g_i'),(c_j,g_j)})$. We can group the $\mathbf{Var}(\tilde{\mathbb{K}}_j)$, in a $(s.g_m)^2 \times s.g_m$ matrix, $\mathbf{Var}^*(\tilde{\mathbb{K}})$, which j^{th} column is the vectorised $\mathbf{Var}(\tilde{\mathbb{K}}_{(c_j,g_j)})$: $\mathbf{Var}^*_j(\tilde{\mathbb{K}}) = \boldsymbol{vec}(\mathbf{Var}(\tilde{\mathbb{K}}_{(c_j,g_j)}))$ where $\boldsymbol{vec}$ is the vec operator, vectorising a matrix into a vector by stacking columns on top of each other (Henderson and Searle, 1981). See SM7.3 for a more detailed description of the variance of kinship.

Distribution The *pgf* of a multivariate process, such as $\tilde{\mathbb{K}}_{(c_j,g_j)}$, is the function of multivariate dummy variable of the same dimensions that the abundance vector $\mathbb{n}$, $\boldsymbol{x} = (x_{1,1}, x_{2,1}, ..., x_{s,1}, x_{1,2}, ..., x_{s,g_m})$, $\varphi_{\tilde{\mathbb{K}}_{(c_j,g_j)}}(\boldsymbol{x}) = E(x_{1,1}^{\tilde{\mathbb{K}}_{(1,1),(c_j,g_j)}} x_{1,2}^{\tilde{\mathbb{K}}_{(1,2),(c_j,g_j)}} ... x_{s,g_m}^{\tilde{\mathbb{K}}_{(s,g_m),(c_j,g_j)}})$.The *pgf* of a such a stochastic vector allows to compute all probabilities $\forall \boldsymbol{n}, P(\tilde{\mathbb{K}}_{(c_j,g_j)} = \boldsymbol{n})$, that is, the full distribution of the number of kin. The *pgf*s of the various $\tilde{\mathbb{K}}_j$ can be grouped as a *pgf* vector: $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}} = (\varphi_{\tilde{\mathbb{K}}_{(1,1)}}, \varphi_{\tilde{\mathbb{K}}_{(2,1)}}, ..., \varphi_{\tilde{\mathbb{K}}_{(s,g_m)}})$. From $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}}$, one can also obtain the (uni-variate) *pgf* $\varphi_{\tilde{\mathbb{K}}_{(i,j)}}$ of $\tilde{\mathbb{K}}_{i,j}$, by setting respective arguments of $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}}$ equal to 1 (see sec. 7.3). Moment generating functions are better suited to compute moments, however, from a *pgf.* one can also compute all mixed factorial moments, see SM7.3.

3 Results

In order to be able to compute $\mathbb{K}$, $Var^*(\tilde{\mathbb{K}})$ and most importantly $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}}$ which contains all the information about the distribution of the number of kin, we need the same information about the building blocks (which we provide in sec. 7.5) and to write the stochastic process $\tilde{\mathbb{K}}$ as a succession and composition of independent processes, which we do in the next section.

3.1 Process

We write that $\tilde{\mathbb{K}}_j$, the number of kin of *ego* in state $j = (c_j, g_j)$, is the number of descendants alive today of the g_j-ancestor of *ego*, considered when it was last known alive (as it gave birth the $g_j - 1$ ancestor if $g_j > 1$, or now as *ego* itself if $g_j = 1$) at time t in the past, and in class c_a (Eq.4a). This event has probability $P_j(c_a, t)$ which can be calculated from $\mathbb{P}$ and $\mathbb{P}_{\mathbb{F}}$: it corresponds to the $(c_a, 1), (c_j, g_j)$ entry of matrix $\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}$ which projects individuals back in time with the earliest process being a reproduction if $t > 0$ and for $t = 0$, it corresponds to case where *ego* is the ancestor and therefore $g_j = 1$ and $c_a = c_j$ (Eq.4c). It constitutes a complete set ($\forall j, \sum_{c_a, t\geq 0} P_j(c_a, t) = 1$, see SM7.11), and the following system of stochastic equations describes a converging process, see SM7.9.

$$\tilde{\mathbb{K}}_j = \tilde{\mathbb{K}}_{c_j,g_j} = \tilde{\mathbb{D}}_{c_a}^{(t)} \text{ with probability } P_j(c_a, t) \tag{4a}$$

$$\tilde{\mathbb{D}}^{(t)} = \begin{cases} \underbrace{\sum_{\alpha>0} \tilde{\mathbb{M}}^{\alpha-1}\tilde{\mathbb{F}}\mathbb{J}^*\tilde{\mathbf{P_S}}^{\alpha}}_{\tilde{\mathbb{D}}_{\mathbb{o}}^{(0)}} \text{ if t=0} \\ \underbrace{\tilde{\mathbb{M}}^{t-1}\tilde{\mathbb{S}}\mathbb{J}^*}_{\tilde{\mathbb{D}}_{\mathbb{y}}^{(t)}} + \underbrace{\tilde{\mathbb{M}}^{t-1}\tilde{\mathbb{F}^*}\mathbb{J}^*}_{\tilde{\mathbb{D}}_{\mathbb{s}}^{(t)}} + \underbrace{\sum_{\alpha>0} \tilde{\mathbb{M}}^{t-1+\alpha}\tilde{\mathbb{F}}\mathbb{J}^*\tilde{\mathbf{P_S}}^{\alpha}}_{\tilde{\mathbb{D}}_{\mathbb{o}}^{(t)}=\tilde{\mathbb{M}}^{(t)}\tilde{\mathbb{D}}_{\mathbb{o}}^{(0)}} \text{ if t>0} \end{cases} \tag{4b}$$

$$P_{c_j,g_j}(c_a, t) = P_j(c_a, t) = \begin{cases} \boldsymbol{e}_{c_a}^T \mathbb{J}^{*T} \mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}\boldsymbol{e}_{(c_j,g_j)} & \text{if } t > 0 \\ \boldsymbol{e}_{c_a}^T \mathbb{J}^{*T} \boldsymbol{e}_{(c_j,g_j)} & \text{if } t = 0 \end{cases} , \tag{4c}$$

where $\mathbb{J}^*$ is a dummy matrix turning a vector of dimension s into one of dimension sg_m and $\boldsymbol{e}_j$ is the zero-vector but for a 1 in position j and ^{T} denotes transposition (see Table 1) for the nomenclature used in this article.

The number of descendants of the g_j-ancestor of *ego*, alive today, can be, in the three common cases we focus on here, decomposed as the sum of three independent processes: the descendants via younger sisters of the $g_j - 1$-ancestor (process $\tilde{\mathbb{D}}\mathbb{y}^{(t)}$), the descendants via older sisters (process $\tilde{\mathbb{D}}\mathbb{o}^{(t)}$) and the descendants via same-litter sisters (process $\tilde{\mathbb{D}}\mathbb{s}^{(t)}$). Indeed in these three cases, the number/class of same-litter sisters only depends on the class of the mother, via process $\tilde{\mathbb{F}^*}$ which provides the number of sisters of an offspring already born (a byproduct, easily described, of $\tilde{\mathbb{F}}$, see SMs7.4 and 7.5; in the general case, the number of same-litter sisters may also depend on the class of the $g_j - 1$ ancestor, see sec.3.7).

Process $\tilde{\mathbb{D}}\mathbb{y}^{(t)}$ corresponds to t forward projections, the first one being survival. Process $\tilde{\mathbb{D}}\mathbb{s}^{(t)}$ to t forward projections, the first one being $\tilde{\mathbb{F}^*}$. And process $\tilde{\mathbb{D}}\mathbb{o}^{(t)}$ to t forward projections from $\tilde{\mathbb{D}}\mathbb{o}^{(0)}$, the number of descendants of an individual,alive today. $\tilde{\mathbb{D}}\mathbb{o}^{(0)}$ sums, for all time-steps α in the past, the number of descendants born then and alive now (first line of Eq 4b) . See Table 1 for all notations used in this article. For a detailed step by step approach to these formulas and the next, see SM7.7)

We also have (see SM7.7),

$$\tilde{\mathbb{D}}\mathbb{o}^{(t)} = \left(\tilde{\mathbb{M}}^t\tilde{\mathbb{F}}\mathbb{J}^* + \tilde{\mathbb{D}}\mathbb{o}^{(t+1)}\right)\tilde{\mathbf{P}}_{\mathbf{S}} \quad \text{with } \lim_{t\to\infty} P(\tilde{\mathbb{D}}\mathbb{o}^{(t)} = \mathbf{0}) = 1 \tag{5a}$$

$$\tilde{\mathbb{D}}\mathbb{o}^{(0)} = \left(\tilde{\mathbb{F}}\mathbb{J}^* + \tilde{\mathbb{M}}\mathbb{D}\mathbb{o}^{(0)}\right)\tilde{\mathbf{P}}_{\mathbf{S}} \tag{5b}$$

Whilst $\tilde{\mathbb{D}}\mathbb{y}^{(t)}$ and $\tilde{\mathbb{D}}\mathbb{s}^{(t)}$ correspond to successions of stochastic processes (that is, here, to products of stochastic matrices) that explicitly say that the descendants t time-steps after the g_j-ancestor gave birth to the $g_j - 1$-ancestor are the descendants of the descendants of the descendants .... of respectively, the g_j-ancestor provided it has survived one more time-step (via $\tilde{\mathbb{S}}$) and of the other daughters that it may have add at the same time it produces the $g_j - 1$-ancestor (via $\tilde{\mathbb{F}^*}$), $\tilde{\mathbb{D}}\mathbb{o}^{(t)}$ cannot be written in such a way. It is the sum over all α time-steps before the g_j-ancestor gave birth to the $g_j - 1$-ancestor of the descendants of the daughters produced by the g_j-ancestor $t + \alpha$ time-step ago (see formula for $\tilde{\mathbb{D}}\mathbb{o}^{(t)}$ in Eq. 4b).

However, the components of this sum are not independent from one another: the class of the ancestor at time $-t - \alpha$ depends on the class it was in at time $-t - \alpha + 1$. For that reason, this formulation, while suitable to compute the expectation of the number of kin, will be useless for other moments and for the computation of the general distribution of the number of kin via the *pgf*. We therefore provide, in Eq 5a, a backward induction allowing to generate approximations of $\tilde{\mathbb{D}}\mathbb{o}^{(t)}$ from a given $\tilde{\mathbb{D}}\mathbb{o}^{(t')} = \mathbf{0}$ for a large t'; this will be the method used in practice, for instance in the calculation of the variance of the number of kin. However it is also possible to study $\tilde{\mathbb{D}}\mathbb{o}^{(t)}$ exactly by considering that it is solution of Eq 5b, a property less central to this article but that will allow to study the theoretical properties of $\tilde{\mathbb{K}}$.

3.2 Expected number of any kin

We compute $\mathbb{K}$, that we call the "Kinship Matrix", as the sum over all ts of $\mathbb{Dy}^{(t)} + \mathbb{Ds}^{(t)} + \mathbb{Do}^{(t)}$ weighted accordingly by the probabilities of eq. 4c: $\mathbb{K} = (\sum_{\alpha>0} \mathbb{M}^{\alpha-1}\mathbb{F}\mathbb{P_S}^{\alpha})\mathbb{J} + \sum_{t>0}\Big(\mathbb{M}^{t-1}\mathbb{S} + \mathbb{M}^{t-1}\mathbb{F}^{*} + \mathbb{M}^{t}(\sum_{\alpha>0} \mathbb{M}^{\alpha-1}\mathbb{F}\mathbb{P_S}^{\alpha})\Big)\mathbb{J}\mathbb{P_F}\mathbb{P}^{t-1}$. This yields after some simplifications (see sec 7.7.6):

$$\mathbb{K} = \sum_{t=0}^{\infty} \mathbb{M}^{t}\left(\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}} + \mathbb{F}^{*}\mathbb{J}\mathbb{P}_{\mathbb{F}}\right)\mathbb{P}^{t} \tag{6}$$

This formulation can be also be obtained directly by focusing on the expectations of the building blocks, as shown in SM7.7.6

improvement This formula (Eq.6), corresponds exactly and therefore yields the same output than the series of formulas at the core of Coste et al. (2021). The approach is the same but for the crucial extension of the structure to include trait *generation*. This "small" change in the structure (the new input matrices $\mathbb{M}$ and $\mathbb{P}$ and their components are effortlessly built from $\mathbf{M}$ and $\mathbf{P}$ and their components) leads to radical simplification of the formula. The new formula does not require intermediary computations, nor does it need a doubly infinite sum. Most importantly, this formula makes the computation of the expected number of kin extremely fast, easy to implement (see sec 3.6) and its theoretical analysis possible. Furthermore, it will allow to compute the sensitivity of kinship to demography and to extend the framework by allowing environmental stochasticity and two sexes in the model, endeavours that will follow up this article.

3.3 Distribution of the number of kin

We turn equation 4 into its *pgf* equivalent:

$$\varphi_{\tilde{\mathbb{K}}_j} = \sum_{c_a,t} P_j(c_a,t)\varphi_{\tilde{\mathbb{D}}^{(t)}_{c_a}}, \qquad \text{with} \tag{7a}$$

$$\begin{cases} \boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(0)}} = \boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(0)}} \\ \boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(t)}} = \underbrace{\left(\boldsymbol{\varphi}_{\tilde{\mathbb{S}}\mathbb{J}^{*}} \circ \bigcirc_{t-1}\boldsymbol{\varphi}_{\tilde{\mathbb{M}}}\right)}_{\varphi_{\tilde{\mathbb{D}}\mathbb{y}^{(t)}}} \bullet \underbrace{\left(\boldsymbol{\varphi}_{\tilde{\mathbb{F}^{*}}\mathbb{J}^{*}} \circ \bigcirc_{t-1}\boldsymbol{\varphi}_{\tilde{\mathbb{M}}}\right)}_{\boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{s}^{(t)}}} \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}} \qquad \text{if } t > 0, \end{cases} \tag{7b}$$

where $\bullet$ is the termwise product of vectors, $\bigcirc_k$ denotes k compositions of the same function and $\boldsymbol{\varphi}_{\tilde{\mathbb{S}}\mathbb{J}^{*}}$ is the first s elements of $\boldsymbol{\varphi}_{\tilde{\mathbb{S}}}$. For $\varphi_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}}$, we have

$$\boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}} = \boldsymbol{\varphi}_{\tilde{\mathbf{P_S}}} \circ \left(\left(\boldsymbol{\varphi}_{\tilde{\mathbb{F}}\mathbb{J}^{*}} \circ \boldsymbol{\varphi}_{\tilde{\mathbb{M}}^{t}}\right) \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(t+1)}}\right) \qquad \text{with } \lim_{t\to\infty} \boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}} = \mathbf{1} \tag{8a}$$

$$\boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(0)}} = \boldsymbol{\varphi}_{\tilde{\mathbf{P_S}}} \circ \left(\boldsymbol{\varphi}_{\tilde{\mathbb{F}}\mathbb{J}^{*}} \bullet \left(\boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(0)}} \circ \boldsymbol{\varphi}_{\tilde{\mathbb{M}}}\right)\right) \tag{8b}$$

In the implementation part of this article (sec 3.6) and its illustration (sc 4), we will use Eq 21a and reserve use and study of Eq8b) for future theoretical work.

3.4 Variance-covariance of the number of kin

We decompose the variance of $\tilde{\mathbb{K}}_j$ according to (c_a, t):

$$\boxed{\boldsymbol{Var}_j^*(\tilde{\mathbb{K}}) = \underbrace{\sum_{c_a,t} P_j(c_a,t)\boldsymbol{Var}^*_{(c_a)}(\tilde{\mathbb{D}}^{(t)})}_{\boldsymbol{Var}_{des}(\tilde{\mathbb{K}}_j)} + \underbrace{\left(\sum_{c_a,t} P_j(c_a,t)\mathbb{D}^{(t)}_{c_a}\mathbb{D}^{(t)T}_{c_a}\right) - \mathbb{K}_j\mathbb{K}_j^T}_{\boldsymbol{Var}_{anc}(\tilde{\mathbb{K}}_j)}} \tag{9}$$

Here $\boldsymbol{Var}^*_{anc}(\tilde{\mathbb{K}}_j)$ is the component of $\boldsymbol{Var}^*(\tilde{\mathbb{K}}_j)$ caused by stochasticity in the (c_a, t) and $\boldsymbol{Var}^*_{des}(\tilde{\mathbb{K}}_j)$ is the component caused by variance of the different types of descendants. It can be computed by:

$$\mathbf{Var}^*(\tilde{\mathbb{D}}^{(t)}) = \mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{y}^{(t)}) + \mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{s}^{(t)}) + \mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) \tag{10}$$

which can itself be computed via induction over t, or via $\boldsymbol{Var^*}(\tilde{\mathbb{M}}^t)$ itself obtained by induction (Eq 66):

$$\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{y}^{(t)}) = (\mathbb{M}\otimes\mathbb{M})^{t-1}\boldsymbol{Var}^*(\tilde{\mathbb{S}})\mathbb{J}^* + \boldsymbol{Var}^*(\tilde{\mathbb{M}}^{t-1})\mathbb{S}\mathbb{J}^* \quad \text{if } t > 0 \text{ and } \mathbf{0} \text{ if } t = 0 \tag{11a}$$

$$\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{s}^{(t)}) = (\mathbb{M}\otimes\mathbb{M})^{t-1}\boldsymbol{Var}^*(\tilde{\mathbb{F}^*})\mathbb{J}^* + \boldsymbol{Var}^*(\tilde{\mathbb{M}}^{t-1})\mathbb{F}^*\mathbb{J}^* \quad \text{if } t > 0 \text{ and } \mathbf{0} \text{ if } t = 0 \tag{11b}$$

$$\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) = (\boldsymbol{Var}^*(\tilde{\mathbb{M}}^t\tilde{\mathbb{F}})\mathbb{J}^* + \boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t+1)}))\mathbf{P_S} + \big((\mathbb{M}^t(\mathbb{F}\mathbb{J}^* + \mathbb{M}\mathbb{D}\mathbb{o}^{(0)}))\otimes(\mathbb{M}^t(\mathbb{F}\mathbb{J}^* + \mathbb{M}\mathbb{D}\mathbb{o}^{(0)}))\big)\boldsymbol{Var}^*\big(\tilde{\mathbf{P_S}}\big) \tag{11c}$$

with $\lim_{t\to\infty}\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) = \mathbf{0}$. The inductions in Eq.11a and 11b are classical forward inductions, whilst Eq.11c is a backward induction that requires setting $\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t')}) = \mathbf{0}$ for some large t'.

3.4.1 Decomposition of Variance into projective and genealogical stochasticity components

We can decompose Eq.11c further according to whether the variance stems from stochasticity in the forward projectors, we call it the projective component of individual stochasticity, or simply projective stochasticity or from stochasticity in the backward projectors, which we call genealogical stochasticity:

$$\begin{cases} \boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) = \boldsymbol{Var}^*_{(ps)}(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) + \boldsymbol{Var}^*_{(gs)}(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) \text{ with },\\ \boldsymbol{Var}^*_{(ps)}(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) = (\boldsymbol{Var}^*(\tilde{\mathbb{M}}^t\tilde{\mathbb{F}})\mathbb{J}^* + \boldsymbol{Var}^*_{(ps)}(\tilde{\mathbb{D}}\mathbb{o}^{(t+1)}))\mathbf{P_S}\\ \boldsymbol{Var}^*_{(gs)}(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) = \boldsymbol{Var}^*_{(gs)}(\tilde{\mathbb{D}}\mathbb{o}^{(t+1)})\mathbf{P_S} + \big((\mathbb{M}^t(\mathbb{F}\mathbb{J}^* + \mathbb{M}\mathbb{D}\mathbb{o}^{(0)}))\otimes(\mathbb{M}^t(\mathbb{F}\mathbb{J}^* + \mathbb{M}\mathbb{D}\mathbb{o}^{(0)}))\big)\boldsymbol{Var}^*\big(\tilde{\mathbf{P_S}}\big) \end{cases} \tag{12}$$

As $\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{y}^{(t)})$ and $\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{s}^{(t)})$ are caused by projective stochasticity only, this allows to write

$$\begin{cases} \boldsymbol{Var}^*_j(\tilde{\mathbb{K}}) = \boldsymbol{Var}^*_{(ps)j}(\tilde{\mathbb{K}}) + \boldsymbol{Var}^*_{(gs)j}(\tilde{\mathbb{K}}), \text{ with} \\ \boldsymbol{Var}^*_{(ps)j}(\tilde{\mathbb{K}}) = \sum_{c_a,t} P_j(c_a,t)\left(\boldsymbol{Var}^*_{(ps)c_a}(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) + \boldsymbol{Var}^*_{(c_a)}(\tilde{\mathbb{D}}\mathbb{y}^{(t)}) + \boldsymbol{Var}^*_{(c_a)}(\tilde{\mathbb{D}}\mathbb{s}^{(t)})\right) \\ \boldsymbol{Var}^*_{(gs)j}(\tilde{\mathbb{K}}) = \sum_{c_a,t} P_j(c_a,t)\boldsymbol{Var}^*_{(gs)c_a}(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) + \underbrace{\sum_{c_a,t} P_j(c_a,t)(\mathbb{D}^{(t)}_{c_a,1})(\mathbb{D}^{(t)}_{c_a,1})^T - \mathbb{K}_j\mathbb{K}_j^T}_{\boldsymbol{Var}_{anc}(\tilde{\mathbb{K}}_j)} \end{cases} \tag{13}$$

In this decomposition, $\boldsymbol{Var}_{ps}(\tilde{\mathbb{K}}_j)$ corresponds to the variance in the number of kin caused by "projective stochasticity", that is by stochasticity in fertility (both in the number of offspring produced and in the state that they are produced in) and survival (both stochasticity in survival per se, and in the transitions towards future states). $\boldsymbol{Var}_{ps}(\tilde{\mathbb{K}}_j)$ corresponds to what population ecologists call individual stochasticity, which aggregates at the population level as demographic stochasticity and which affects the annual growth rate, as measured by demographic variance, itself affecting (negatively) the long term growth rate.

It is of note that even in a model without individual stochasticity, that is, where all individuals have the same exact life trajectories (age at death, fertility schedule, net reproductive rate etc...), the variance in the number of kin can be non-zero, because of $\boldsymbol{Var}_{gs}(\tilde{\mathbb{K}}_j)$ which embeds "genealogical stochasticity", that is stochasticity in the genealogy of individuals, which causes two individual in the same state at the same time-step to have mothers, grand-mother, etc. of different ages and different states.

3.5 Unstructured Kinship

The kinship stochastic matrix $\tilde{\mathbb{K}}$ and its deterministic version $\mathbb{K}$ are structured by class and generation, that is, by class and type of kin. It is of course also possible to consider the unstructured kinship of this structured population, that is, to consider the number of kin, whatever their class, for *ego* taken at random in the entire population.

3.5.1 Process

Let $\tilde{\mathbb{K}}^u_{g_i,g_j}$ be the number of (g_i,g_j)-kin of an individual taken at random in the population. At the stable-state, the probability for *ego* to be in class c_j is w_{c_j} and the number of its g_i kin is the sum of the number of the c_i, g_i kin over all c_i: $\sum_{c_i}\tilde{\mathbb{K}}_{(c_i,g_i),(c_j,g_j)}$ and therefore

$$\tilde{\mathbb{K}}^u_{g_j} = \tilde{\mathbb{K}}^u\boldsymbol{e}_{g_j} = \mathbb{J}_y\tilde{\mathbb{K}}_{c_j,g_j} = \mathbb{J}_y\tilde{\mathbb{K}}\boldsymbol{e}_{c_j,g_j} \text{ with probability } w_{c_j} \tag{14}$$

where $\mathbb{J}_y = (\mathbf{I}_{g_m} \otimes \mathbf{1}_s^T)$ is the $g_m \times sg_m$ matrix where the ith $g_m \times s$ block is empty but for the ith row filled with 1s.

3.5.2 Expectation

In terms of expectations, we can therefore write directly from Eq. 14: $\mathbb{K}^u_{g_j} = \sum_{c_j} w_{c_j} \mathbb{J}_y \mathbb{K} \boldsymbol{e}_{c_j,g_j}$ and so:

$$\mathbb{K}^u_{g_i,g_j} = \sum_{c_i,c_j} w_{c_j} \mathbb{K}_{(c_i,g_i),(c_j,g_j)} \qquad \text{and} \qquad \mathbb{K}^u = \mathbb{J}_y \mathbb{K} \mathbb{J}_w \tag{15}$$

with $\mathbb{J}_w = (\mathbf{I}_{g_m} \otimes \boldsymbol{w})$, the $sg_m \times g_m$ matrix where the ith $s \times g_m$ block is empty but for the ith column equal to $\boldsymbol{w}$.

3.5.3 *PGF*

We turn the process definition in Eq. 14 into *pgf* form: $\varphi_{\tilde{\mathbb{K}}^u_{g_j}} = \sum_{c_j} w_{c_j} \varphi_{\mathbb{J}_y \tilde{\mathbb{K}}_{c_j,g_j}} = \sum_{c_j} w_{c_j} \varphi_{\tilde{\mathbb{K}}_{c_j,g_j}} \circ \boldsymbol{\varphi}_{\mathbb{J}_y}$ and therefore

$$\boldsymbol{\varphi}_{\tilde{\mathbb{K}}^u} = (\boldsymbol{\varphi}_{\tilde{\mathbb{K}}} \circ \boldsymbol{\varphi}_{\mathbb{J}_y}) \mathbb{J}_w$$

3.5.4 Variance

we apply decomposition according to the probabilities, as per Eq.9:

$$\boldsymbol{Var}(\tilde{\mathbb{K}}_{u_{g_j}}) = \sum_{c_j} w_{c_j} \boldsymbol{Var}(\tilde{\mathbb{K}}_{u\,g_j}) + \sum_{c_j} w_{c_j} \mathbb{K}_{u\,g_j} \mathbb{K}_{u\,g_j}^T - \mathbb{K}_u \mathbb{K}_u^T$$

3.5.5 The unstructured kinship matrix is symmetrical

A key new result from this manuscript, is that the unstructured kinship matrix $\mathbf{K_u}$ is symmetrical: $\mathbf{K_u} = \mathbf{K_u}^T$, that we demonstrate in sec 7.10. This is equivalent to

$$\forall g_i, g_j, \mathbf{K_u}(g_i, g_j) = \mathbf{K_u}(g_j, g_i) \tag{16}$$

Theorem 1. For *ego* drawn at random in the population at stable state, its expected number of (g_i, g_j)-kin is equal to its expected number of (g_j, g_i)-kin.

In other words, the probability for an individual to have its mother alive is equal to the expected number of its daughters, and the expected number of nieces is equal to that of aunts! This result, can be fathomed intuitively: a g_i, g_j kin relationship for *ego* corresponds to a g_j, g_i for the kin. In a population with n individuals, among the $n(n-1)$ (directional) relationship pairs, there will therefore be as many g_i, g_j-kin relationships as g_j, g_i ones: $n_{g_i,g_j} = n_{g_j,g_i}$. At the individual level, focusing on the expected number of kin, this means that $\mathbf{K_u}(g_i, g_j) = \frac{n_{g_i,g_j}}{n(n-1)} = \frac{n_{g_j,g_i}}{n(n-1)} = \mathbf{K_u}(g_i, g_j)$. This symmetry is absent from $\mathbb{K}$ itself, but present in $\mathbb{K}_w$ the matrix of same size than $\mathbb{K}$ and which $(c_i, g_i), (c_j, g_j)$ element corresponds to $\mathbb{K}_{(c_i,g_i),(c_j,g_j)} w_{c_j}$ (see SM7.10)

3.5.6 Unstructured kinship of structured population vs kinship of unstructured population

In the multitrait theoretical framework developed by Coste et al. (2017), see appendix 6.3, we say that $\mathbf{K_u}$ is (*state* × *generation*)-matrix $\mathbb{K}$, *folded* over trait *state*. The kinship matrix is a new object to which one can apply the Trait Level Analysis developed in that paper, which consists in yielding information on the processes embedded in traits (see 6.3) by comparing the model with the trait to its folded version on that trait. In the case of the kinship matrix, by comparing $\mathbf{K_u}$ with the kinship matrix obtained from $\mathbf{M}$ folded over trait *state*, $\mathbf{M_u}$, which is simply the sum of two scalars $\mathbf{M_u} = f_u + s_u$ corresponding to the fertility and survival rates of any individual in the unstructured population (all individuals are identical), we can understand the effect of the structure embedded in *state*, that is, of the individual heterogeneity of that population, on kinship. Indeed, it is important to note that the unstructured kinship $\mathbf{K_u}$ is not equal to the kinship of the unstructured population, as it does account for the structure in its construction. The analysis of the differences between these two outputs corresponds to a "trait-level analysis" of the effects of individual heterogeneity on kinship (see app.6.3).

3.6 In practice/Implementation

The process described in Eq. 4 is an infinite sum (over t, of the descendant processes), which is reflected in the computation of the expectations, *pgf*s and variance (Eqs 6 , 7a and 9). The sum converges in general (see SM7.9), however in practice the expectations, variances and *pgf*s need finite sum to be computed exactly. We first consider the case where there is a maximum age embedded in the model, that is where there exists ω such that $\mathbf{S}^{\omega} = \mathbf{0}$ (but $\mathbf{S}^{\omega-1} \neq \mathbf{0}$). This yields (see SM7.7.7):

3.6.1 Expectations

$$\boxed{\mathbb{K} = \sum_{t=0}^{\omega.(g_m-1)-1} \mathbb{M}^t \left(\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}} + \mathbb{F}^*\mathbb{J}\mathbb{P}_{\mathbb{F}}\right)\mathbb{P}^t} \tag{17}$$

pseudocode

```
• generate $\mathbf{P_S}$ and $\mathbf{P_F}$ (Eq.2) and $\mathbf{F}^*$ (SM7.4) from $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{F}}$ and, in particular their expectations $\mathbf{S}$ and $\mathbf{F}$.
• set $g_m$ the highest generation depth one wishes to explore.
• generate $\mathbb{F} = \mathbf{J_1} \otimes \mathbf{F}$, $\mathbb{F}^* = \mathbf{J_1} \otimes \mathbf{F}^*$, $\tilde{\mathbb{S}} = \mathbf{I}_{g_m} \otimes \tilde{\mathbf{S}}$, $\mathbb{P}_{\mathbb{S}} = \mathbf{I}_{g_m} \otimes \mathbf{P_S}$ and $\mathbb{P}_{\mathbb{F}} = \mathbf{J_1}^T \otimes \mathbf{F}$ from the building blocks (via the kronecker product function embedded in most programs), and then $\mathbb{M} = \mathbb{F} + \mathbb{S}$ and $\mathbb{P} = \mathbb{P}_{\mathbb{F}} + \mathbb{P}_{\mathbb{S}}$ as per Eq 3. Save these matrices as sparse matrices if the program allows it
• Compute $\mathbb{K}$ by
    – Computing $\mathbb{K}(0) = (\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}} + \mathbb{F}^*\mathbb{J}\mathbb{P}_{\mathbb{F}})$
    – Then all $\mathbb{K}(t) = \mathbb{M}\mathbb{K}(t-1)\mathbb{P}$ for $0 \leq t \leq \omega.(g_{max}-1)-1$ iteratively
    – Summing the $\mathbb{K}(t)$, as $\mathbb{K} = \sum_{t=0}^{\omega.(g_{max}-1)-1} \mathbb{K}(t)$ (from Eq.17)
```

3.6.2 Variance

$$\boldsymbol{Var}_{des}(\tilde{\mathbb{K}}_j) = \sum_{t=0}^{\omega.(g_m-1)} \sum_{c_a} P_j(c_a,t)\boldsymbol{Var}^*_{c_a}(\tilde{\mathbb{D}}^{(t)}) \tag{18a}$$

$$\boldsymbol{Var}_{anc}(\tilde{\mathbb{K}}_j) = \sum_{t=0}^{\omega.(g_m-1)} \sum_{c_a} P_j(c_a,t)\mathbb{D}^{(t)}_{c_a,1}{\mathbb{D}^{(t)}_{c_a,1}}^T - \mathbb{K}_j\mathbb{K}_j^T \tag{18b}$$

with

$$\begin{cases} \boldsymbol{Var}^*(\tilde{\mathbb{D}}_{\mathbb{O}}^{(\omega.(g_m-1))}) = \mathbf{0} & \text{,and } \forall 0 \leq t < \omega.(g_{max}-1) \\ \boldsymbol{Var}^*(\tilde{\mathbb{D}}_{\mathbb{O}}^{(t)}) = (\boldsymbol{Var}^*(\tilde{\mathbb{M}}^t\tilde{\mathbb{F}}) + \boldsymbol{Var}^*(\tilde{\mathbb{D}}_{\mathbb{O}}^{(t+1)}))\mathbb{P}_{\mathbb{S}} + \left(\left(\mathbb{M}^t(\mathbb{F}+\mathbb{M}\mathbb{D}_{\mathbb{O}}^{(0)})\right) \otimes \left(\mathbb{M}^t(\mathbb{F}+\mathbb{M}\mathbb{D}_{\mathbb{O}}^{(0)})\right)\right)\boldsymbol{Var}^*\left(\tilde{\mathbb{P}_{\mathbb{S}}}\right) \end{cases} \tag{19}$$

We provide implementation steps/pseudocode for the computation of the variance-covariance matrix of the number if kin in SM.7.8

3.6.3 *Pgf*

$$\varphi_{\tilde{\mathbb{K}}_j} = \sum_{t=0}^{\omega.(g_{max}-1)} \sum_{c_a} P_j(c_a,t)\varphi_{\tilde{\mathbb{D}}^{(t)}_{c_a}}, \qquad \text{with} \tag{20a}$$

$$\boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(t)}} = \underbrace{(\boldsymbol{\varphi}_{\tilde{\mathbb{S}}} \circ \bigcirc_{t-1} \boldsymbol{\varphi}_{\tilde{\mathbb{M}}})}_{\varphi_{\tilde{\mathbb{D}}_{\mathbb{Y}}(t)}} \bullet \underbrace{(\boldsymbol{\varphi}_{\tilde{\mathbb{F}^*}} \circ \bigcirc_{t-1} \boldsymbol{\varphi}_{\tilde{\mathbb{M}}})}_{\boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{S}}(t)}} \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{O}}^{(t)}} \tag{20b}$$

with

$$\boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{O}}^{(\omega.(g_{max}-1))}} = \mathbf{1} \text{ ,and } \forall 0 \leq t < \omega.(g_{max}-1) \tag{21a}$$

$$\boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{O}}^{(t)}} = \boldsymbol{\varphi}_{\tilde{\mathbb{P}_{\mathbb{S}}}} \circ ((\boldsymbol{\varphi}_{\tilde{\mathbb{F}}} \circ \boldsymbol{\varphi}_{\tilde{\mathbb{M}}^t}) \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{O}}^{(t+1)}}) \tag{21b}$$

pseudocode

- compose $\boldsymbol{\varphi}_{\tilde{\mathbb{M}}}$ with itself k times to obtain $\boldsymbol{\varphi}_{\tilde{\mathbb{M}}^{(k)}} = \bigcirc_k \boldsymbol{\varphi}_{\tilde{\mathbb{M}}}$ for $0 \leq k \leq \omega(g_m-1)-1$.
- From this, get $\boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{Y}}^{(t)}} = \boldsymbol{\varphi}_{\tilde{\mathbb{S}}} \circ \bigcirc_{t-1} \boldsymbol{\varphi}_{\tilde{\mathbb{M}}}$ and $\boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{S}}^{(t)}} = \boldsymbol{\varphi}_{\tilde{\mathbb{F}^*}} \circ \bigcirc_{t-1} \boldsymbol{\varphi}_{\tilde{\mathbb{M}}}$ for $1 \leq t \leq \omega(g_m-1)$, adding the boundary values $\boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{Y}}^{(0)}} = \boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{S}}^{(0)}} = \mathbf{1}$.
- From the boundary value $\boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{O}}^{(\omega(g_m-1))}} = \mathbf{1}$ and $\boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{O}}^{(t)}} = \boldsymbol{\varphi}_{\tilde{\mathbb{P}_{\mathbb{S}}}} \circ ((\boldsymbol{\varphi}_{\tilde{\mathbb{F}}} \circ \boldsymbol{\varphi}_{\tilde{\mathbb{M}}^t}) \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{O}}^{(t+1)}})$, get $\boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{O}}^{(t)}}$ for $0 \leq t \leq \omega(g_m-1)$ by backward induction.
- Multiply these functions term-wise to get $\boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(t)}} = \boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{Y}}^{(t)}} \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{S}}^{(t)}} \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{D}}_{\mathbb{O}}^{(t)}}$.
- Then from $\mathbb{J}_5 = ({\boldsymbol{e}_1^{(g_m)}}^T \otimes \mathbf{I}_s)$, obtain $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}} = \boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(0)}}\mathbb{J}_5 + \sum_{t=1}^{\omega(g_m-1)} \boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(t)}}\mathbb{J}_5\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}$.
- We can go further here and also generate the *pgf* of the unstructured number of kin: $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}^u} = (\boldsymbol{\varphi}_{\tilde{\mathbb{K}}} \circ \boldsymbol{\varphi}_{\mathbb{J}_y})\mathbb{J}_w$

The computation of *pgf*s can be made easier by using functional/formal programming softwares (see SM7.12)

3.6.4 General case without embedded maximum age

In the case where there is no maximum age *incorporated* into the model, that is $\forall t$, $\mathbf{S}^t \neq \mathbf{0}$, which will be the case for most models that are not structured by *age*, then the researcher is left with two options.

maximum age known If the individuals in the population are known not to survive past age ω, despite $\mathbf{S}^{\omega} \neq \mathbf{0}$, $\tilde{\mathbf{F}} + \tilde{\mathbf{S}}$ is not a consistent projection model for the population. Instead, one should consider model $\tilde{\mathbb{F}}^{\dagger} + \tilde{\mathbb{S}}^{\dagger}$, structured by the original classes of the model and trait *age*, which construction is described in Appendix6.4. Then the expectation, *pgf* and variance of the number of kin can be computed exactly via equations 17-21 using $\tilde{\mathbb{F}}^{\dagger}$ and $\tilde{\mathbb{S}}\dagger$ as inputs.

maximum age unknown If there is no known maximum age (because for instance, the researcher did not collect the data themselves and is only provided by population dynamics that is not structured by *age*), then the expectations, variance and distribution of number of kin will have to be approximated. For such an approximation, one can use the computations for the embedded maximum age case as a heuristic. This approximation will be based on a parameter ω and corresponds, again, to equations 17-21, as shown in (Coste et al., 2021) focusing on expectations, the error will be all the more reduced that $\mathbf{S}^{\omega}$ is small (for any matrix norm).

3.7 General case for reproduction process

3.7.1 Process

In the general case, we have to account, w.r.t the number of descendants via same litter sisters, for both the state of the g_j-ancestor at it gives birth and the state of the new born $g_j - 1$ ancestor. This therefore requires accounting for the process yielding the number of same-litter sisters, providing one birth in class c_b, $\tilde{\mathbb{F}}^{(c_b)}$ which is described and the distribution of provided in SM7.4. From this, the process is

$$\tilde{\mathbb{K}}_j = \tilde{\mathbb{K}}_{c_j,g_j} = \begin{cases} \tilde{\mathbb{D}}_{\mathbb{o}}^{(0)} \text{ with probability } P_j(c_a, c_b, 0) \\ \underbrace{\tilde{\mathbb{M}}^{t-1}\tilde{\mathbb{S}}}_{\tilde{\mathbb{D}}_{\mathbb{y}}^{(t)}} + \underbrace{\tilde{\mathbb{M}}^{t-1}\tilde{\mathbb{F}}^{(c_b)}}_{\tilde{\mathbb{D}}_{\mathbb{s}}^{(c_b)(t)}} + \underbrace{\sum_{\alpha>0} \tilde{\mathbb{M}}^{t-1+\alpha}\tilde{\mathbb{F}}\tilde{\mathbb{P}}_{\mathbb{S}}^{\alpha}}_{\tilde{\mathbb{D}}_{\mathbb{o}}^{(t)}} \text{ with probability } P_j(c_a, c_b, t), \text{ for } t > 0 \end{cases}, \text{ with} \tag{22}$$

$$P_{c_j,g_j}(c_a, c_b, t) = P_j(c_a, c_b, t) = \begin{cases} \boldsymbol{e}_{c_a}^T \mathbb{P}_{\mathbb{F}} \boldsymbol{e}_{(c_b,2)} \boldsymbol{e}_{(c_b,2)}^T \mathbb{P}^{t-1} \boldsymbol{e}_{(c_j,g_j)} & \text{if } t > 0 \\ \boldsymbol{e}_{c_a}^T \boldsymbol{e}_{(c_j,g_j)} & \text{if } t = 0 \end{cases}, \tag{23}$$

3.7.2 Expectation

Similarly to sec.3.2 we write: $\mathbb{K} = \underbrace{\left(\sum_{\alpha>0} \mathbb{M}^{\alpha-1}\mathbb{F}\mathbb{P}_{\mathbb{S}}{}^{\alpha}\right)}_{\mathbb{D}\mathbb{o}^{(0)}} \mathbb{J} + \sum_{c_a,c_b,t>0}\left(\mathbb{M}^{t-1}\mathbb{S}+\mathbb{M}^{t-1}\mathbb{F}^{(c_b)}+\mathbb{M}^{t}\mathbb{D}\mathbb{o}^{(0)}\right)\boldsymbol{e}_{c_a}^{T}\mathbb{P}_{\mathbb{F}}\boldsymbol{e}_{(c_b,2)}\boldsymbol{e}_{(c_b,2)}^{T}\mathbb{P}^{t-1}$.

This yields (see SM7.7.8):

$$\mathbb{K} = \sum_{t=0}^{\infty} \mathbb{M}^{t}\left(\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}} + \mathbb{Z}\right)\mathbb{P}^{t}, \tag{24}$$

noting that in the three common cases, we have $\mathbb{Z} = \mathbb{F}^{*}\mathbb{J}\mathbb{P}_{\mathbb{F}}$.

3.7.3 Distribution

From Eqs.22 and 23 turned into *pgf* form, we get:

$$\varphi_{\tilde{\mathbb{K}}_j} = \sum_{c_a,c_b,t} P_j(c_a,c_b,t)\varphi_{\tilde{\mathbb{D}}_{c_a}^{(c_b),(t)}} \tag{25}$$

with

$$\varphi_{\tilde{\mathbb{D}}_{c_a}^{(c_b),(t)}} = \varphi_{\tilde{\mathbb{D}}\mathbb{y}_{c_a}^{(t)}} \bullet \varphi_{\tilde{\mathbb{D}}\mathbb{s}_{c_a}^{(c_b),(t)}} \bullet \varphi_{\tilde{\mathbb{D}}\mathbb{o}_{c_a}^{(t)}} \tag{26}$$

In this expression, the only new term is

$$\varphi_{\tilde{\mathbb{D}}\mathbb{s}_{c_a}^{(c_b),(t)}} = \left(\boldsymbol{\varphi}_{\mathbb{F}^{\tilde{c}_b}} \circ \bigcirc_{t-1} \boldsymbol{\varphi}_{\tilde{\mathbb{M}}}\right)_{\boldsymbol{\varphi}} \tag{27}$$

where $\boldsymbol{\varphi}_{\mathbb{F}^{\tilde{c}_b}}$, the *pgf* of $\mathbb{F}^{\tilde{c_b}}$ is required. It can computed from Eq.40 in SM7.4.

4 Illustration

Here we illustrate our method on a population of ground squirrels *Spermophilus dauricus*, structured by stages. It is an extended Leslie model, i.e. an age-structured model with a class that regroups all individuals older than a certain age. Here, classes 1 and 2 correspond to individuals that are about to turn 1 and 2 year(s) old, respectively, and class 3 corresponds to older individuals. We want to compute the expectation, variance and distribution of the number of all $(g_i - q_j)$-kin for all g_j and $g_i \leq g_m = 3$ and for all pair of classes; there are $g_m^2 - 1 = 8$ such kin: the mother (1,2), grand-mother (1,3), the daughters (2,1), grand-daughters (3,1), the sisters (2,2), maternal aunts (2,3), nieces (3,2) and cousins (3,3). From the life table for that population rom Luo and Fox (1990), we get the following projection matrix: $\mathbf{M} = \mathbf{S} + \mathbf{F} = \begin{bmatrix} 0.23 & 0.51 & 0.84 \\ 0.57 & 0 & 0 \\ 0 & 0.54 & 0.46 \end{bmatrix}$.

From, the asymptotic growth rate $\lambda = 1.00$ and the stable class distribution $\mathbf{w} = (0.47, 0.27, 0.27)$, we can construct $\mathbf{P}$,$\mathbf{P_F} = \begin{bmatrix} 0.23 & 0 & 0 \\ 0.29 & 0 & 0 \\ 0.48 & 0 & 0 \end{bmatrix}$ and $\mathbf{P_S} = \begin{bmatrix} 0 & 1.00 & 0 \\ 0 & 0 & 0.54 \\ 0 & 0 & 0.46 \end{bmatrix}$. Without further information, we assume that the fertility is Poisson and therefore $\tilde{\mathbb{F}^*} = \tilde{\mathbb{F}}$ and in particular $\mathbb{F}^* = \mathbb{F}$. This also implies that from $\mathbf{M} = \mathbf{F} + \mathbf{S}$, we know $\tilde{\mathbf{M}} = \tilde{\mathbf{F}} + \tilde{\mathbf{S}}$ (see SM7.1).

4.1 Building blocks

Expectations From these, we can construct $\mathbb{S} = \mathbf{I}_{g_m} \otimes \mathbf{S}$, $\mathbb{F} = \mathbf{J_1} \otimes \mathbf{F}$, $\mathbb{P}_{\mathbb{S}} = \mathbf{I} \otimes \mathbf{P_S}$ and $\mathbb{P}_{\mathbb{F}} = \mathbf{J_1}^T \otimes \mathbf{F}$. That is, $\mathbb{M} = \mathbb{S} + \mathbb{F} = \left[\begin{array}{c|c|c} \mathbf{S} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{F} & \mathbf{S} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{F} & \mathbf{S} \end{array}\right]$ and $\mathbb{P} = \mathbb{P}_{\mathbb{S}} + \mathbb{P}_{\mathbb{F}} = \left[\begin{array}{c|c|c} \mathbf{P_S} & \mathbf{P_F} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{P_S} & \mathbf{P_F} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{P_S} \end{array}\right]$

Pgf From Eq.46 (SM7.5), we have $\varphi_{\tilde{\mathbb{S}}_j}(\boldsymbol{x}) = \varphi_{\tilde{\mathbb{S}}_{c_j,g_j}}(\boldsymbol{x}) = \sum_{c_i} \mathbf{S}_{c_i,c_j}(x_{c_i,g_j} - 1) + 1$. This yields for instance $\varphi_{\tilde{\mathbb{S}}_{1,1}}(\boldsymbol{x}) = \sum_{c_i} \mathbf{S}_{c_i,1}(x_{c_i,1} - 1) + 1 = \mathbf{S}_{2,1}(x_{2,1} - 1) + 1 = 0.57x_{2,1} + 0.43$. We regroup these into $\boldsymbol{\varphi}_{\tilde{\mathbb{S}}} = (\varphi_{\tilde{\mathbb{S}}_{1,1}}, \varphi_{\tilde{\mathbb{S}}_{2,1}}, \dots, \varphi_{\tilde{\mathbb{S}}_{s,gm}})$ and perform the exact same operation with $\mathbf{P_S}$ to obtain $\boldsymbol{\varphi}_{\tilde{\mathbf{P_S}}}$. From equation 48, we have $\varphi_{\tilde{\mathbb{F}}_{c_j,g_j}}(\boldsymbol{x}) = exp(\sum_{c_i} F_{c_i,c_j}(x_{c_i,g_j+1} - 1))$, which in our case means that $\varphi_{\tilde{\mathbb{F}}_{c_j,g_m=3}}(\boldsymbol{x}) = 1$ and $\varphi_{\tilde{\mathbb{F}}_{c_j,g_j<3}}(\boldsymbol{x}) = exp((F_{1,c_j}(x_{1,g_j+1} - 1)))$. And thus, for instance, $\varphi_{\tilde{\mathbb{F}}_{2,2}}(\boldsymbol{x}) = exp(F_{1,2}(x_{1,3} - 1)) = e^{0.51x_{1,3} - 0.51}$. We have $\boldsymbol{\varphi}_{\tilde{\mathbb{F}^*}} = \boldsymbol{\varphi}_{\tilde{\mathbb{F}}}$. Remember that we index, for simplicity, according to the 2 dimensions, class and generations, of the population vector, even if this one is vectorized, where $(c_i, g_i) = i$ iff $i = c_i + sg_i$. For the structure of the population vectors in this illustration this means:

i	$g_i = 1$	$g_i = 2$	$g_i = 3$
$c_i = 1$	1	4	7
$c_i = 2$	2	5	8
$c_i = 3$	3	6	9

Therefore $\varphi_{\tilde{\mathbb{F}}_{2,2}}$ is the 5th entry of $\boldsymbol{\varphi}_{\tilde{\mathbb{F}}}$ and $x_{1,3}$ corresponds to x_7. From $\boldsymbol{\varphi}_{\tilde{\mathbb{S}}}$ and $\boldsymbol{\varphi}_{\tilde{\mathbb{F}}}$, we get $\boldsymbol{\varphi}_{\tilde{\mathbb{M}}} = \boldsymbol{\varphi}_{\tilde{\mathbb{S}}} \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{F}}}$, that is $\boldsymbol{\varphi}_{\tilde{\mathbb{M}}}$ is such that $\varphi_{\tilde{\mathbb{M}}_j}(\boldsymbol{x}) = \varphi_{\tilde{\mathbb{S}}_j}(\boldsymbol{x})\varphi_{\tilde{\mathbb{F}}_j}(\boldsymbol{x})$.

Variance From equation 47, we have $\boldsymbol{Var}(\tilde{\mathbb{S}}_j)_{i,k} = \mathbb{S}_{i,j}(1 - \mathbb{S}_{i,j})$ if $i = k$ and $\mathbb{S}_{i,j}\mathbb{S}_{k,j}$ otherwise. In our case, this means that $\boldsymbol{Var}(\tilde{\mathbb{S}}_j)$ is diagonal for non-zero elements only for states with generation g_j and $\boldsymbol{Var}(\tilde{\mathbb{S}}_j)_{(c_i,g_j),(c_i,g_j)} = \mathbb{S}_{(c_i,g_j),(c_j,g_j)}(1 - \mathbb{S}_{(c_i,g_j),(c_j,g_j)}) = \mathbf{S}_{c_i,c_j}(1 - \mathbf{S}_{c_i,c_j})$. We then vectorize each $\boldsymbol{Var}(\tilde{\mathbb{S}}_j)$ as the jth column of $\boldsymbol{Var}^*(\tilde{\mathbb{S}})$. We operate similarly to get $\boldsymbol{Var}^*(\tilde{\mathbf{P_S}})$. From equation 49, we have $\boldsymbol{Var}(\tilde{\mathbb{F}}_j) = F_{1,c_j}\boldsymbol{e}_{(1,g_j+1)}\boldsymbol{e}^T_{(1,g_j+1)}$ for $g_j < 3$ and 0 if $g_j = 3$. This allows to quickly build $\boldsymbol{Var}^*(\tilde{\mathbb{F}^*}) = \boldsymbol{Var}^*(\tilde{\mathbb{F}})$. And from these we get $\boldsymbol{Var}^*(\tilde{\mathbb{M}}) = \boldsymbol{Var}^*(\tilde{\mathbb{S}}) + \boldsymbol{Var}^*(\tilde{\mathbb{F}})$.

4.2 Expectations

Here, we consider that no individual survives beyond the age of $\omega = 6$, the maximum age reported in Luo and Fox (1990). In order to be able to display some results, we opt to keep the structure with 3 classes, instead of "correcting" the projection models by zeroing survival after age 6. This means that we use the suggested heuristic of section 3.6 from which we obtain approximations for the expectations, variance and *pgf*. For expectations, Eq.3.6.3 yields the kinship matrix:

$$\mathbb{K} = \sum_{t=0}^{\omega.(g_m-1)=12} \mathbb{M}^t \left(\mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}} + \mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \mathbb{F}^*\mathbb{J}\mathbb{P}_{\mathbb{F}}\right)\mathbb{P}^t = \begin{array}{c} \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & \color{red}{0} & \color{red}{0} & \color{red}{0} & 0 & 0 & 0 \\ 0 & 0 & 0 & \color{red}{0.13} & \color{red}{0} & \color{red}{0} & 0 & 0 & 0 \\ 0 & 0 & 0 & \color{red}{0.38} & \color{red}{0.24} & \color{red}{0.08} & 0.11 & 0.05 & 0.02 \\ \hline 0 & 0.23 & 0.66 & 0.60 & 0.38 & 0.14 & 0.18 & 0.09 & 0.03 \\ 0 & 0 & 0.24 & 0.22 & 0.34 & 0.16 & 0.18 & 0.10 & 0.03 \\ 0 & 0 & 0.08 & 0.08 & 0.16 & 0.22 & \color{green}{0.20} & 0.19 & 0.11 \\ \hline 0 & 0 & 0.19 & 0.18 & 0.32 & 0.35 & \color{blue}{0.33} & \color{blue}{0.30} & \color{blue}{0.18} \\ 0 & 0 & 0.05 & 0.05 & 0.10 & 0.19 & \color{blue}{0.17} & \color{blue}{0.19} & \color{blue}{0.14} \\ 0 & 0 & 0.02 & 0.02 & 0.03 & 0.11 & \color{blue}{0.10} & \color{blue}{0.14} & \color{blue}{0.16} \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\}\text{kin is ca} \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\}\text{kin is daughter of ca} \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\}\text{kin is gdaughter of ca} \end{array} \\ \underbrace{\hphantom{0\quad 0.23\quad 0.66}}_{\textit{ego}\text{ is ca}} \quad \underbrace{\hphantom{0.60\quad 0.38\quad 0.14}}_{\textit{ego}\text{ is daughter of ca}} \quad \underbrace{\hphantom{0.18\quad 0.09\quad 0.03}}_{\textit{ego}\text{ is gdaughter of ca}} \end{array} \tag{28}$$

This matrix displays the expected number of all $(g_i - g_j)$-kin, structured by states, for g_i and g_j up to $g_m = 3$. In $\mathbb{K}(1,2)$, for instance (in red, the block in the first row and second column), we have the expected number of mothers, that is, the probability for *ego* to have its mother alive. This matrix indicates, for instance, that a yearling has a 13% probability that its mother is alive in class 2, and 38% that it is alive in class 3. In $\mathbb{K}(3,3)$, for instance (in blue, the bottom right block), we have the expected number of cousins. This matrix indicates, for instance, that a yearling will have on average 0.19 cousins in the yearling state.

Unstructured kinship From equation 15, we get the unstructured kinship matrix, which gives the expected number of any kin for an individual at random in the population:

$$\mathbf{K_u} = \begin{bmatrix} 0 & 0.32 & 0.07 \\ 0.32 & 0.80 & 0.41 \\ 0.07 & 0.41 & 0.58 \end{bmatrix} \tag{29}$$

Reading $\mathbf{K_u}$, we know that *ego*, at random in the population, is expected to have 0.80 sisters, 0.58 cousins, 0.41 nieces and aunts and 0.07 grand-daughters and grandmothers (that is, the probability that its grandmother is alive is 0.07)

Kinship of unstructured population In order to illustrate one of the uses of these calculations in ecology and evolution, we generate $\mathbf{M}^{fold} = \mathbf{F}^{fold} + \mathbf{S}^{fold}$ the unstructured model equivalent to $\mathbf{M} = \mathbf{F} + \mathbf{S}$ (see app.6.3). $\mathbf{M}^{fold}$ is a scalar, that is a 1×1 matrix which says that all individuals in the population have the same probability to survive $\mathbf{S}^{fold} = 0.53$ and the same expected number of offspring $\mathbf{S}^{fold} = 0.47$. This model has the same asymptotic growth rate than $\mathbf{M}$: $\lambda = \mathbf{M}^{fold} = 1.00$, as they model the same population. However, the kinship matrix of $\mathbf{M}^{fold}$, generated assuming similarly Poisson reproduction and $\omega = 6$, yields a different output:

$$\mathbf{K}(\mathbf{M}^{fold}) = \begin{bmatrix} 0 & 0.35 & 0.12 \\ 0.35 & 0.55 & 0.34 \\ 0.12 & 0.34 & 0.40 \end{bmatrix} \tag{30}$$

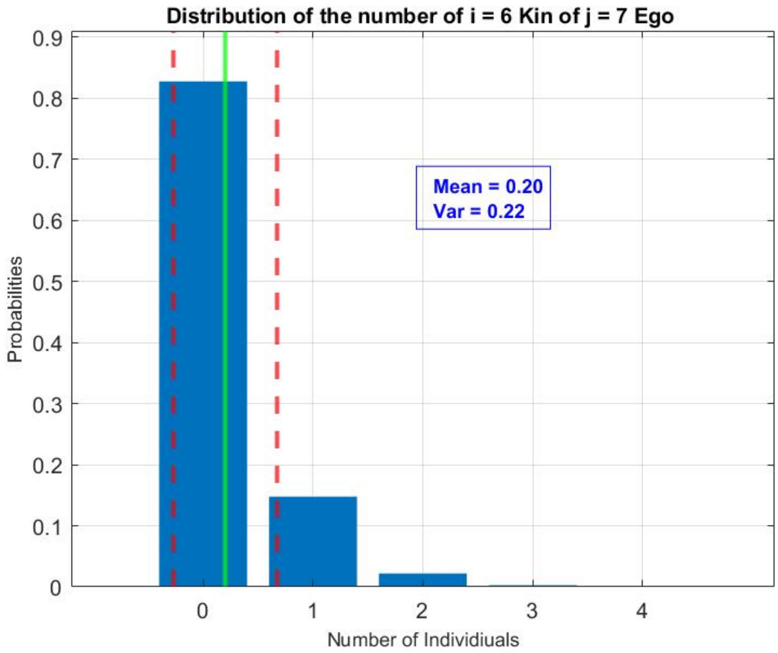

Figure 1: Probability distribution of $\tilde{\mathbb{K}}_{6,7}$

By comparing the kinship matrix of the unstructured population (Eq.30) and the unstructured kinship of the structured population (Eq.29), we can draw conclusions about the effect of the processes embedded in the classes of the model on kinship. Here the classes correspond to age classes (broadly speaking, the last one being open ended and maybe more akin to a stage), and they embed mostly, an increase in fertility with age and a decline in survival with age (senescence). The combination of these effects implies that the probabilities that your direct ancestors (mother, grandmother) will be alive and the number of your direct descendant (daughters, grand-daughters) is less than for the equivalent model without individual heterogeneity, where all individuals have the same vital rates (App.6.3). To the contrary, it implies that the number of other kin, that descendants of ancestors (the sisters, nieces, cousins and aunts) are more numerous. In other words, there is a trade-off between the number of direct ancestors/descendants and the number of other kin as there is a trade-offs between various life-histories, and a movement on the axis of the latter involves a change in the position in the former.

4.3 Distributions

structured kinship We follow the steps described in sec.3.6.3, to build $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}}$ from the *pgf*s of the building blocks (SM7.5). It is a vector of functions of the $x_{(c_i,g_i)}$ of dimension sg_m. By focusing on the c_j, g_j entry of $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}}$ and by setting all other unknowns to 1, but for a specific $x_{(c_i,g_i)}$, we get the uni-variate *pgf*, function of $x_{(c_i,g_i)}$, $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}_{i,j}}$. For $i = 6$ and $j = 7$, which corresponds to $g_i = 2$,$g_j = 3$,$c_i = 3$ and $c_j = 1$, $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}_{6,7}}$ yields, by successive derivations, the probabilities $P(\tilde{\mathbb{K}}_{6,7}) = n$, that is, the distribution of the number of aunts ($2-3$ kin) that are adults (age 3 or older) for *ego* in class 1. We compute these up to $n = 7$, which corresponds to 99.99% of the set and display this probability distribution in Fig.1, together with its mean (which corresponds to $\mathbb{K}_{6,7}$, in green in Eq.28) and its variance.

unstructured kinship From $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}^u}$, a vector of dimension g_m of functions of $\boldsymbol{x} = (x_1, x_2, x_{3=g_m})$ where the g_j^{th} element is function $\varphi_{\tilde{\mathbb{K}}^u_{g_j}}(\boldsymbol{x}) = E(x_1^{\tilde{\mathbb{K}}^u_{1,g_j}} x_2^{\tilde{\mathbb{K}}^u_{2,g_j}} x_3^{\tilde{\mathbb{K}}^u_{3,g_j}})$, we can compute all (mixed factorial) moments and distributions. In particular one can get the (uni-variate) *pgf* $\varphi_{\tilde{\mathbb{K}}^u_{(g_i,g_j)}}$ of $\tilde{\mathbb{K}}^u_{g_i,g_j}$, the unstructured number of g_i, g_j-kin of *ego*, by setting respective arguments of $\varphi_{\tilde{\mathbb{K}}^u_{g_j}}$ equal to 1: For instance for $g_i = 2$, $\tilde{\mathbb{K}}^u_{2,g_j}(x) = \varphi_{\tilde{\mathbb{K}}^u_{g_j}} \underbrace{(1,x,1)}_{x \text{in the } g_i^{th} position}$. (see SM7.3). We generate these *pgf* for all $\leq 1 g_i \leq 3$ and $\leq 1 g_j \leq 3$, and from each, extract the probability distribution of the number of (unstructured) (g_i, g_j)-kin, that we display in Fig.3. Here again, those are computed up to $n = 7$, which corresponds to 99.99% of the set. For that reason, rounded at 2 digits, we get the exact same expectations for these distributions as for the full computation of $\mathbb{K}_u$ in Eq.29. Variances are more sensitive to distribution tails and therefore variances of truncated distributions as displayed in Fig.3, will be marginally smaller that the real variances (such as displayed in Fig.2).

4.4 Variances

We follow the procedure in SM7.8, and get $\boldsymbol{Var}^*(\tilde{\mathbb{K}}) = \boldsymbol{Var}^*_{ps}(\tilde{\mathbb{K}}) + \boldsymbol{Var}^*_{gs}(\tilde{\mathbb{K}})$. We display in Fig 2 Variance/Covariance of $\tilde{\mathbb{K}}_j$ for $j = 7$, that is for *ego* in generation 3 and class 1, decomposed into projective and genealogical stochasticity components. The covarying kin for an individual in j, correspond to the $\tilde{\mathbb{K}}_{i,j}$ for all $i = (c_i, g_i)$. In other words, for $g_j = 3$, the covarying kin are the $g_i - 3$ kin of all classes and all g_i, that is the grand mother, the aunts and the cousins. The realisation of the number of cousins is indeed related to the life trajectory of my grand mother and to the performances of my aunts, but not to how old my mother lived nor how many sisters she gave me. For the particular decomposition of variance for $j = 7$, we can see that the genealogical stochasticity has very little impact compared on the variance compared to projective stochasticity.

It is expected that projective stochasticity will have more impact in general than genealogical stochasticity, as the former corresponds to stochasticity in class, identity and number, whilst the latter corresponds to stochasticity in class and identity only. Indeed, via $\mathbf{P}$ every individual has exactly one ancestor at the previous time-step. The only stochasticity concerns its identity, it is the same individual (via $\mathbf{P_S}$) or its parent (via $\mathbf{P_F}$), and its class.

We also perform the computation of $\boldsymbol{Var}\mathbb{K}_u$, and we provide here the Variance-Covariance Matrix for $\tilde{\mathbb{K}}_{u\,g_j}$ for $g_j = 2$. This corresponds, to the variance-covariance of the number g_i-kin for $g_j = 2$, that is to the variance-covariance of the number of mothers (i.e., the probability that the mother is alive), of sisters and of nieces:

$$\boldsymbol{Var}(\tilde{\mathbb{K}}_{u\,g_j}) = \begin{bmatrix} 0.22 & 0.06 & -0.04 \\ 0.06 & 1.08 & 0.27 \\ -0.04 & 0.27 & 0.86 \end{bmatrix} \tag{31}$$

We can read in Eq.31 that the number of mother and sisters covary positively, as do the number of sisters and nieces, but that the number of mother and nieces covary negatively. The positive covariances are expected in general,when generation numbers are close, because of both numbers of kin covary positively with the fitness (survival and reproduction) of the ancestor: if my mother is alive, she may have given me more sisters until recently, and the more sisters i have, the more nieces i have.' This intuitive relationship gets eroded as generation difference increases, and can then be reversed. If my mother is still alive, it may indicate that she had my younf, and that my sisters are therefore very young or not born yet, which in turn implies that they have not produced a significant amount of their expected reproductive success.

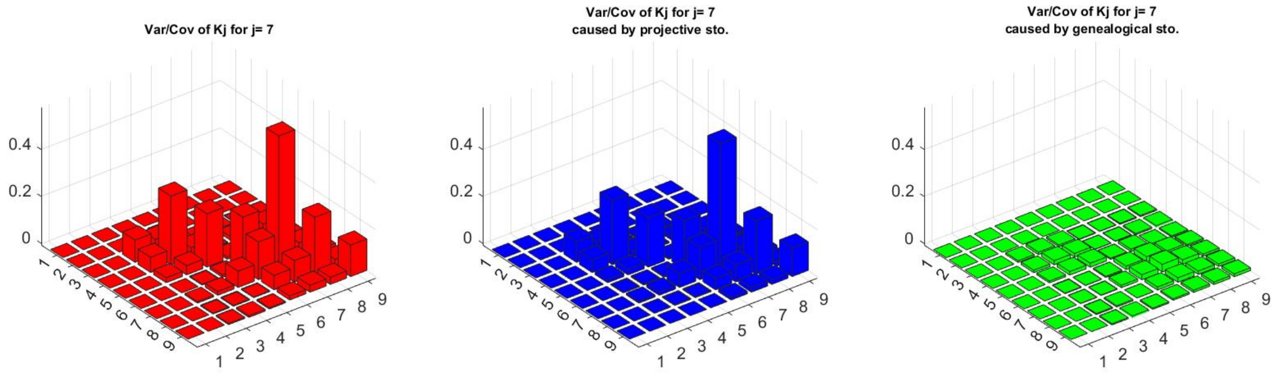

Figure 2: Variance/Covariance of $\tilde{\mathbb{K}}_j$ for $j = 4$, decomposed into projective and genealogical stochasticity components.

5 Conclusion / Discussion

In this article we provide formulas for the expectation, the variance/covariance and more generally the probability generating function for the number of kin in a structured population, for any kin. This number of kin is structured of kin is structured by the class of *ego*, the focal individual, and the class of kin but we show how to obtain the unstructured Kinship Matrix also which the number of kin (for any kin) for an individual taken at random in the population.

For the expectations this yields a formula that computes the Kinship Matrix, containing all kin structured by kin and class, for any arbitrary generation depth. This "kinship formula" is closed-form, it does not require interpedent equations representing particular kin relationships such as in (Goodman et al., 1974) or (Caswell, 2019), and is much simpler than that in (Coste et al., 2021) thanks to the novel generation structure approach. This not only allows easy and fast computation of the expected number of kin for any, even very complex, population projection matrix; this will also provide a path towards a possible theory of kinship, as it decomposed simply the number of kin via projection matrices of survival and fertility processes and their genealogical Markov chains.

The novel approach also allows to provide a formula for the variance-covariance of the number of all kin in all classes of *ego* in a given class and to decompose this variance-covariance according to projective stochasticity (called individual stochasticity in population ecology) and to genealogical stochasticity which characterises uncertainty about the ancestors of *ego*. It is important to note that, while in projective demography individual stochasticity, is only important for small populations, via its effect on the stochastic long term growth rate and demographic variance, it impacts the number of kin for populations of any size. This is because kinship demography focuses on the number of kin of *ego*, which is itself a "small" population. To the contrary, the formula provided here will only be valid for populations that are not "very small". Indeed for "very small" populations, even if vital rates (transitions) are constant in time, and have been for a while, it is unlikely that the population will be at the stable-state relative abundances because, here again, of individual stochasticity. The special stochasticity of kinship for very small populations is therefore a problem in itself that is yet to be tackled.

Finally the generation-structure approach, combined with a reasoning typical of multitype branching processes, allows us to provide a formula for the probability generating function of any kin, which provides the full distribution of the number of kin. The arguments for the interest of having a distribution (and the co-distributions amongst kin and classes) of the number of kin are similar to these provided by (Tuljapurkar et al., 2020) as they extend the computation of the expected lifetime reproductive output to its full distribution: expectations and variance-covariance provide useful information for kinship theory, but one needs many further moments to be able to understand and study phenonema that are crucial for kinship theory. One crucial such phenomenom is that, as for the distribution of the number of offspring in (Tuljapurkar et al., 2020), the number of kin, in general, seems to be zero-inflated with very large skew (see figure 3), with vast consequences for popualtion genetics and behavioral ecology for instance.

The simplicity of the kinship formula (Eq 6) allow a relatively easy extension of the model to 2-sex models and to changing environments (manuscripts in preparation) for the expected number of kin. It will certainly prove harder to provide simple outputs for further moments and the *pgf*s for models including both environmental and individual stochasticity in a 2-sex framework. However focusing on ad-hoc problematic, it will be possible to use the general approach developed in this manuscript, to more complex models.

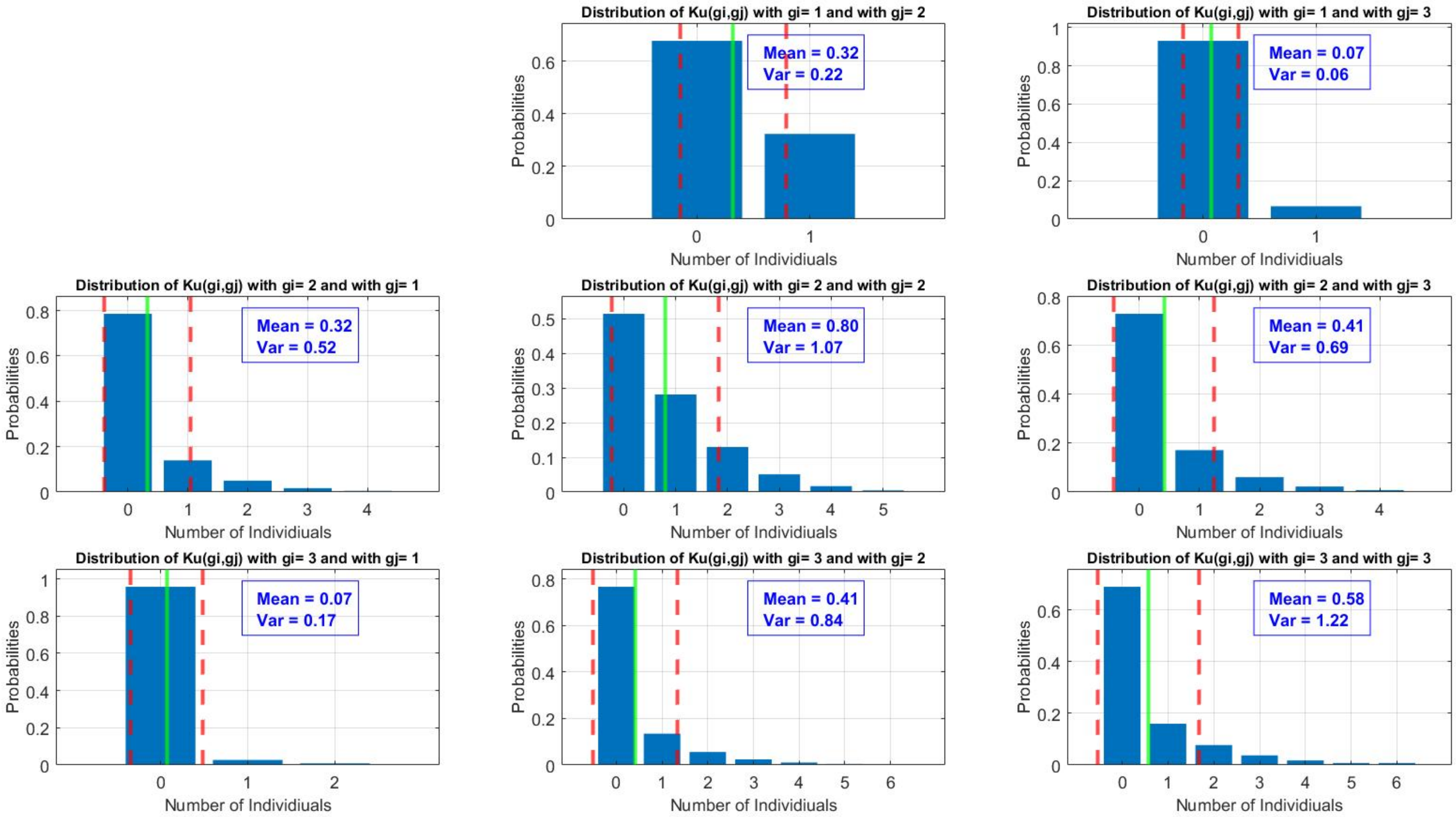

Figure 3: Distribution of the (unstructured) number of g_i, g_j-kin for $g_i, g_j \leq gm = 3$

Variable	Dimension	Description
s	1	number of classes in the structured population
$\tilde{\boldsymbol{n}}$	s	stochastic population vector
$\tilde{\mathbf{M}} = \tilde{\mathbf{F}} + \tilde{\mathbf{S}}$	$s \times s$	stochastic population projection model, decomposed into fertility and survival components.
$\tilde{\mathbb{F}}^{\star}$	$sg_m \times sg_m$	number of sisters of an offspring already born, as a function of the state of the mother
$\mathbf{M} = \mathbf{E}(\tilde{\mathbf{M}}) = \mathbf{F} + \mathbf{S}$	$s \times s$	deterministic population projection model, decomposed into fertility and survival components.
λ and $\boldsymbol{w}$	1 and $s \times 1$	maximum eigenvalue and associated right-eigenvector of $\mathbf{M}$ (such that $\sum_i w_i = 1$).
$\mathbf{P} = \mathbf{P_F} + \mathbf{P_S}$	$s \times s$	transition matrix of the genealogical Markov chain of $\mathbf{M}$, that is $P_{ij} = \frac{M_{ji}\, w_i}{\lambda\, w_j}$, decomposed into fertility and survival components (see Eq 2).
g_m	1	number of generations (investigated in the model)
ω	1	maximum age which means that $\mathbf{S}^{\omega} = \mathbf{0}$ (or is considered to as such).
$\otimes$	-	tensor (aka kronecker) product
$\boldsymbol{vec}()$	-	denotes the vec operator, stacking all columns of a matrix on top of one another
$\bullet$	-	denotes termwise product of vectors
$\bigcirc_k$	-	denotes k compositions of the same function (if $k = 0$, it is the identity)
$\mathbf{I}_n{=}\mathbf{I}$	$n \times n$	identity matrix, if obvious the index will be avoided.
$\mathbf{1}_n$	$n \times 1$	vector of ones .
$\mathbf{J_1}$	$g_m \times g_m$	matrix with ones in the sub-diagonal.
$\boldsymbol{e}_{(c_j, g_j)}$	$sg_m \times 1$	vector of 0s but for a 1 in position (c_j, g_j).
$\boldsymbol{e}_{g_j}$	$g_m \times 1$	vector of 0s but for a 1 in position g_j.
$\boldsymbol{e}_{c_j}$	$s \times 1$	vector of 0s but for a 1 in position c_j.
$\mathbb{J}^{*} = (\boldsymbol{e}_1^{(g_m)} \otimes \mathbf{I}_s)$	$sg_m \times s$	zero-matrix but for the diagonal of the first block made of 1s
$\mathbb{J}^{*T} = J5 = ({\boldsymbol{e}_1^{(g_m)}}^T \otimes \mathbf{I}_s) = \mathbb{J}_4 = \sum_{c_j} \boldsymbol{e}_{c_j} \boldsymbol{e}_{(c_j,1)}^T$	$s \times sg_m$	zero-matrix but for the diagonal of the first block made of 1s
$\mathbb{J} = \mathbb{J}^{*}\mathbb{J}^{*T} = \left(\boldsymbol{e}_1^{(g_m)} \boldsymbol{e}_1^{(g_m)T}\right) \otimes \mathbf{I}_s$	$sg_m \times sg_m$	zero-matrix but for the diagonal of the first block made of 1s
$\mathbb{J}_1 = (\mathbf{I}_{g_m} \otimes \mathbf{1}_s^T)$	$g_m \times sg_m$	matrix where the ith $g_m \times s$ block is empty but for the ith row filled with 1s.
$\mathbb{J}_w = (\mathbf{I}_{g_m} \otimes \boldsymbol{w})$	$sg_m \times g_m$	matrix where the ith $s \times g_m$ block is empty but for the ith column equal to $\boldsymbol{w}$.
$\mathbb{J}_2 = \mathbb{J}_6 = \sum_{c_a} \boldsymbol{vec}(\boldsymbol{e}_{c_a} \boldsymbol{e}_{c_a}^T) \boldsymbol{e}_{c_a}^T \mathbb{J}^{*T}$	$(s^2 \times sg_m)$	
$\mathbb{J}_3 = \mathbb{J}_7 = \sum_j \boldsymbol{vec}(\boldsymbol{e}_j \boldsymbol{e}_j^T) \boldsymbol{e}_j^T$	$((sg_m)^2 \times sg_m)$	
for any $s \times s$ $\tilde{\mathbf{A}}$,		
$\tilde{A}_{i,j}$	1	stochastic number of class i individual produced by a class j individual, via $\tilde{\mathbf{A}}$.
$\tilde{\mathbf{A}}_j$	$s \times 1$	multivariate stochastic process of the production of individual (in all classes) by a class j individual, via $\tilde{\mathbf{A}}$ (i.e., $\tilde{\mathbf{A}} = \{\tilde{\mathbf{A}}_j\}_j$)
$\tilde{\mathbb{n}}$	sg_m	stochastic population vector in the generation-structured model. Is equivalently considered as a $s \times g_m$ 2-dimensional vector or its vectorised $sg_m \times 1$ version.
$\tilde{\mathbb{n}}_i = \tilde{\mathbb{n}}_{i_c, i_g}$	1	stochastic number of state i individuals in the generation-structured population. Corresponds to class i_c and generation number i_g, and to entry i_c, i_g in the 2-dimensional form of $\tilde{\mathbb{n}}$ or to entry i of its vectorised version, such that $i = i_c + s(i_g - 1)$.
$\tilde{\mathbb{M}} = \tilde{\mathbb{S}} + \tilde{\mathbb{F}}$	$sg_m \times sg_m$	stochastic population projection model, decomposed into fertility and survival components, for the generation-structured model.
$\tilde{\mathbb{S}} = \mathbf{I}_{g_m} \otimes \tilde{\mathbf{S}}$	$sg_m \times sg_m$	stochastic generation-structure survival projector.
$\tilde{\mathbb{F}} = \mathbf{J_1} \otimes \tilde{\mathbf{F}}$	$sg_m \times sg_m$	stochastic generation-structure fertility projector.
$\mathbb{M} = \mathbf{E}(\tilde{\mathbb{M}}) = \mathbb{S} + \mathbb{F}$	$sg_m \times sg_m$	deterministic population projection model, decomposed into fertility and survival components, for the generation-structured model.
$\mathbb{P}_{\mathbb{S}} = \mathbf{I}_{g_m} \otimes \mathbf{P_S}$	$sg_m \times sg_m$	generation-structure survival genealogical transition matrix.
$\mathbb{P}_{\mathbb{F}} = \mathbf{J_1}^T \otimes \mathbf{F}$	$sg_m \times sg_m$	generation-structure survival genealogical fertility matrix.
^{T}	-	denotes matrix transposition
$\tilde{\mathbb{K}} = \{\tilde{\mathbb{K}}_{i,j}\}_{i,j}$	$sg_m \times sg_m$	stochastic process of the number of kin.
$\tilde{\mathbb{K}}_{i,j} = \tilde{\mathbb{K}}_{(c_i,g_i),(c_j,g_j)}$	1	stochastic process of the number of kin in state i of *ego* in state j, that is **number of** (g_i, g_j)**-kin of class** c_i **of *ego* in class** c_j.
$\varphi_{\tilde{\mathbb{D}}^t_{c_j}}$	1	c_j^{th} element of $\boldsymbol{\varphi}_{\tilde{\mathbb{D}}^t}$ which has s elements (contrary to $\varphi_{\tilde{\mathbb{K}}}$ or $\varphi_{\tilde{\mathbb{M}}}$ which have sg_m)
$\tilde{\mathbb{D}}^{(t)}$	$sg_m \times s$	Number of descendants, alive today, of an individual that was alive t time-steps ago but for the descendants of the $g_j - 1$ ancestor.$\tilde{\mathbb{D}}^{(t)} = \tilde{\mathbb{D}}\mathbb{y}^{(t)} + \tilde{\mathbb{D}}\mathbb{s}^{(t)} + \tilde{\mathbb{D}}\mathbb{o}^{(t)}$.
$\tilde{\mathbb{D}}\mathbb{y}^{(t)}$	$sg_m \times s$	Number of descendants, alive today, of an individual that was alive t time-steps, stemming from daughters not born yet then.
$\tilde{\mathbb{D}}\mathbb{s}^{(t)}$	$sg_m \times s$	Number of descendants, alive today, of an individual that was alive t time-steps, stemming from daughters produced at that time, but for the descendants of the $g_j - 1$ ancestor.
$\tilde{\mathbb{D}}\mathbb{o}^{(t)}$	$sg_m \times s$	Number of descendants, alive today, of an individual that was alive t time-steps, stemming from daughters already produced then.
$\tilde{\mathbb{K}}^u$	$g_m \times g_m$	Number of kin, strucutured by generation/kin only .
for any $sg_m \times sg_m$ $\tilde{\mathbb{A}}$,		
$\tilde{\mathbb{A}}_{i,j} = \tilde{\mathbb{A}}_{(c_i,g_i),(c_j,g_j)}$	1	stochastic number of state i individual (individuals in class c_i and generation g_i) produced by a state j individual (an individual in class c_j and generation g_j), via $\tilde{\mathbb{A}}$.
$\tilde{\mathbb{A}}_j = \tilde{\mathbb{A}}_{(c_j,g_j)}$	$sg_m \times 1$	stochastic vector, describing the production of individual (in all classes and generations) by an individual in class c_j and generation g_j, via $\tilde{\mathbb{A}}$
$\boldsymbol{Var}(\tilde{\mathbb{A}}_j)$	$sg_m \times sg_m$	Variance-Covariance matrix of stochastic vector $\tilde{\mathbb{A}}_j$
$\boldsymbol{Var}^{*}(\tilde{\mathbb{A}})$	$(sg_m)^2 \times sg_m$	Variance of process $\tilde{\mathbb{A}}$, where $\boldsymbol{Var}^{*}_j(\tilde{\mathbb{A}}) = \boldsymbol{vec}(\boldsymbol{Var}(\tilde{\mathbb{A}}_j))$
$\varphi_{\tilde{\mathbb{A}}_j}$	1	j^{th} element of $\boldsymbol{\varphi}_{\tilde{\mathbb{A}}}$, contains only s elements (contrary to $\varphi_{\tilde{\mathbb{K}}}$ which contains sg_m)

Table 1: Nomenclature and definitions

6 Appendices

6.1 Appendix: Discrete time and state space mode

In many models, such as those of population ecology, the structure space cannot be made continuous (e.g. for a population structured by life history morphs). At the possible cost of discretisation, the structure consists of s classes, and the population can be described by a stochastic vector $\tilde{\boldsymbol{n}}(t)$ of size s. Structure discretisation is commonly used in population ecology, when accounting for continuous traits such as body mass, via, for instance, integral PMs (Rees and Ellner, 2009; Rees et al., 2014; Ellner and Rees, 2006). For many populations also, the time structure cannot be structured continuously. In population ecology again, for instance, many species, such as most birds or ungulates, reproduce at a specific period (or at specific periods) of the year. In case case, the models are built from various "seasonal" projectors that are periodic matrices (they are same year after year, see Bacaër, 2009), that are then combined (that is, multiplied, if in matrix form) to get an annual (or another time period) projector that does not depend on the year (Caswell, 2001). If, for instance, reproduction occurs at a specific time-period of the year (modeled by $\mathbf{F}'$), and the rest of the year the birds "only" survive (modeled by $\mathbf{S}$), one can build an annual model by combining these, either by having reproduction first: $\mathbf{M}_{pre} = \mathbf{S}(\mathbf{I}_s + \mathbf{F}') = \mathbf{S} + \underbrace{\mathbf{SF}'}_{\mathbf{F}_{pre}}$ or last $\mathbf{M}_{post} = (\mathbf{I}_s + \mathbf{F}')\mathbf{S} = \underbrace{\mathbf{F}'\mathbf{S}}_{\mathbf{F}_{post}} + \mathbf{S}$. $\mathbf{M}_{pre}$ is called a pre-breeding model and $\mathbf{M}_{post}$ a post-breeding model, they share the same survival component but differ in their reproduction components. As they model the same populations these two models have the same asymptotic properties (same asymptotic growth rate) (see, for instance Beissinger and Westphal, 1998; Caswell, 2001).

6.2 Appendix: I.i.d. processes of survival and reproduction

We consider that the process $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{S}}$ are i.i.d., that is $\forall(c_i, c_j), Var(\tilde{\mathbf{F}}_{c_i}, \tilde{\mathbf{S}}_{c_j}) = \mathbf{0}$, between and within individuals. In other words, we consider that (1) all individuals in class c_i follow the same processes of reproduction ($\tilde{\mathbf{F}}_{c_i}$) and survival ($\tilde{\mathbf{S}}_{c_i}$), (2) that the individual realisations of survival and fertility are independent from that of other individuals in that same ($c_i = c_j$) or other ($c_i \neq c_j$) classes and (3) that for a given individual in c_i, the realisation of survival and reproduction are independent: $Var(\tilde{\mathbf{F}}_{c_i}, \tilde{\mathbf{S}}_{c_i}) = \mathbf{0}$. The first assumption of identity is classic in structured population models: the population is categorised into as many classes, differing in vital rates (the $\widetilde{m}_{i,j}(t)$), as one knows of, and individuals in the same class are therefore considered identical, their fate only differing because of stochasticity. The second assumption of independence of individual realisations between individuals within and between states is also common: it relates to the fact that, at a given time-step t, the fate of an individual is entirely determined by its class. The third assumption, of independence of individual realisation of reproduction and survival, is not that common, and for two main reasons, but this can be dealt with. The first reason preventing this independence is the case of "seasonal" matrices described in Appendix 6.1. As is obvious from the formulas of $\mathbf{F}_{pre}$ and $\mathbf{F}_{post}$, the reproductive process depends, in general, on the survival process. However, while $\tilde{\mathbf{F}}_{post_{c_j}}$ depends in general on $\tilde{\mathbf{S}_{c_j}}$ of the same individual because of the first projection of $\tilde{\mathbf{F}}_{post}$ is $\tilde{\mathbf{S}}$, it is not the case for $\tilde{\mathbf{F}}_{pre_{c_j}}$ where the first projection is $\mathbf{I}_s + \tilde{\mathbf{F}}'$, where the first component is constant and the second one yields different individuals. Therefore, in the case of seasonal models, using pre-breeding formulation, allows to respects the third assumption. The second reason preventing this independence is the case of trade-offs between *realisations* of survival and reproduction. In (Coste and Pavard, 2020), such trade-offs are called "individual" trade-offs (which differ from "individual" trade-offs which occur between *expectations* of survival and reproduction, across classes). Those can be modelled by having, for instance, for an individual in class c_j in a given year, survival $\tilde{\mathbf{S}}_{c_j}$ that year be a function of reproduction $\tilde{\mathbf{F}}_{c_j}$ that year, which violates the third assumption above. This can also be circumvented by applying the relationship between survival and reproduction at the previous time-step. This is commonly done in bird population ecology, where models with "breeders" and "non-breeders" segregate individuals producing offspring or not that year. ¨ In such a model, the stochasticity in the realisation of fertility is therefore embedded in the survival of the previous time-step (in a structured model, survival involves both survival proper, and the transition of survivors to the various stages). The trade-off is then incorporated by having "breeders" and "non-breeders" have different survival processes. In that case, there is independence, for a given individual in class c_j at a given time-step, between the realisation of its fertility (which can be constant, or depend on offspring survival or breeding success, but does not influence survival) and its survival. See (Jenouvrier et al., 2017) for an example of such a model. Therefore, in the case of individual trade-offs, using "breeders"/"non-breeders" model construction allows to respects the third assumption.

6.3 Appendix: Concept of trait

The concept of multi-trait projection models arises for the increased amount and quality of data in all fields of population dynamics, which allows an individual in the population to be described by different traits influencing its fate. In population ecology for instance, we now have models where individuals are structured by *age* and *stage*, or *stage*,*body mass* or *stage* and *location* (this would be a so called metapopulation model, see, Layton-Matthews et al., 2021, , for an example of such a model). In such a model, these categories, themselves categorising the various *states* in the structured population, are called traits. In (Coste et al., 2017), the authors develop a theory of such models where they call $ts = \{t_1, t_2, .., t_r\}$, where t_i is the number of *classes* of the ith trait t_i, the trait structure of the r-trait model. Such a model is a tensor, that can be projected over time, like any projection model, by vectorising its r-dimensional population vector. The result is a $q \times q$ (stochastic or determinsitic) matrix model with $q = t_1 t_2 ... t_r$, where the ith state ($\forall 1 \leq i \leq q$) corresponds to multitrait state $(c_1, c_2, ..., c_r)$ (where $\forall i, 1 \leq c_r \leq t_r$) such that $i = c_1 + \sum_{i=2}^{r} (c_r - 1) \prod_{j=1}^{i-1} t_j$. In tensor notation, this is simply $\boldsymbol{e}_i = \boldsymbol{e}_{c_r} \otimes \boldsymbol{e}_{c_{r-1}} \otimes ... \otimes \boldsymbol{e}_{c_2} \otimes \boldsymbol{e}_{c_1}$, which allows to see the projection matrix as a nested block matrix of dimensions t_1,t_2, .., t_r.

Such models allow to analyse the effect of specific traits (and the processes they embed) on the demography of a population comparing the model with "equivalent" models from which the traits in question are absent. The procedure to obtain the "equivalent" model, called folding in (Coste et al., 2017), consists at computing, at the demographic stable state, the transition rates between states of the models without the trait(s) under scrutiny by merging all states with the same classes for all the other states. This merger is simply performed by summing all incoming flows and summing all outgoing flows, weighted by the relative abundances, of the states to be merged into one. In other words $\mathbf{M}^{fold}_{trait_r}(c_1, c_2, .., c_{r-1}, c'_1, c'_2, .., c'_{r-1}) = \sum_{1 \leq c_r, c'_r \leq t_r} w_{c'_r} \mathbf{M}(c_1, c_2, .., c_r, c'_1, c'_2, .., c'_r)$ is $\mathbf{M}$ folded over the $r^t h$ (and last) trait (see, Coste et al., 2017). $\mathbf{M}^{fold}_{trait_r}$ and $\mathbf{M}$ correspond to the same population: if trait r is the colour of the wings of butterfly, then two entomologists, one colour-blind one not, will obtain, from the exact same data and analysis, respectively models $\mathbf{M}^{fold}_{trait_r}$ and $\mathbf{M}$. These have therefore the same asymptotic growth rate and relative abundances but will differ in reproductive values and therefore sensitivities. The analysis of these differences will, called the trait-level analysis, by comparison of these two models, describe the effects of trait r on the dynamics (and evolution) of the studied population.

In this framework, $\mathbb{K}_u$, as computed in equation xxxx, corresponds to $\mathbb{K}$ folded over trait *class*. And therefore comparing it with the kinship matrix obtained from $\mathbf{M}^{fold}_{class} = f + s$ which is the unstructured version of $\mathbf{M}$ where all individuals have the same probability to survive s and expected number of offspring f, allows to measure the effect of the processes embedded in the class structure of the population on its kinship.

6.4 Appendix: Class by Age model

Because matrices project populations over a given time-step, time and therefore age, are inherent parameters. We aim at making *age* an additional explicit trait, by building such a *class* x *age* matrix $\tilde{\mathbb{M}}^{\dagger}$. In such a model, trait *age*, like trait *generation* from $\mathbf{M}$ to $\mathbb{M}$, has no effect on the fate of individuals, but for the particular age class ω. Therefore, in this 2 trait-model, $\forall c_i, c_j$ classes and $\forall a_i, a_j \leq \omega$ ages, we have $\tilde{\mathbb{F}}^{\dagger}((c_i, a_i), (c_j, a_j)) = \tilde{\mathbf{F}}(c_i, c_j)$ if $a_i = 1$ and 0 otherwise, and $\tilde{\mathbb{S}}^{\dagger}((c_i, a_i), (c_j, a_j)) = \tilde{\mathbf{S}}(c_i, c_j)$ if $a_i = a_j + 1$ and $a_j < \omega$ and 0 otherwise.

As for the generation structured model, this is easier to describe with tensor notation: $\tilde{\mathbb{M}}^{\dagger} = \tilde{\mathbb{S}}^{\dagger} + \tilde{\mathbb{F}}^{\dagger}$ with $\tilde{\mathbb{S}}^{\dagger} = \mathbf{J}_{\omega} \otimes \mathbf{S}$ and $\tilde{\mathbb{F}}^{\dagger} = \mathbf{1}^{T}_{\omega} \otimes \mathbf{F}$, with $\mathbf{J}_{\omega}$ the $\omega \times \omega$ matrix with ones on the sub-diagonal and $\mathbf{1}_{\omega}$ a vertical vector, of size ω, of ones. For $\omega = 4$, this is

$$\mathbb{M}^{\dagger} = \mathbb{F}^{\dagger} + \mathbb{S}^{\dagger} = \left[\begin{array}{c|c|c|c} \mathbf{F} & \mathbf{F} & \mathbf{F} & \mathbf{F} \\ \hline \mathbf{S} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{S} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{S} & \mathbf{0} \end{array}\right]$$

7 Supplementary Materials

7.1 SM: Building blocks S and F for three common type of reproduction

From $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{F}}$ one can infer the projection matrices $\mathbf{S}$ and $\mathbf{F}$. It is,however, always the case for survival and commonly the case for fertility that the reverse is also true.

Survival process Indeed, via survival, there are only two outcomes for the fate, at time-step t, of an individual in state j: it survives if $\sum_{i=1}^{s} \tilde{S}_{i,j}(t) = 1$ and dies if this is r.v. is 0. In other words, $\forall j, \tilde{\mathbf{S}}_j$ describes a generalised Bernoulli process of parameters the elements of $\mathbf{S}_j$.

Reproduction process In general, the stochastic reproductive process $\tilde{\mathbf{F}}$ cannot be directly inferred by the elements in its expectation $\mathbf{F}$. In most cases in practice however, the relationship between $\tilde{\mathbf{F}}$ and $\mathbf{F}$ is straightforward and we will focus, in this article, on such three cases, two of them for which $\tilde{\mathbf{F}}$ can be inferred by $\mathbf{F}$,These are Bernoulli reproduction, (independent) Poisson reproduction and One-offspring-class reproduction. (we provide formulas for the general case in sec. xxx). :

Bernoulli Reproduction In that case, an individual can produce at most one offspring per time-step. This case is similar to the survival process: $\forall j, \tilde{\mathbf{F}}_j$ describes a generalised Bernoulli process of parameters the elements of $\mathbf{F}_j$. We denote this process B-reproduction.

Independent Poissons reproduction Here, the $\tilde{F}_{i,j}$ are considered independent Poisson processes, of parameters the $F_{i,j}$. In that case $\sum_i \tilde{F}_{i,j}$ is itself a Poisson of parameter $\sum_i F_{i,j}$. We denote this process P-reproduction.

One offspring class If there is only one class of offspring (we consider it to be class 1), then $\tilde{\mathbf{F}}$ is fully described by the s univariate distributions of the $\tilde{F}_{1,j}$. We denote this process 1-reproduction. Note that the reproduction process can be both 1-reproduction and F-reproduction or B-reproduction.

7.2 SM: Generation structured population model

All tensor matrices of this article, such as $\mathbb{M}$, can be described as block-matrices made of $g_m \times g_m$ blocks corresponding to all pairs of generation numbers, where each block, such as for any $1 \le g_i, g_j \le g_m$, the (g_i, g_j) block, we denote $\mathbb{M}(g_i, g_j)$, is of size $s \times s$ with entries corresponding to all pairs of classes. And therefore the expected number of individuals in class c_i and generation g_i produced by an individual in class c_j and generation g_j, that we denote $\mathbb{M}_{(c_i,g_i),(c_j,g_j)}$ or simply $\mathbb{M}_{i,j}$ is the (c_i, c_j) entry of $\mathbb{M}(g_i, g_j)$.

generation modulo model Consider, first, projection model, $\mathbb{m}(t+1) = \mathbb{M}^*\mathbb{m}(t)$, where an individual is categorised according to its class c_i corresponding to that in the input model $\mathbf{M}$, and a "relative generation number" g_i, a fixed trait, that is a trait constant throughout life, inherited from the mother with an increment of 1 modulo g_m. Since trait *relative generation number* does not affect the demographic processes, is constant via survival and increases by 1 modulo g_m via reproduction, $\tilde{\mathbb{M}}^*$ is the zero matrix, with $\tilde{\mathbf{S}}$ as each diagonal block , and $\tilde{\mathbf{F}}$ as each sub-diagonal block as well as as the top right block. For $g_m = 4$, this is:

$$\tilde{\mathbb{M}}^* = \tilde{\mathbb{F}}^* + \tilde{\mathbb{S}}^* = \underbrace{\left[\begin{array}{c|c|c|c} \tilde{\mathbf{S}} & \mathbf{0} & \mathbf{0} & \tilde{\mathbf{F}} \\ \hline \tilde{\mathbf{F}} & \tilde{\mathbf{S}} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \tilde{\mathbf{F}} & \tilde{\mathbf{S}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \tilde{\mathbf{F}} & \tilde{\mathbf{S}} \end{array}\right]}_{g_m \times g_m \text{ blocks of size } s \times s}$$

which we can also write $\tilde{\mathbb{S}}^* = \mathbf{I} \otimes \tilde{\mathbf{S}}$ and $\tilde{\mathbb{F}}^* = \mathbf{J_2} \otimes \tilde{\mathbf{F}}$ with $\mathbf{J_2}$ the $s \times s$ matrix with ones in the subdiagonal and the top right entry. And similarly for expectations:$\mathbb{S}^* = \mathbf{I} \otimes \mathbf{S}$ and $\mathbb{F}^* = \mathbf{J_2} \otimes \mathbf{F}$. We show in 7.2 below that $\mathbb{P}_{\mathbb{S}}{}^* = \mathbf{I} \otimes \mathbf{P_S}$ and $\mathbb{P}_{\mathbb{F}}{}^* = \mathbf{J_2}^T \otimes \mathbf{F}$ and for $g_m = 4$ this yields:

$$\mathbb{P}^* = \mathbb{P}_{\mathbb{F}}{}^* + \mathbb{P}_{\mathbb{S}}{}^* = \left[\begin{array}{c|c|c|c} \mathbf{P_S} & \mathbf{P_F} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{P_S} & \mathbf{P_F} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{P_S} & \mathbf{P_F} \\ \hline \mathbf{P_F} & \mathbf{0} & \mathbf{0} & \mathbf{P_S} \end{array}\right]$$

absolute generation model To build a model for kinship, we are actually interested in absolute generation numbers. To that end, we fix one end of the spectrum: generation 1 corresponds to the generation number of the ancestor of *ego* and the kin, and we limit ourselves to the study of the first g_m generations.

We can construct such a model directly, by considering that trait *generation* does not affect the demographic processes, is constant via survival and increases by 1 via reproduction. This implies that $\tilde{\mathbb{S}}_{(c_i,g_i),(c_j,g_j)} = \delta(g_i, g_j)\tilde{\mathbf{S}}_{c_i,c_j}$ and $\tilde{\mathbb{F}}_{(c_i,g_i),(c_j,g_j)} = \delta(g_i, g_j + 1)\tilde{\mathbf{F}}_{c_i,c_j}$ (where $\delta(i,j) = 1$ if $i = j$, 0 otherwise). In block-matrix form, this implies that stochastic matrix $\tilde{\mathbb{M}}$ is the empty matrix but for all diagonal blocks being $\tilde{\mathbf{S}}$ and all sub-diagonal blocks being $\tilde{\mathbf{F}}$. These matrices can be denoted by using classical tensor product $\otimes$ (also referred to as Kronecker product), we can write $\tilde{\mathbb{S}} = \mathbf{I} \otimes \tilde{\mathbf{S}}$ and $\tilde{\mathbb{F}} = \mathbf{J_1} \otimes \tilde{\mathbf{F}}$, with $\mathbf{J_1}$ the $s \times s$ matrix with ones in the sub-diagonal, and similarly for expectations $\mathbb{S} = \mathbf{I} \otimes \mathbf{S}$ and $\mathbb{F} = \mathbf{J_1} \otimes \mathbf{F}$ which can be built very quickly in most programs where the Kronecker product is implemented as a function.

We can also consider it as a truncation of generation modulo model $\mathbb{M}*$, where reproduction towards generation 1 is not allowed as well as ancestry by reproduction in generation g_m. With regards to forward projection, this mean that $\tilde{\mathbb{M}}$ has the exact same properties thatn $\tilde{\mathbb{M}}^*$, but for generation 1 which cannot be reached by reproduction, reducing the top-right block of $\tilde{\mathbb{M}}$ to 0. With regards to backwards projection, this mean that $\mathbb{P}_{\tilde{\mathbb{M}}}$ has the exact same properties than $\mathbb{P}_{\tilde{\mathbb{M}}}{}^*$, but for generation 1 which does not yield any ancestor by (backward) reproduction, reducing the bottom-left block of $\tilde{\mathbb{M}}$ to 0.

This models yield the exact same models $\mathbb{M} = \mathbb{F} + \mathbb{S}$ and $\mathbb{F} = \mathbf{J_1} \otimes \mathbf{F}$ and $\mathbb{S} = \mathbf{I} \otimes \mathbf{S}$ that we have just seen. Most importantly it allows to build $\mathbb{P}$, the transition matrix of the genealogical Markov chain of $\mathbb{M}$. Indeed, $\mathbb{M}$ model does not have the asymptotic properties of $\tilde{\mathbf{M}}$: the population will eventually go extinct as individuals in generation g_m do not reproduce. This implies $\mathbb{P}_{\mathbb{S}} = \mathbf{I} \otimes \mathbf{P_S}$ and $\mathbb{P}_{\mathbb{F}} = \mathbf{J_1}^T \otimes \mathbf{F}$. If one chooses the arbitrary g_m to be $g_m = 4$, we get:

$$\mathbb{M} = \mathbb{F} + \mathbb{S} = \left[\begin{array}{c|c|c|c} \mathbf{S} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{F} & \mathbf{S} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{F} & \mathbf{S} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{F} & \mathbf{S} \end{array}\right] \text{ and } \mathbb{P} = \mathbb{P}_{\mathbb{F}} + \mathbb{P}_{\mathbb{S}} = \left[\begin{array}{c|c|c|c} \mathbf{P_S} & \mathbf{P_F} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{P_S} & \mathbf{P_F} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{P_S} & \mathbf{P_F} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{P_S} \end{array}\right] \tag{32}$$

Transition matrix of the genealogical Markov chain of $\mathbb{M}^*$ In model $\tilde{\mathbb{M}}^*$, the fate of an individual in class c_j is independent from its generation (trait "relative generation" does is affect the dynamics) and is the same than that of an individual in class c_j in model $\tilde{\mathbf{M}}$, contrary to model $\tilde{\mathbb{M}}$ where individuals in generation g_m have no fertility. This implies that $\mathbb{M}^*$ and $\mathbf{M}$ have the same assymptotic growth rate λ and that the corresponding vector of relative abundances for $\mathbb{M}^*$ is such that $w^*(c_i, g_i) = \frac{w(c_i)}{g_m}$. This allow us to construct the transition matrix of the genealogical markov chain of $\mathbb{M}^*$: ${\mathbb{P}_{\mathbb{S}}}^*((c_i, g_i), (c_j, g_j)) = \mathbb{S}^*((c_j, g_j), (c_i, g_i))\frac{w^*(c_i,g_i)}{w^*(c_j,g_j)} = \mathbb{S}^*((c_j, g_j), (c_i, g_i))\frac{w(c_i)}{w(c_j)}$ which is 0 for $g_i \neq g_j$ and ${\mathbb{P}_{\mathbb{S}}}^*_{(c_i,g_j),(c_j,g_j)} = \mathbb{S}^*((c_j, g_j), (c_i, g_j))\frac{w(c_i)}{w(c_j)} = \mathbf{S}(c_j, c_i))\frac{w(c_i)}{w(c_j)}$, this is independent of g_j, and therefore $\forall g_j, {\mathbb{P}_{\mathbb{S}}}^*(g_i, g_j) = \mathbf{P_S}$. Similarly, ${\mathbb{P}_{\mathbb{F}}}^*((c_i, g_i), (c_j, g_j)) = \mathbb{F}^*((c_j, g_j), (c_i, g_i))\frac{w^*(c_i,g_i)}{w^*(c_j,g_j)} = \mathbb{F}^*((c_j, g_j), (c_i, g_i))\frac{w(c_i)}{w(c_j)}$ which is 0 for $g_j - 1 \neq g_i[g_m]$ and ${\mathbb{P}_{\mathbb{F}}}^*_{(c_i,g_j-1[g_m]),(c_j,g_j)} = \mathbb{F}^*((c_j, g_j), (c_i, g_j - 1[g_m]))\frac{w(c_i)}{w(c_j)} = \mathbf{F}(c_j, c_i))\frac{w(c_i)}{w(c_j)}$, this is independent of g_j, and therefore $\forall g_j, {\mathbb{P}_{\mathbb{F}}}^*(g_j - 1[g_m], g_j) = \mathbf{P_F}$. We can write this as ${\mathbb{P}_{\mathbb{S}}}^* = \mathbf{I} \otimes \mathbf{P_S}$ and ${\mathbb{P}_{\mathbb{F}}}^* = {\mathbf{J_2}}^T \otimes \mathbf{F}$ and for $g_m = 4$ this yields:

$$\mathbb{P}^* = {\mathbb{P}_{\mathbb{F}}}^* + {\mathbb{P}_{\mathbb{S}}}^* = \left[\begin{array}{c|c|c|c} \mathbf{P_S} & \mathbf{P_F} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{P_S} & \mathbf{P_F} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{P_S} & \mathbf{P_F} \\ \hline \mathbf{P_F} & \mathbf{0} & \mathbf{0} & \mathbf{P_S} \end{array}\right]$$

Euler-Lotka generation matrix It follows from the above definitions, that the probability for an individual in c_j, g_j that its mother was in state c_i, g_i, when it gave birth to it, any number of time-steps ago, is the (i, j) entry of $\mathbb{G}^* = {\mathbb{P}_{\mathbb{F}}}^*(\mathbb{I} - {\mathbb{P}_{\mathbb{S}}}^*)^{-1} = \mathbb{J}_2^T \otimes (\mathbf{F}(\mathbf{I} - \mathbf{P_S})^{-1})$, which corresponds to the transition matrix of the genealogical Markov chain of $\mathbb{E}^* = (\lambda\mathbb{I} - \mathbb{S}^*)^{-1}\mathbb{F}^* = \mathbb{J}_2 \otimes ((\lambda\mathbf{I} - \mathbf{S})^{-1}\mathbf{F})$. We call $\mathbb{E}^*$, the Euler-Lotka generation matrix, as it generalises the Euler-Lotka equation to any structure. Indeed $\mathbb{E}^* = \sum_{t\geq1} \lambda^t {\mathbb{S}^*}^{t-1}\mathbb{F}^* = \mathbb{J}_2 \otimes (\sum_{t\geq1} \lambda^t \mathbf{S}^{t-1}\mathbf{F})$. See SM5 of (Coste et al., 2021) for proofs and discussion.

7.3 SM: Expectation, Variance and Probability Generating Functions

From the expectation of process $\tilde{\mathbb{K}}$, $\mathbb{K} = \mathbf{E}(\tilde{\mathbb{K}})$, we can get the expected number of (g_i, g_j)-kin in class c_i of *ego* in class c_j, directly from entry $\mathbb{K}_{(c_i,g_i),(c_j,g_j)}$. We can also expand our investigation by going beyond the mean towards the general distribution of the number of kin and its moments, of which the expectation is only a particular case. Tools stemming from probability theory and in particular branching processes, allow to investigate $\tilde{\mathbb{K}}_{(c_j,g_j)}$ via various approaches. One can compute its moments directly, or via moment generation functions, or use the related tool of probability-generating functions (*pgf*) to compute the distributions of the number of kin. In this paper, we will focus on the first moments - expectation and variance - and on the *pgf.*

As for the other stochastic vectors encountered, the variance of stochastic vector $\tilde{\mathbb{K}}_{(c_j,g_j)}$,$\mathbf{Var}(\tilde{\mathbb{K}}_{(c_j,g_j)})$, is a variance-covariance matrix of size $s.g_m \times s.g_m$, which provides, on the diagonal, the variance of the number of specific each kin in state (c_i, g_i) for *ego* in class (c_j, g_j), $Var(\tilde{\mathbb{K}}_{(c_i,g_i)}, (c_j, g_j))$ and away from the diagonal, the covariances $Cov(\tilde{\mathbb{K}}_{(c_i,g_i),(c_j,g_j)}, \tilde{\mathbb{K}}_{(c_i',g_i'),(c_j,g_j)})$. We can group the $\tilde{\mathbb{K}}_{(c_j,g_j)}$, in a $s.g_m.s.g_m \times s.g_m$ matrix, $\mathbf{Var}^*(\tilde{\mathbb{K}})$, which $(c_j, g_j)^{th}$ column is the vectorised $\mathbf{Var}(\tilde{\mathbb{K}}_{(c_j,g_j)})$.

The *pgf* of a multivariate process, such as $\tilde{\mathbb{K}}_{(c_j,g_j)}$, is the function of multivariate dummy variable of the same dimensions that the abundance vector $\ltimes$, $\boldsymbol{x} = (x_{1,1}, x_{2,1}, ..., x_{s,1}, x_{1,2}, ..., x_{s,g_m})$, $\varphi_{\tilde{\mathbb{K}}_{(c_j,g_j)}}(\boldsymbol{x}) = E(x_{1,1}^{\tilde{\mathbb{K}}_{(1,1),(c_j,g_j)}} x_{1,2}^{\tilde{\mathbb{K}}_{(1,2),(c_j,g_j)}} ... x_{s,g_m}^{\tilde{\mathbb{K}}_{(s,g_m),(c_j}}$
$\sum_{\boldsymbol{n}=(n_{1,1},n_{1,2},...,n_{s,g_m})} P(\tilde{\mathbb{K}}_{(c_j,g_j)} = \boldsymbol{n}) x_{1,1}^{n_{1,1}} x_{1,2}^{n_{1,2}} ... x_{s,g_m}^{n_{s,g_m}}$. The *pgf* of a stochastic vector $\tilde{\boldsymbol{v}}$ has interesting properties, such as $\varphi_{\tilde{\boldsymbol{v}}}(\mathbf{1}) = 1$, and $\varphi_{\tilde{\boldsymbol{v}}}(\mathbf{0}) = P(\tilde{\boldsymbol{v}} = \mathbf{0})$ and in general $P(\tilde{\boldsymbol{v}} = \boldsymbol{n}) = \frac{1}{n_{1,1}! n_{1,2}! ... n_{s,g_m}!} \frac{\partial \varphi_{\tilde{\boldsymbol{v}}}^{(n_{1,1}+n_{1,2}+...+n_{s,g_m})}}{\partial x_{1,1}^{(n_{1,1})} \partial x_{1,2}^{(n_{1,2})} ... \partial x_{s,g_m}^{(n_{s,g_m})}}(\mathbf{0})$, a property which justifies the name of the function.

The pgfs of the various $\tilde{\mathbb{K}}_{(c_j,g_j)}$ can be grouped as a vector of pgf, which is the multivariate pgf of $\tilde{\mathbb{K}}$: $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}} = \left(\varphi_{\tilde{\mathbb{K}}_{(1,1)}}, \varphi_{\tilde{\mathbb{K}}_{(2,1)}}, ..., \varphi_{\tilde{\mathbb{K}}_{(s,g_m)}}\right)$ From $\varphi_{\tilde{\mathbb{K}}_{(c_j,g_j)}}$, one can also obtain the (uni-variate) *pgf* $\varphi_{\tilde{\mathbb{K}}_{(c_i,g_i),(c_j,g_j)}}$ of $\tilde{\mathbb{K}}_{(c_i,g_i),(c_j,g_j)}$, by setting respective arguments of $\varphi_{\tilde{\mathbb{K}}_{(c_j,g_j)}}$ equal to 1:

$$\varphi_{\tilde{\mathbb{K}}_{(c_i,g_i),(c_j,g_j)}}(x_{c_i,g_i}) = \varphi_{\tilde{\mathbb{K}}_{(c_j,g_j)}}(1, 1, ..., 1, x_{c_i,g_i}, 1, ..., 1) \tag{33}$$

From $\varphi_{\tilde{\mathbb{K}}_{(c_i,g_i),(c_j,g_j)}}$, one can compute the full distribution of the number of (g_i, g_j)-kin in class c_i for *ego* in class c_j:

$$P(\tilde{\mathbb{K}}_{(c_i,g_i),(c_j,g_j)} = n) = \frac{\varphi^{(n)}_{\tilde{\mathbb{K}}_{(c_i,g_i),(c_j,g_j)}}(0)}{n!}, \tag{34}$$

where $\varphi^{(n)}_{\tilde{\mathbb{K}}_{(c_i,g_i),(c_j,g_j)}}$ is the n-th derivative of $\varphi_{\tilde{\mathbb{K}}_{(c_i,g_i),(c_j,g_j)}}$. Moment generating functions are better suited to compute moments, however, from $\varphi_{\tilde{\boldsymbol{v}}}$. one can compute all mixed factorial moments: $\mu_{n_{1,1},n_{2,1},...,n_{s,g_m}} = E(\prod_{(c_i,g_i)=(1,1)}^{(s,g_m)} \prod_{m=0}^{n_{s.g_m}-1} (\tilde{v}_{c_i,g_i} - m)) = \frac{\partial \varphi_{\tilde{\boldsymbol{v}}}^{(n_{1,1}+n_{2,1}+...+n_{s,g_m})}}{n_{1,1}! n_{2,1}! ... n_{s,g_m}! \partial x_{1,1}^{(n_{1,1})} \partial x_{2,1}^{(n_{2,1})} ... \partial x_{s,g_m}^{(n_{s,g_m})}}(\mathbf{1})$. And for instance, $\mu_{1,0,0,...,0} = E(\tilde{v}_{1,1}) = \frac{\partial \varphi_{\tilde{\boldsymbol{v}}}}{\partial x_{1,1}}(\mathbf{1})$, $\mu_{1,1,0,...,0} = E(\tilde{v}_{1,1}\tilde{v}_{2,1}) = Cov(\tilde{v}_{1,1}, \tilde{v}_{2,1}) + E(\tilde{v}_{1,1})E(\tilde{v}_{2,1}) = \frac{\partial \varphi_{\tilde{\boldsymbol{v}}}^{(2)}}{\partial x_{1,1} \partial x_{2,1}}(\mathbf{1})$ and $\mu_{2,0,0,...,0} = E((\tilde{v}_{1,1})^2) - E(\tilde{v}_{1,1}) = Var(\tilde{v}_{1,1}) + (E(\tilde{v}_{1,1}))^2 - E(\tilde{v}_{1,1}) = \frac{\partial \varphi_{\tilde{\boldsymbol{v}}}^{(2)}}{\partial x_{1,1}^{(2)}}(\mathbf{1})$.

In order to be able to compute $\mathbb{K}$, $Var^*(\tilde{\mathbb{K}})$ and $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}}$, which contains all the information about the process, we need to write the stochastic process $\tilde{\mathbb{K}}$ as a succession and composition of independent processes, which we do in the next section.

7.4 SM: Same Litter Fertility

general case In order to count the number of kin, we count the descendants of the ancestors of *ego*. These descendants can be reached directly via successive multiplication by $\tilde{\mathbf{M}}$ as in common in such multi-type branching processes. However when counting the descendants via sisters of the ancestor born at the same time as the ancestor in the generation before, it proves necessary, in the general case, to consider a further building-block, the same-litter-sister matrix $\tilde{\mathbf{Z}}$, which element $\tilde{Z}_{i,j}$ is the number of sisters of the same-litter in class j of an individual in class i that has just been born.

$\tilde{\mathbf{Z}}$ can be inferred from $\tilde{\mathbf{F}}$; indeed,

$$\tilde{Z}_{i,j} = \sum_k \tilde{F}^{*(j)}(i,k) P_F(k,j), \tag{35}$$

where $\tilde{F}^{*(j)}(i,k)$ is the number of same-litter sisters in class i of an individual born in class j born from a mother in class k. That is

$$\tilde{F}^{*(j)}(i,k) = (\tilde{F}^{\dagger(j)}(i,k) - \delta(i,j)), \tag{36}$$

where $\tilde{F}^{\dagger(j)}(i,k)$ has the distribution of $\tilde{F}(i,k)$ biased by $\tilde{F}(j,k)$, that is

$$P(\tilde{\mathbf{F}}^{\dagger(j)}(k) = \boldsymbol{n}) = \frac{n_j}{F_{j,k}} P(\tilde{\mathbf{F}}_k = \boldsymbol{n}) \tag{37}$$

From this we can compute the expectation, $E(\tilde{F}^{\dagger j}(i,k)) = \sum_{n_i} n_i P(\tilde{F}^{\dagger j}(i,k) = n_i)$, by noting that

$$P(\tilde{F}^{\dagger j}(i,k) = n*_i) = \sum_{\boldsymbol{n}=n_1,n_2,\ldots,n*_i,\ldots,n_s} P(\tilde{\mathbf{F}}^{\dagger j}(k) = \boldsymbol{n}) = \sum_{\boldsymbol{n}=n_1,n_2,\ldots,n*_i,\ldots,n_s} \frac{n_j}{F_{j,k}} P(\tilde{\mathbf{F}}(k) = \boldsymbol{n}) = \sum_{n_j} \frac{n_j}{F_{j,k}} P(\tilde{F}(i,k) = n*_i, \tilde{F}(j,k) = n_j)$$

and therefore $E(\tilde{F}^{\dagger j}(i,k)) = \sum_{n_i,n_j} \frac{n_i n_j}{F_{j,k}} P(\tilde{F}(i,k) = n_i, \tilde{F}(j,k) = n_j) = \frac{1}{F_{j,k}} E(\tilde{F}(i,k)\tilde{F}(j,k))$. That is

$$E(\tilde{F}^{\dagger j}(i,k)) = \frac{Cov(\tilde{F}(i,k), \tilde{F}(j,k)) + F_{i,k}F_{j,k}}{F_{j,k}} \tag{38}$$

From there, we have

$$Z_{i,j} = \sum_k \left(\frac{Cov(\tilde{F}(i,k), \tilde{F}(j,k))}{F_{j,k}} + F_{i,k} - \delta(i,j) \right) P_F(k,j) \tag{39}$$

Note that this formula differs from and corrects Eq(9) of (Coste et al., 2021) which was erroneous despite the calculations being "elementary".

In terms of *pgf*, in the general case, we get, from Eqs.36 and 37 $\varphi_{\tilde{\mathbf{F}}^{*(c_j)}_{c_k}} = \sum_{\boldsymbol{n}} P(\mathbf{F}^{*(c_j)}_{c_k} = \boldsymbol{n}) x_1^{n_1} x_2^{n_2} .. x_s^{n_s}$ where $P(\mathbf{F}^{*(c_j)}_{c_k} = \boldsymbol{n}) = P(\mathbf{F}^{\dagger(c_j)}_{c_k} = (n_1, n_2, \ldots, n_{c_j} + 1, \ldots, n_s)) = \frac{n_{c_j}+1}{F_{c_j,c_k}} P(\mathbf{F}_{c_k} = (n_1, n_2, \ldots, n_{c_j} + 1, \ldots, n_s))$, that is

$$\varphi_{\tilde{\mathbf{F}}^{*(c_j)}_{c_k}} = \sum_{n_1,n_2,\ldots n_s} \frac{n_{c_j} + 1}{F_{c_j,c_k}} P(\mathbf{F}_{c_k} = (n_1, n_2, \ldots, n_{c_j} + 1, \ldots, n_s)) x_1^{n_1} x_2^{n_2} .. x_s^{n_s} \tag{40}$$

From which one gets the $\varphi_{\tilde{\mathbb{F}}^{*(j)}_k}$ necessary to compute $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}}$ with general fertility.

common cases In the three common cases we have set aside however (see sec 7.1), this step is not necessary, as, for different reasons, in these frameworks, the number of same-litter sisters does not depend on the class of the daughter ancestor of *ego*, that is $\tilde{F}^{*(j)}(i,k) = \tilde{F}^{*}(i,k)$ is independent from j. Then one can obtain the number of same-litter sisters by projecting the ancestor directly by $\tilde{\mathbf{F}}^*$, where:

Bernoulli Reproduction In that case, individuals can produce at most 1 offspring per time step, then we simply have

$$\tilde{\mathbf{F}}^* = \mathbf{0}. \tag{41}$$

Independent Poissons reproduction Here, the number and state of supplemental offspring is independent from the number and states of offspring produced and therefore

$$\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}} \tag{42}$$

This can be checked directly by noting that $\tilde{F}^{*j}(i,k) = \tilde{F}(i,k)$ when $j \neq i$, as these processes are independent and that $\tilde{F}^{\dagger i}(i,k)$ is such that $P(\tilde{\mathbf{F}}^{\dagger i}(i,k) = n_i) = \frac{n_i}{F_{i,k}} P(\tilde{F}_{i,k} = n_i) = \frac{n_i}{F_{i,k}} \frac{F_{i,k}^{n_i} e^{-F_{i,k}}}{n_i!} = \frac{F_{i,k}^{n_i-1} e^{-F_{i,k}}}{(n_i-1)!}$, and therefore

$\tilde{F}^{*i}(i,k) = (\tilde{F}^{\dagger i}(i,k) - 1$ is such that $P(\tilde{F}^{*i}(i,k) = n_i) = P(\tilde{F}^{\dagger i}(i,k) = n_i + 1) = \frac{F_{i,k}^{n_i} e^{-F_{i,k}}}{(n_i)!} = P(\tilde{F}_{i,k} = n_i)$. Therefore, $\tilde{\mathbf{F}}^* = \tilde{\mathbf{F}}$.

One offspring class If there is only one class of offspring (say class 1), the distribution and expectations of $\tilde{\mathbf{F}}^*$ are such that

$$\begin{cases} P(\tilde{F}^*_{1,k} = n_1) = \frac{n_1+1}{F_{1,k}} P(\tilde{F}_{1,k} = n_1 + 1) \text{ and} \\ E(\tilde{F}^*_{1,i}) = \frac{Var(\tilde{F}_{1,i})}{F_{1,i}} + F_{1,i} - 1 \end{cases} \tag{43}$$

Indeed, then $\tilde{F}^{*j}(i,k)$ is only non zero for i=1 and j=1, then $\tilde{F}^{*1}(1,k) = \tilde{F}^{\dagger 1}(1,k) - 1$ with $P(\tilde{F}^{\dagger 1}_{1,k} = n_1) = \frac{n_1}{F_{1,k}} P(\tilde{F}_{1,k} = n_1)$ and therefore $P(\tilde{F}^{*1}_{1,k} = n_1) = P(\tilde{F}^{\dagger 1}_{1,k} = n_1 + 1) = \frac{n_1+1}{F_{1,k}} P(\tilde{F}_{1,k} = n_1 + 1)$. One can verify, that $\sum_x P(\tilde{F}^*_{1,i} = x) = \sum_x \frac{x+1}{F_{1,i}} P(\tilde{F}_{1,i} = x + 1)) = \frac{1}{F_{1,i}} \sum_x (x+1) P(\tilde{F}_{1,i} = x + 1) = \frac{F_{1,i}}{F_{1,i}} = 1$. From this one can compute $E(\tilde{F}^*_{1,i}) = \frac{1}{F_{1,i}} \sum_{y=x-1>0} (y-1) y P(\tilde{F}_{1,i} = y) = \frac{1}{F_{1,i}} \left(\sum_y y^2 P(\tilde{F}_{1,i} = y) - \sum_y y P(\tilde{F}_{1,i} = y) \right) = \frac{1}{F_{1,i}} \left(Var(\tilde{F}_{1,i}) + F^2_{1,i} - F_{1,i} \right) = \frac{Var(\tilde{F}_{1,i}}{F_{1,i}} + F_{1,i} - 1$, as per Eqs 38 and 39.
If $\tilde{F}_{1,i}$ is Bernoulli, we get $P(\tilde{F}^*_{1,i} = 0) = \frac{1}{F_{1,i}} P(\tilde{F}_{1,i} = 1) = \frac{F_{1,i}}{F_{1,i}} = 1$ as per the result above.
If $\tilde{F}_{1,i}$ is Poisson, we get $P(\tilde{F}^*_{1,i} = x) = \frac{x+1}{\lambda} \frac{\lambda^{x+1} e^{-\lambda}}{(x+1)!} = \frac{\lambda^x e^{-\lambda}}{x!}$, which is a Poisson of same parameter, as per the result above.

We can relate these common cases with the general case, as $\tilde{\mathbf{Z}} = \tilde{\mathbf{F}}^* \mathbf{P_F}$

7.5 SM: Expectation, Variance and *PGF* of the building blocks

The building blocks of the processes are the stochastic projectors $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{S}}$ from which one can generate $\mathbf{P_F}$ and $\mathbf{P_S}$ (eq 2).

7.5.1 Expectations of the building blocks

From the definition of the generation structured population stochastic projectors $\tilde{\mathbb{S}} = \mathbf{I} \otimes \tilde{\mathbf{S}}$ and $\tilde{\mathbb{F}} = \mathbf{J_1} \otimes \tilde{\mathbf{F}}$ we can get the expectations $\mathbb{S} = \mathbf{I} \otimes \mathbf{S}$ and $\mathbb{F} = \mathbf{J_1} \otimes \mathbf{F}$ as well as $\mathbb{P}_{\mathbb{S}} = \mathbf{I} \otimes \mathbf{P_S}$ and $\mathbb{P}_{\mathbb{F}} = \mathbf{J_1}^T \otimes \mathbf{F}$ and from these we have $\mathbb{M} = \mathbb{F} + \mathbb{S}, \mathbb{P} = \mathbb{P}_{\mathbb{F}} + \mathbb{P}_{\mathbb{S}}$.

7.5.2 *Pgf* and Variance of the building blocks

Since the components of $\tilde{\mathbb{M}} = \tilde{\mathbb{S}} + \tilde{\mathbb{F}}$ are independent, we have:

$$\boldsymbol{\varphi}_{\tilde{\mathbb{M}}} = \boldsymbol{\varphi}_{\tilde{\mathbb{S}}} \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{F}}} \tag{44}$$

$$\boldsymbol{Var}^*(\tilde{\mathbb{M}}) = \boldsymbol{Var}^*(\tilde{\mathbb{S}}) + \boldsymbol{Var}^*(\tilde{\mathbb{F}}). \tag{45}$$

Survival processes Survival processes, either projecting over time via $\tilde{\mathbb{S}}$, or backwards in time, via $\tilde{\mathbb{P}_{\mathbb{S}}}$, are multinoulli (or generalised Bernoulli) processes. Therefore they *pgf* and variance can be obtained directly from $\mathbb{S}$ and $\mathbb{P}_{\mathbb{S}}$:

$$\varphi_{\tilde{\mathbb{S}_j}}(\boldsymbol{x}) = \sum_i \mathbb{S}_{i,j}(x_i - 1) + 1. \tag{46}$$

Note that this implies that $\varphi_{\tilde{\mathbb{S}_i}}(\boldsymbol{x}) = \varphi_{\tilde{\mathbb{S}}_{(c_i,g_i)}}(\boldsymbol{x}) = \sum_{c_j,g_j} \mathbb{S}_{(c_j,g_j),(c_i,g_i)}(x_{cj,g_j} - 1) + 1 = \sum_{c_j} \mathbf{S}_{c_j,c_j}(x_{cj,g_i} - 1) + 1$, which allows to construct $\varphi_{\tilde{\mathbb{S}_i}}$ directly from $\mathbf{S}$. From this, we can generate $\boldsymbol{\varphi}_{\tilde{\mathbb{S}}} = (\varphi_{\tilde{\mathbb{S}_1}}, \varphi_{\tilde{\mathbb{S}_2}}, \ldots \varphi_{\tilde{\mathbb{S}}_{sgm}})$. Similarly, we have $\varphi_{\mathbb{P}_{\tilde{\mathbb{S}}_j}}(\boldsymbol{x}) = \sum_i \mathbb{P}_{\mathbb{S}i,j}(x_i - 1) + 1$, from which we get $\boldsymbol{\varphi}_{\tilde{\mathbb{P}_{\mathbb{S}}}}$. In most calculations, we will "only" need $\boldsymbol{\varphi}_{\tilde{\mathbf{P_S}}}$, which can be obtained similarly; $\varphi_{\tilde{\mathbf{P_S}}_{c_j}}(x_1, x_2, ..., x_s) = \sum_{c_i} \mathbf{P_S}_{c_i,c_j}(x_{c_i} - 1) + 1$.

For the variances, we have that the (k,i) element of $\boldsymbol{Var}(\tilde{\mathbb{S}}_j)$ (in square matrix form), which correspond to $Cov(\tilde{\mathbb{S}}_{i,j}, \tilde{\mathbb{S}}_{k,j})$ is such that

$$\begin{cases} \boldsymbol{Var}(\tilde{\mathbb{S}}_j)_{i,k} = \mathbb{S}_{i,j}(1 - \mathbb{S}_{i,j}) \text{ , if } i = k \\ \mathbb{S}_{i,j}\mathbb{S}_{k,j} \text{ , if } i \neq k. \end{cases} \tag{47}$$

From these, we have $\boldsymbol{Var}^*(\tilde{\mathbb{S}})$, which j^{th} column is $\boldsymbol{vec}(\boldsymbol{Var}(\tilde{\mathbb{S}}_j))$. Similarly $\boldsymbol{Var}(\tilde{\mathbb{P}_{\mathbb{S}}}_j)_{i,k} = \mathbb{P}_{\mathbb{S}i,j}(1 - \mathbb{P}_{\mathbb{S}i,j})$ if $i = k$ and $\mathbb{P}_{\mathbb{S}i,j}\mathbb{P}_{\mathbb{S}k,j}$ otherwise. And finally, for $\tilde{\mathbf{P_S}}$, we have $\boldsymbol{Var}(\tilde{\mathbf{P_S}}_{c_j})_{c_i,c_k} = \mathbf{P_S}_{c_i,c_j}(1 - \mathbf{P_S}_{c_i,c_j})$ if $c_i = c_k$ and $\mathbf{P_S}_{c_i,c_j}\mathbf{P_S}_{c_k,c_j}$ otherwise.

Reproduction Process

Bernoulli Reproduction In that case, $\tilde{\mathbb{F}}$ is a multinomial process and its variance and *pfg* can be obtained with the above formulas that for $\tilde{\mathbb{S}}$: $\varphi_{\tilde{\mathbb{F}}_i}(\boldsymbol{x}) = \sum_j \mathbb{F}_{j,i}(x_j - 1) + 1$. and $\boldsymbol{Var}(\tilde{\mathbb{F}}_i)_{j,k} = \mathbb{F}_{j,i}(1 - \mathbb{F}_{j,i})$ if $j = k$ and $\mathbb{F}_{j,i}\mathbb{F}_{k,i}$ otherwise. In that case there is no biased reproduction (see sec 7.4) and therefore $\tilde{\mathbb{F}^*} = \mathbf{0}$.

Poisson reproduction We write that $\tilde{\mathbb{F}}_j$ is the sum of independent Poisson processes, and get

$$\varphi_{\tilde{\mathbb{F}}_j}(\boldsymbol{x}) = \varphi_{\tilde{\mathbb{F}}_{c_j,g_j}}(\boldsymbol{x}) = exp(\sum_{c_i} F_{c_i,c_j}(x_{c_i,g_j+1} - 1)) \tag{48}$$

and

$$\boldsymbol{Var}^*_j(\tilde{\mathbb{F}}) = \boldsymbol{vec}(\boldsymbol{Var}(\tilde{\mathbb{F}}_j)) = \boldsymbol{vec}(\boldsymbol{Var}(\tilde{\mathbb{F}}_{c_j,g_j})) = \boldsymbol{vec}(\sum_{c_i} F_{c_i,c_j}\boldsymbol{e}_{(c_i,g_j+1)}\boldsymbol{e}^T_{(c_i,g_j+1)}) \tag{49}$$

In this case the biased reproduction is the same than reproduction (see sec 7.4) and $\boldsymbol{\varphi}_{\tilde{\mathbb{F}^*}} = \boldsymbol{\varphi}_{\tilde{\mathbb{F}}}$ and $\boldsymbol{Var}^*(\tilde{\mathbb{F}^*}) = \boldsymbol{Var}^*(\tilde{\mathbb{F}})$.

One offspring class reproduction If there is only one class of offspring (class 1), then

$$\boldsymbol{Var}^*_j(\tilde{\mathbb{F}}) = \boldsymbol{vec}(\boldsymbol{Var}(\tilde{\mathbb{F}}_j)) = \boldsymbol{vec}(Var(\tilde{F}_{1,c_j})\boldsymbol{e}_{(1,g_j+1)}\boldsymbol{e}^T_{(1,g_j+1)}) = Var(\tilde{F}_{1,c_j})\boldsymbol{vec}(\boldsymbol{e}_{(1,g_j+1)}\boldsymbol{e}^T_{(1,g_j+1)}) \tag{50}$$

and

$$\varphi_{\tilde{\mathbb{F}}_j} = \sum_l P(\tilde{F}_{1,c_j} = l)x^l_{1,g_j+1}. \tag{51}$$

Similarly for the biased reproduction, we have

$$\begin{cases} \boldsymbol{Var}^*_j(\tilde{\mathbb{F}^*}) = Var(\tilde{F}^*_{1,c_j})\boldsymbol{vec}(\boldsymbol{e}_{(1,g_j+1)}\boldsymbol{e}^T_{(1,g_j+1)}) \text{ with} \\ Var(\tilde{F}^*_{1,c_j}) = \sum_{n=2} \frac{n(n-1)^2}{F_{1,c_j}} P(F_{1,c_j} = n) \end{cases} \tag{52}$$

and

$$\varphi_{\tilde{\mathbb{F}}^*_j} = \sum_l P(\tilde{F}^*_{1,c_j} = l)x^l_{1,g_j+1} = \sum_l \frac{l+1}{F_{1,c_j}} P(\tilde{F}_{1,c_j} = l+1)x^l_{1,g_j+1} \tag{53}$$

7.6 SM: Description of Building Blocks

The building blocks of the processes are the stochastic projectors $\tilde{\mathbb{M}}$ and $\tilde{\mathbb{P}}$ and their components $\tilde{\mathbb{S}}$,$\tilde{\mathbb{F}}$, $\tilde{\mathbb{P}_{\mathbb{F}}}$ and $\tilde{\mathbb{P}_{\mathbb{S}}}$ of which we need the variance and the *pgf*. Since the components of $\tilde{\mathbb{M}} = \tilde{\mathbb{S}} + \tilde{\mathbb{F}}$ are independent, we have: $\boldsymbol{Var}^*(\tilde{\mathbb{M}}) = \boldsymbol{Var}^*(\tilde{\mathbb{S}}) + \boldsymbol{Var}^*(\tilde{\mathbb{F}})$ and $\boldsymbol{\varphi}_{\tilde{\mathbb{M}}} = \boldsymbol{\varphi}_{\tilde{\mathbb{S}}} \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{F}}}$.

survival Survival, either projecting over time via $\tilde{\mathbb{S}}$, or backwards in time, via $\tilde{\mathbb{P}_{\mathbb{S}}}$, is a multinoulli process: There is at most one descendant over a time-step via survival, and maybe zero if the individual dies and there is a most one "ancestor" via survival at the previous time-step and potentially none if the probability of it having been produced via fertility is 1, that is, if the individual could not have been "already born" at the previous time-step. This implies that $\boldsymbol{Var}(\tilde{\mathbf{S}}_i) = \{Cov(\tilde{\mathbf{S}}_{j,i}, \tilde{\mathbf{S}}_{k,i})\}_{j,k}$ and similarly $\boldsymbol{Var}(\tilde{\mathbb{S}}_i) = \{Cov(\tilde{\mathbb{S}}_{j,i}, \tilde{\mathbb{S}}_{k,i})\}_{j,k}$ has the following structure $Cov(\tilde{\mathbb{S}}_{j,i}, \tilde{\mathbb{S}}_{k,i}) = \begin{cases} \mathbb{S}_{j,i}(1-\mathbb{S}_{j,i}) & \text{if } j = k \\ -\mathbb{S}_{j,i}\mathbb{S}_{k,i} & \text{if } j \neq k \end{cases}$ and similarly for $Cov(\tilde{\mathbb{P}_{\mathbb{S}}}_{j,i}, \tilde{\mathbb{P}_{\mathbb{S}}}_{k,i})$.

With regards to the *pgf*, this means that $\varphi_{\tilde{\mathbb{S}_i}}(\boldsymbol{x}) = P(\tilde{\mathbb{S}}_i = \mathbf{0}) + \sum_j P(\tilde{\mathbb{S}}_i = \boldsymbol{e}_j)x_j = \sum_j \mathbb{S}_{j,i}(x_j - 1) + 1 =$, from which we can compute $\boldsymbol{\varphi}_{\tilde{\mathbb{S}}} = (\varphi_{\tilde{\mathbb{S}_1}}, \varphi_{\tilde{\mathbb{S}_2}}, ...\varphi_{\tilde{\mathbb{S}_k}k})$, and similarly for $\boldsymbol{\varphi}_{\tilde{\mathbb{P}_{\mathbb{S}}}}$. Note that one can simplify this by writing the *pgf*s with a tensor notation: $\varphi_{\tilde{\mathbb{S}_i}}(\boldsymbol{x}) = \varphi_{\tilde{\mathbb{S}}_{i_s,i_g}}(\boldsymbol{x}) = \sum_{j_s,j_g} \mathbb{S}_{(j_s,j_g),(i_s,i_g)}(x_{j_s,j_g} - 1) + 1 = \sum_{j_s} \mathbb{S}_{(j_s,i_g),(i_s,i_g)}(x_{j_s,i_g} - 1) + 1 = \sum_{j_s} \mathbf{S}_{j_s,i_s}(x_{j_s,i_g} - 1) + 1 = \varphi_{\tilde{\mathbf{S}_i}}(x_{1,i_g}, x_{2,i_g}, ..., x_{s,i_g})$

Reproduction Let us consider the properties of the reproduction processes in the 3 cases we have highlighted:

Bernoulli Reproduction In that case, $\tilde{\mathbb{S}}$ is a multinomial process and its Variance and *pfg* can be obtained with the above formulas sdfsds. In that case there is no biased reproduction.

Independent Poissons reproduction Here, for all i, $\tilde{\mathbb{F}}_{i_s,i_g} = \sum_{(k_s,k_g)} \tilde{\mathbb{F}}_{(k_s,k_g),(i_s,i_g)}\boldsymbol{e_{k_s,k_g}} = \sum_{k_s} \tilde{\mathbb{F}}_{(k_s,i_g+1),(i_s,i_g)}\boldsymbol{e_{k_s,i_g+1}}$. That is, $\tilde{\mathbb{F}}_{i_s,i_g<g_{max}} = \sum_{k_s} \tilde{\mathbb{F}}_{(k_s,i_g+1),(i_s,i_g)}\boldsymbol{e_{k_s,i_g+1}}$ and $\tilde{\mathbb{F}}_{i_s,g_{max}} = 0$. Further, $\tilde{\mathbb{F}}_{i_s,i_g<g_{max}} = \sum_{k_s} \tilde{\mathbb{F}}_{(k_s,i_g+1),(i_s,i_g)}\boldsymbol{e_{k_s,i_g+1}} = \sum_{k_s} \tilde{\mathbf{F}}_{k_s,i_s}\boldsymbol{e_{k_s,i_g+1}}$

, sum of independent processes, with $\tilde{\mathbf{F}}_{k_s,i_s)}$ a Poisson of parameter $\mathbf{F}_{k_s,i_s)}$. This implies that $\boldsymbol{Var}(\tilde{\mathbb{F}}_{i_s,i_g<g_{max}}) = \sum_{k_s} \boldsymbol{Var}(\tilde{\mathbf{F}}_{k_s,i_s}\boldsymbol{e_{k_s,i_g+1}})$ with $\boldsymbol{Var}(\tilde{\mathbf{F}}_{k_s,i_s}\boldsymbol{e_{k_s,i_g+1}}) = \boldsymbol{E}(\tilde{\mathbf{F}}_{k_s,i_s}\boldsymbol{e_{k_s,i_g+1}}\tilde{\mathbf{F}}_{k_s,i_s}\boldsymbol{e}^T_{\boldsymbol{k_s,i_g+1}}) - \boldsymbol{E}(\tilde{\mathbf{F}}_{k_s,i_s}\boldsymbol{e_{k_s,i_g+1}})\boldsymbol{E}(\tilde{\mathbf{F}}_{k_s,i_s}\boldsymbol{e_{k_s,i_g+1}})^T = \boldsymbol{E}(\tilde{F}^2_{k_s,i_s}\mathbf{J_{k_s,i_g+1}}) - F^2_{k,i}\mathbf{J_{k_s,i_g+1}}$, where $\mathbf{J_{k_s,i_g+1}} = \boldsymbol{e_{k_s,i_g+1}}\boldsymbol{e}^T_{\boldsymbol{k_s,i_g+1}}$ is the matrix with 0s everywhere but for $\mathbf{J_{k_s,i_g+1}}((k_s, i_g+1), (k_s, i_g+1)) = 1$. This yields $\boldsymbol{Var}(\tilde{\mathbf{F}}_{k_s,i_s}\boldsymbol{e_{k_s,i_g+1}}) = (E(\tilde{F}^2_{k_s,i_s}) - F^2_{k_s,i_s})\mathbf{J_{k_s,i_g+1}} = Var(\tilde{F}_{k_s,i_s})\mathbf{J_{k_s,i_g+1}} = F_{k_s,i_s}\mathbf{J_{k_s,i_g+1}}$.

Therefore $\boldsymbol{Var}(\tilde{\mathbb{F}}_{i_s,i_g<g_{max}}) = \sum_{k_s} F_{k_s,i_s}\mathbf{J_{k_s,i_g+1}}$

The *pgf* of $\tilde{\mathbb{F}}_{i_s,i_g}$, $\varphi_{\tilde{\mathbb{F}_{i_s,i_g}}}$ is such that $\varphi_{\tilde{\mathbb{F}}_{i_s,i_g}}(\boldsymbol{x}) = \prod_{k_s} \varphi_{\tilde{F}_{k_s,i_s}\boldsymbol{e_{k_s,i_g+1}}}$ with $\varphi_{\tilde{F}_{k_s,i_s}\boldsymbol{e_{k_s,i_g+1}}} = \varphi_{\tilde{F}_{k_s,i_s}}(x_{k_s,i_g+1})$and $\varphi_{\tilde{F}_{k_s,i_s}}(x) = \sum_l P(\tilde{F}_{k_s,i_s} = l)x^l = \sum_l \frac{e^{-F_{k_s,i_s}}F^l_{k_s,i_s}}{l!}x^l = e^{-F_{k_s,i_s}}\sum_l \frac{(F_{k_s,i_s}x)^l}{l!} = e^{-F_{k_s,i_s}}e^{F_{k_s,i_s}x} = e^{F_{k_s,i_s}(x-1)}$ This implies that $\varphi_{\tilde{\mathbb{F}}_{i_s,i_g}}(\boldsymbol{x}) = \prod_{k_s} \varphi_{\tilde{F}_{k_s,i_s}}(x_{k_s,i_g+1}) = \prod_{k_s} e^{F_{k_s,i_s}(x_{k_s,i_g+1}-1)} = exp(\sum_{k_s} F_{k_s,i_s}(x_{k_s,i_g+1} - 1))$

In this case the biased reproduction is the same than reproduction. Note that because in a Poisson, the probability of any number of offspring, however large, is never zero, which implies that the associated *pgf* may have to be saved (for some programs) as the truncation of an infinite polynom. In that case, we recomend to resort to right-truncated poisson distribution, truncated at a number of offspring k_{max} over which the the probability is considered to be, in practice, zero. A right-truncated poisson of paramter λ at k_{max} has probability $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!(1-\sum_{j=0}^{k_{max}} \frac{e^{-\lambda}\lambda^j}{j!}}$

One offspring class If there is only one class of offspring (say class 1), then $\tilde{\mathbb{F}}_i = \tilde{F}_{1,i}\boldsymbol{e_1}$ and $\boldsymbol{Var}(\tilde{\mathbb{F}}_i) = \boldsymbol{Var}(\tilde{F}_{1,i}\boldsymbol{e_1}) = Var(\tilde{F}_{1,i})\mathbf{J}_1$ from the computations just above. We also have $\varphi_{\tilde{\mathbb{F}_i}} = \varphi_{\tilde{F}_{1,i}\boldsymbol{e_1}} = \sum_l P(\tilde{F}_{1,i} = l)x^l_1$. We can do the same for the biased reproduction process:$\boldsymbol{Var}(\tilde{\mathbb{F}^*}_i) = Var(\tilde{F^*}_{1,i})\mathbf{J}_1$ and $\varphi_{\tilde{\mathbb{F}_i}} = \varphi_{\tilde{F}_{1,i}\boldsymbol{e_1}} = \sum_l P(\tilde{F}^*_{1,i} = l)x^l_1 = \sum_l \frac{l+1}{F_{1,i}}P(\tilde{F}_{1,i} = l+1)x^l_1$.

7.7 SM: Top-down description of the process

In this section, we develop the process, compute expectations, variances and pgfs, with a top down approach, starting from the all process and decomposing its elements until reaching the building blocks.

7.7.1 Decomposing according to the state and time-step of the ancestor

Process

distribution of g_j-ancestor

Any kin is related to *ego* via a unique common ancestor, and the last time-step at which its aliveness is known t describes a complete set. For a given *ego* in state j, corresponding to class c_j and generation g_j, we consider two cases. If $g_j > 1$, this unique common ancestor is *ego*'s g_j-ancestor (therefore in generation 1), and t correspond to the time-step at which the g_j-ancestor gave birth to the $g_j - 1$-ancestor (if $g_j = 2$, this $g_j - 1$-ancestor is *ego* itself). We call the class of that ancestor, t time-step age c_a. If $g_j = 1$, this unique common ancestor is *ego* itself and therefore $t = 0$ and $c_a = c_j$.

If $g_j > 1$, the probability that *ego*'s g_j-ancestor gave birth to *ego*'s $g_j - 1$-ancestor t time-steps ago while in class c_a is the $((c_a, 1), (c_j, g_j))$ entry of $\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}$, which we can write $\boldsymbol{e}_{c_a,1}^T \mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}\boldsymbol{e}_{c_j,g_j}$, where $\boldsymbol{e}_{c_i,g_i} = \boldsymbol{e}_i$ is the vector of 0s expect for a 1 in the jth entry and ^{T} denotes matrix transposition. Indeed as $\mathbb{P}\boldsymbol{e}_{c_i,g_i}$ is the vector of probabilities (it sums to 1 as $\mathbb{P}$ is column stochastic) that the unique "ancestor" of *ego* (we denote "ancestors", ancestors via survival or reproduction, by contrast to ancestors which are ancestors via reproduction), was in a given state at the previous time-step, $\mathbb{P}^t\boldsymbol{e}_{c_i,g_i}$ is the vector of probabilities (it sums to 1 as $\mathbb{P}^t$ is also column stochastic) that the unique "ancestor" of *ego*, via survival or reproduction, was in a given state t time-step ago. This "ancestor" is unique as $\mathbb{P}$ tracks the "ancestors" in succession from the time they give birth to the next ancestor to their birth; once an "ancestor" has given birth to the next ancestor, it is not in the ancestry line any longer and not an "ancestor" anymore.

Therefore $\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}\boldsymbol{e}_{c_j,g_j}$ is the vector of probabilities that the unique "ancestor" of *ego* alive t time-step ago was in a given state as it gave birth the following ancestor. These do not sum to one, as it is also possible that this ancestor did not give birth but just "survived" during that time-step. In particular, $P_{c_j,g_j}(c_a, t) = \boldsymbol{e}_{c_a,1}^T \mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}\boldsymbol{e}_{c_j,g_j}$, is therefore the probability that the g_j-ancestor of *ego* in class c_j was alive t time-step ago and in class c_a as it gave birth the $g_j - 1$-ancestor. If $g_j = 1$, this unique common ancestor is *ego* itself and therefore $t = 0$ and $c_a = c_j$ and $P_{c_j,g_j}(c_a, 0) = \delta(c_j, c_a)\delta(g_j, 1)$ is the probability that the 1-ancestor of *ego*, that is *ego* itself, is in class c_a now. In other words, for *ego* in state (c_j, g_j), the probability its g_j-ancestor was last know alive t time-step ago and in state c_a (as it gave birth the $g_j - 1$-ancestor if $g_j > 1$, or as *ego*, now,if $g_j = 1$) is

$$P_{c_j,g_j}(c_a, t) = P_j(c_a, t) = \begin{cases} \boldsymbol{e}_{\boldsymbol{c_a}}^T \mathbb{J}^{*T}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}\boldsymbol{e}_{\boldsymbol{(c_j,g_j)}} & \text{if } t > 0 \\ \boldsymbol{e}_{\boldsymbol{c_a}}^T \mathbb{J}^{*T}\boldsymbol{e}_{\boldsymbol{(c_j,g_j)}} & \text{if } t = 0 \end{cases} \tag{4c}$$

where $\mathbb{J}^* = (\boldsymbol{e}_1^{(g_m)} \otimes \mathbf{I}_s)$ is a $s \times sg_m$ zero-matrix but for the diagonal of the first block. For a given state j of *ego*,

$\{P_j(c_a,t)\}_{c_a,t}$ describes a complete set as we show in 7.11. From this we can write the stochastic process $\tilde{\mathbb{K}}_j$ as follows:

$$\tilde{\mathbb{K}}_j = \tilde{\mathbb{K}}_{c_j,g_j} = \tilde{\mathbb{D}}_{c_a}^{(t)} \text{ with probability } P_j(c_a,t) \tag{4a}$$

where $\tilde{\mathbb{D}}_{c_a}^{(t)}$ is the number of descendants of an individual that was in class c_a (and generation 1), t time-steps ago, but for the ancestors of *ego* (but for the g_j-ancestor). This process is well defined (see 7.9). Note that $\tilde{\mathbb{D}}_{c_a}^{(t)}$ is only defined on $1 \leq c_a \leq s$ (it can be extended to all states, but we are only concerned on the first s ones which correspond to the descendants of the ancestor in generation 1)

Expectation

We decompose the expectations of $\tilde{\mathbb{K}}_j$ according to (c_a,t): $\mathbb{K}_j = \boldsymbol{E}(\tilde{\mathbb{K}}_j) = \boldsymbol{E}\left(\boldsymbol{E}\left(\tilde{\mathbb{K}}_j|(c_a,t)\right)\right) = \boldsymbol{E}_{(c_a,t)}\left(\boldsymbol{E}\left(\tilde{\mathbb{D}}_{c_a}^{(t)}\right)\right) = \boldsymbol{E}_{(c_a,t)}(\mathbb{D}_{c_a}^{(t)}) = \sum_{c_a,t}\mathbb{D}_{c_a}^{(t)}P_j(c_a,t) = \sum_t\sum_{c_a}\mathbb{D}^{(t)}\boldsymbol{e}_{c_a}\boldsymbol{e}_{c_a}^T\mathbb{A}\boldsymbol{e}_{(\boldsymbol{c_j},\boldsymbol{g_j})}$ with $\mathbb{A} = \mathbb{J}^{*T}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}$ if $t > 0$ and $\mathbb{A} = \mathbb{J}^{*T}$ if $t = 0$. We note that $\sum_{c_a}\boldsymbol{e}_{c_a}\boldsymbol{e}_{c_a}^T = \mathbf{I}$ and therefore

$$\mathbb{K} = \mathbb{D}^{(0)}\mathbb{J}^{*T} + \sum_{t>0}\mathbb{D}^{(t)}\mathbb{J}^{*T}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1} \tag{54}$$

PGF

We turn Eq 4a into a pgf:

$$\varphi_{\tilde{\mathbb{K}}_j} = \sum_{c_a,t}P_j(c_a,t)\varphi_{\tilde{\mathbb{D}}_{c_a}^{(t)}} \tag{7a}$$

and from this we have $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}} = \{\varphi_{\tilde{\mathbb{K}}_j}\}_j$. We can embed the probabilities of Eq.4c into Eq. **??** and write directly

$$\boldsymbol{\varphi}_{\tilde{\mathbb{K}}} = \boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(0)}}\mathbb{J}^* + \sum_{t>0}\boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(t)}}\mathbb{J}^*\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1} \tag{55}$$

The first component populates the first s elements of $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}}$, corresponding to the $\varphi_{\tilde{\mathbb{K}}_{c_j,1}}$ that is to the direct descendants of *ego*. The second component populates the remaining $s(g_m-1)$ elements of $\boldsymbol{\varphi}_{\tilde{\mathbb{K}}}$, corresponding to the $\varphi_{\tilde{\mathbb{K}}_{c_j,g_j\geq 2}}$ (the first $s \times s$ block of $(\boldsymbol{e}_1^{(g_m)} \otimes \mathbf{I}_s)\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}$ is by construction $\mathbf{0}$).

Variance

Similarly to the expectations, we decompose the variance of $\tilde{\mathbb{K}}_j$ according to (c_a,t):

$$\boldsymbol{Var}(\tilde{\mathbb{K}}_j) = \underbrace{\boldsymbol{E}_{(c_a,t)}\left(\boldsymbol{Var}(\tilde{\mathbb{K}}_j|(c_a,t)\right)}_{\boldsymbol{Var}_{des}(\tilde{\mathbb{K}}_j)} + \underbrace{\boldsymbol{Var}_{(c_a,t)}\left(\boldsymbol{E}(\tilde{\mathbb{K}}_j|(c_a,t)\right)}_{\boldsymbol{Var}_{anc}(\tilde{\mathbb{K}}_j)}$$

In this decomposition, $\boldsymbol{Var}_{des}(\tilde{\mathbb{K}}_j)$ corresponds to the variance in the number of kin caused by variance in the number of descendants and $\boldsymbol{Var}_{anc}(\tilde{\mathbb{K}}_j)$ corresponds to the variance in the number of kin caused by stochasticity in the time and state at/in which the ancestor gave birth. We have $\left(\tilde{\mathbb{K}}_j|(c_a,t)\right) = \tilde{\mathbb{D}}_{c_a}^{(t)}$, therefore, from the general formula of the covariance of a random vector, we can write $\boldsymbol{Var}_{gs}(\tilde{\mathbb{K}}_j) = \boldsymbol{Var}_{(c_a,t)}(\mathbb{D}_{c_a}^{(t)}) = \boldsymbol{E}_{(c_a,t)}(\mathbb{D}_{c_a}^{(t)}\mathbb{D}_{c_a}^{(t)T})) - \boldsymbol{E}_{(c_a,t)}(\mathbb{D}_{c_a}^{(t)}).\boldsymbol{E}_{(c_a,t)}(\mathbb{D}_{c_a}^{(t)})^T$ in which we recognise $\boldsymbol{E}_{(c_a,t)}\left(\mathbb{D}_{c_a}^{(t)}\right) = \mathbb{K}_j$. And therefore $\boldsymbol{Var}_{gs}(\tilde{\mathbb{K}}_j) = \sum_{c_a,t}P_j(c_a,t)\mathbb{D}_{c_a}^{(t)}\mathbb{D}_{c_a}^{(t)T} - \mathbb{K}_j\mathbb{K}_j^T$, and from there:

$$\boldsymbol{Var}(\tilde{\mathbb{K}}_j) = \underbrace{\sum_{c_a,t}P_j(c_a,t)\boldsymbol{Var}(\tilde{\mathbb{D}}_{c_a}^{(t)})}_{\boldsymbol{Var}_{des}(\tilde{\mathbb{K}}_j)} + \underbrace{\sum_{c_a,t}P_j(c_a,t)\mathbb{D}_{c_a}^{(t)}\mathbb{D}_{c_a}^{(t)T} - \mathbb{K}_j\mathbb{K}_j^T}_{\boldsymbol{Var}_{anc}(\tilde{\mathbb{K}}_j)} \tag{9}$$

We can integrate the probabilities inside the computation and generalise to all j by writing that $\boldsymbol{vec(Var}_{des}(\tilde{\mathbb{K}}_j)) = \boldsymbol{Var}^*_{j\,des}(\tilde{\mathbb{K}}) = \boldsymbol{Var}^*_{c_j,g_j\,des}(\tilde{\mathbb{K}}) = \sum_{c_a,t} P_j(c_a,t)\boldsymbol{Var}^*_{c_a}(\tilde{\mathbb{D}}^{(t)}) = \sum_{c_a} P_j(c_a,0)\boldsymbol{Var}^*_{c_a}(\tilde{\mathbb{D}}^{(0)}) + \sum_{c_a,t>0} P_j(c_a,t)\boldsymbol{Var}^*_{c_a}(\tilde{\mathbb{D}}^{(t)}) = \boldsymbol{Var}^*(\tilde{\mathbb{D}}^{(0)})\mathbb{J}^{*T} + \sum_{t>0}\boldsymbol{Var}^*(\tilde{\mathbb{D}}^{(t)})\mathbb{J}^{*T}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}$.

We also have that $\boldsymbol{vec(Var}_{anc}(\tilde{\mathbb{K}}_j)) = \boldsymbol{Var}^*_{j\,anc}(\tilde{\mathbb{K}}) = \boldsymbol{Var}^*_{c_j,g_j\,gs}(\tilde{\mathbb{K}}) = \sum_{c_a,t} P_j(c_a,t)\boldsymbol{vec(}\mathbb{D}^{(t)}_{c_a}\mathbb{D}^{(t)T}_{c_a}) - \boldsymbol{vec(}\mathbb{K}_j\mathbb{K}_j^T)$.

We can write $\boldsymbol{vec(}\mathbb{K}_j\mathbb{K}_j^T) = \boldsymbol{vec(}\mathbb{K}\boldsymbol{e}_j\boldsymbol{e}_j^T\mathbb{K}_j^T) = (\mathbb{K}\otimes\mathbb{K})\boldsymbol{vec(e}_j\boldsymbol{e}_j^T) = (\mathbb{K}\otimes\mathbb{K})\boldsymbol{vec(e}_j\boldsymbol{e}_j^T)$. Then $\boldsymbol{vec(e}_j\boldsymbol{e}_j^T) = \boldsymbol{e}_j\otimes\boldsymbol{e}_j$. Similarly $\boldsymbol{vec(}\mathbb{D}^{(t)}_{c_a}\mathbb{D}^{(t)T}_{c_a}) = (\mathbb{D}^{(t)}\otimes\mathbb{D}^{(t)})\boldsymbol{vec(e}_{c_a}\boldsymbol{e}_{c_a}^T)$.

Therefore $\boldsymbol{Var}^*_{j\,anc}(\tilde{\mathbb{K}}) = \sum_{c_a,t} P_j(c_a,t)(\mathbb{D}^{(t)}\otimes\mathbb{D}^{(t)})\boldsymbol{vec(e}_{c_a}\boldsymbol{e}_{c_a}^T) - (\mathbb{K}\otimes\mathbb{K})\boldsymbol{vec(e}_j\boldsymbol{e}_j^T)$ which is equal to $\sum_{c_a}(\mathbb{D}^{(0)}\otimes\mathbb{D}^{(0)})\boldsymbol{vec(e}_{c_a}\boldsymbol{e}_{c_a}^T)\boldsymbol{e}_{c_a}^T\mathbb{J}^{*T}\boldsymbol{e}_j + \sum_{t>0}\sum_{c_a}(\mathbb{D}^{(t)}\otimes\mathbb{D}^{(t)})\boldsymbol{vec(e}_{c_a}\boldsymbol{e}_{c_a}^T)\boldsymbol{e}_{c_a}^T\mathbb{J}^{*T}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}\boldsymbol{e}_j - (\mathbb{K}\otimes\mathbb{K})\boldsymbol{vec(e}_j\boldsymbol{e}_j^T)$.

Call $\mathbb{J}_2 = \sum_{c_a}\boldsymbol{vec(e}_{c_a}\boldsymbol{e}_{c_a}^T)\boldsymbol{e}_{c_a}^T\mathbb{J}^{*T}$ (a $(s^2\times sg_m)$ matrix) and $\mathbb{J}_3 = \sum_j\boldsymbol{vec(e}_j\boldsymbol{e}_j^T)\boldsymbol{e}_j^T$ (a $((sg_m)^2\times sg_m)$ matrix). Then $\boldsymbol{Var}^*_{j\,gs}(\tilde{\mathbb{K}}) = (\mathbb{D}^{(0)}\otimes\mathbb{D}^{(0)})\mathbb{J}_6\boldsymbol{e}_j + \sum_{t>0}(\mathbb{D}^{(t)}\otimes\mathbb{D}^{(t)})\mathbb{J}_6\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}\boldsymbol{e}_j - (\mathbb{K}\otimes\mathbb{K})\mathbb{J}_7\boldsymbol{e}_j$ and therefore

$$\boldsymbol{Var}^*(\tilde{\mathbb{K}}) = \underbrace{\boldsymbol{Var}^*(\tilde{\mathbb{D}}^{(0)})\mathbb{J}^{*T} + \sum_{t>0}\boldsymbol{Var}^*(\tilde{\mathbb{D}}^{(t)})\mathbb{J}^{*T}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}}_{\boldsymbol{Var}^*_{des}(\tilde{\mathbb{K}})} + \underbrace{(\mathbb{D}^{(0)}\bigotimes\mathbb{D}^{(0)})\mathbb{J}_2 + \sum_{t>0}(\mathbb{D}^{(t)}\bigotimes\mathbb{D}^{(t)})\mathbb{J}_2\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1} - (\mathbb{K}\bigotimes\mathbb{K})\mathbb{J}_3}_{\boldsymbol{Var}^*_{anc}(\tilde{\mathbb{K}})} \tag{56}$$

7.7.2 Decomposing according to older, younger and same-litter sisters

Process

We now turn our attention to the number of descendants, $\tilde{\mathbb{D}}^{(t)}_{c_a}$, which is the sum of three independent processes (once one has conditioned on (c_a,t)), the number of descendants via younger sisters, same-litter sisters and older sisters, of the (g_j-1)-ancestor. We decompose, $\tilde{\mathbb{D}}^{(t)}$ – the number of descendants alive today of an individual that was last know alive t time-steps ago (and therefore giving birth to *ego*'s g_j-1 ancestor if $g_j>1$– according to whether these descendants come from older, younger or same-litter sisters of the $(g-1)$-ancestor.

$$\tilde{\mathbb{D}}^{(t)} = \tilde{\mathbb{D}}\mathbb{y}^{(t)} + \tilde{\mathbb{D}}\mathbb{s}^{(t)} + \tilde{\mathbb{D}}\mathbb{o}^{(t)} \tag{57}$$

Note that, in the particular case where $t=0$, all kin stem from older sisters, and therefore $\tilde{\mathbb{D}}\mathbb{y}^{(t)} = \tilde{\mathbb{D}}\mathbb{s}^{(t)} = \mathbf{0}$ and $\tilde{\mathbb{D}}^{(0)} = \tilde{\mathbb{D}}\mathbb{o}^{(0)}$.

Expectation

We simply have

$$\mathbb{D}^{(t)} = \boldsymbol{E}(\tilde{\mathbb{D}}^{(t)}) = \boldsymbol{E}(\tilde{\mathbb{D}}\mathbb{y}^{(t)}) + \boldsymbol{E}(\tilde{\mathbb{D}}\mathbb{s}^{(t)}) + \boldsymbol{E}(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) = \mathbb{D}\mathbb{y}^{(t)} + \mathbb{D}\mathbb{s}^{(t)} + \mathbb{D}\mathbb{o}^{(t)}. \tag{58}$$

PGF

A classic result on the *pgf* of the sum of independent random variables, is that it is the product of the *pgf*s of the r.v.s, therefore:

$$\forall c_a, \varphi_{\tilde{\mathbb{D}}^{(t)}_{c_a}}(\boldsymbol{x}) = \varphi_{\tilde{\mathbb{D}}\mathbb{y}^{(t)}_{c_a}}(\boldsymbol{x})\varphi_{\tilde{\mathbb{D}}\mathbb{s}^{(t)}_{c_a}}(\boldsymbol{x})\varphi_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}_{c_a}}(\boldsymbol{x}).$$

which we can generalise to:

$$\varphi_{\tilde{\mathbb{D}}^{(t)}} = \varphi_{\tilde{\mathbb{D}}\mathbb{y}^{(t)}} \bullet \varphi_{\tilde{\mathbb{D}}\mathbb{s}^{(t)}} \bullet \varphi_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}}, \tag{59}$$

where $\bullet$ is the termwise product of vectors. Note that in the particular case where $t=0$, we have $\varphi_{\tilde{\mathbb{D}}^{(0)}} = \varphi_{\tilde{\mathbb{D}}\mathbb{o}^{(0)}}$

Variance

Since these are independent random variables – for an individual alive now its fate and future offspring only depends on its current state, not on the realisations of past reproductions – we also have:

$$\mathbf{Var}^*(\tilde{\mathbb{D}}^{(t)}) = \mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{y}^{(t)}) + \mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{s}^{(t)}) + \mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)}), \tag{10}$$

and in particular,

$$\boldsymbol{Var}(\tilde{\mathbb{D}}_{c_a}^{(t)}) = \boldsymbol{Var}(\tilde{\mathbb{D}}\mathbb{y}_{c_a}^{(t)}) + \boldsymbol{Var}(\tilde{\mathbb{D}}\mathbb{s}_{c_a}^{(t)}) + \boldsymbol{Var}(\tilde{\mathbb{D}}\mathbb{o}_{c_a}^{(t)}). \tag{60}$$

Note that in the particular case where $t = 0$, we have $\mathbf{Var}^*(\tilde{\mathbb{D}}^{(0)}) = \mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(0)})$

7.7.3 Number of descendants via younger sisters

Process

Descendants of younger sisters $\tilde{\mathbb{D}}\mathbb{y}^{(t)}$ are produced by having an individual (the ancestor), first survive (stochastically, we denote the stochastic counterpart of deterministic $\mathbb{S}$ as $\tilde{\mathbb{S}}$ and do the same for the other matrices) and then be projected over time:

$$\tilde{\mathbb{D}}\mathbb{y}^{(t)} = \tilde{\mathbb{M}}^{t-1}\tilde{\mathbb{S}}\mathbb{J}^* \quad \text{if } t > 0 \text{ and } \mathbf{0} \text{ otherwise} \tag{61}$$

We recognise here a classic multitype branching process. This process is well defined (see 7.9). We have the dummy matrix $\mathbb{J}^*$ since this process is only defined for inputs in generation 1.,

Expectation

We get the expectation, by using the deterministic version of the matrices:

$$\mathbb{D}\mathbb{y}^{(t)} = \mathbb{M}^{t-1}\mathbb{S}\mathbb{J}^* \quad \text{if } t > 0 \text{ and } \mathbf{0} \text{ otherwise} \tag{62}$$

PGF

The *pgf* of the product of two stochastic matrices is the composition of the *pgf*s of these matrices (in reverse order).

$$\boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{y}^{(t)}} = (\boldsymbol{\varphi}_{\tilde{\mathbb{S}}\mathbb{J}^*} \circ \underbrace{\boldsymbol{\varphi}_{\tilde{\mathbb{M}}} \circ \boldsymbol{\varphi}_{\tilde{\mathbb{M}}} \circ \ldots \circ \boldsymbol{\varphi}_{\tilde{\mathbb{M}}}}_{t-1 \text{ times}}) = (\boldsymbol{\varphi}_{\tilde{\mathbb{S}}\mathbb{J}^*} \circ \bigcirc_{t-1} \boldsymbol{\varphi}_{\tilde{\mathbb{M}}}) \quad \text{if } t > 0 \text{ and } \mathbf{1} \text{ if } t = 0 \ , \tag{63}$$

where $\bigcirc_k$ denotes k compositions of the same function (if $k = 0$, it is the identity). Note that $\varphi_{\tilde{\mathbb{D}}\mathbb{y}_{c_a}^{(t)}}(\boldsymbol{x})$ is the c_a element of $\boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{y}^{(t)}}$. $\boldsymbol{\varphi}_{\tilde{\mathbb{S}}\mathbb{J}^*}$ is simply the first s elements of $\boldsymbol{\varphi}_{\tilde{\mathbb{S}}}$.

Variance

Variance of $\tilde{\mathbb{A}}\tilde{\boldsymbol{n}}$, $\tilde{\mathbb{A}}\tilde{\mathbb{B}}$ and $\tilde{\mathbb{M}}^t$

Let us first consider the variance, for any stochastic matrix $\tilde{\mathbb{A}}$ and stochastic vector $\tilde{\boldsymbol{n}}$ focus on the variance of $\tilde{\mathbb{A}}\tilde{\boldsymbol{n}}$. We decompose it, according to the realisations of $\tilde{\boldsymbol{n}}$: $\boldsymbol{Var}(\tilde{\mathbb{A}}\tilde{\boldsymbol{n}}) = \boldsymbol{Var}_{\tilde{\boldsymbol{n}}}\boldsymbol{E}(\tilde{\mathbb{A}}\tilde{\boldsymbol{n}}|\tilde{\boldsymbol{n}} = \boldsymbol{n}) + \boldsymbol{E}_{\tilde{\boldsymbol{n}}}\boldsymbol{Var}(\tilde{\mathbb{A}}\tilde{\boldsymbol{n}}|\tilde{\boldsymbol{n}} = \boldsymbol{n}) = \boldsymbol{Var}_{\tilde{\boldsymbol{n}}}(\mathbb{A}\tilde{\boldsymbol{n}}) + \boldsymbol{E}_{\tilde{\boldsymbol{n}}}\boldsymbol{Var}(\tilde{\mathbb{A}}\boldsymbol{n})$. We have $\boldsymbol{Var}_{\tilde{\boldsymbol{n}}}(\mathbb{A}\tilde{\boldsymbol{n}}) = \boldsymbol{E}((\mathbb{A}\tilde{\boldsymbol{n}})(\mathbb{A}\tilde{\boldsymbol{n}})^T) - \boldsymbol{E}(\mathbb{A}\tilde{\boldsymbol{n}}).\boldsymbol{E}(\mathbb{A}\tilde{\boldsymbol{n}})^T = \mathbb{A}(\tilde{\boldsymbol{n}}\tilde{\boldsymbol{n}}^T)\mathbb{M}^T - \mathbb{A}(\boldsymbol{n}\boldsymbol{n}^T)\mathbb{A}^T =$

$\mathbb{A}(\tilde{\boldsymbol{n}}\tilde{\boldsymbol{n}}^T - \boldsymbol{n}\boldsymbol{n}^T)\mathbb{A}^T = \mathbb{A}\boldsymbol{Var}(\tilde{\boldsymbol{n}})\mathbb{A}^T$. The realisations of $\tilde{\mathbb{M}}$ for the various individuals in $\boldsymbol{n}$ is independent among and within states, and therefore by writing $\boldsymbol{n} = \sum n_i \boldsymbol{e_i}$, we have $\boldsymbol{Var}(\tilde{\mathbb{A}}\boldsymbol{n}) = \sum_i n_i \boldsymbol{Var}(\tilde{\mathbb{A}}\boldsymbol{e_i})$. Remember that $\boldsymbol{Var}^*_i(\tilde{\mathbb{A}}) = \boldsymbol{Var}^*(\tilde{\mathbb{A}}_i) = \boldsymbol{vec}(\boldsymbol{Var}(\tilde{\mathbb{A}}\boldsymbol{e_i}))$, the ith column of $\boldsymbol{Var}^*(\tilde{\mathbb{A}})$, is the variance-covariance matrix of the production of individuals, via $\tilde{\mathbb{A}}$ by an individual in state i (in a vectorised format). Then $\boldsymbol{E_{\tilde{n}}}\boldsymbol{Var}(\tilde{\mathbb{A}}\boldsymbol{n}) = \boldsymbol{E_{\tilde{n}}}\sum_i \tilde{n}_i \boldsymbol{Var}(\tilde{\mathbb{A}}\boldsymbol{e_i}) = \sum_i n_i \boldsymbol{Var}(\tilde{\mathbb{A}}\boldsymbol{e_i}) = \boldsymbol{vec}^{\mathbf{-1}}(\boldsymbol{Var}^*(\tilde{\mathbb{M}})\boldsymbol{n})$ and we have

$$\boldsymbol{Var}(\tilde{\mathbb{A}}\tilde{\boldsymbol{n}}) = \mathbb{A}\boldsymbol{Var}(\tilde{\boldsymbol{n}})\mathbb{A}^T + \boldsymbol{vec}^{\mathbf{-1}}(\boldsymbol{Var}^*(\tilde{\mathbb{A}})\boldsymbol{n}) \tag{64}$$

By setting $\tilde{\boldsymbol{n}} = \tilde{\mathbb{B}}\tilde{\boldsymbol{e}}_i$ in Eq.64, we compute the variance of the product of two stochastic matrices: $\boldsymbol{Var}(\tilde{\mathbb{A}}\tilde{\mathbb{B}}\boldsymbol{e}_i) = \mathbb{A}\boldsymbol{Var}(\tilde{\mathbb{B}}\boldsymbol{e}_i)\mathbb{A}^T + \boldsymbol{vec}^{\mathbf{-1}}(\boldsymbol{Var}^*(\tilde{\mathbb{A}})\mathbb{B}\boldsymbol{e}_i)$. Therefore

$$\boldsymbol{Var}^*_i(\tilde{\mathbb{A}}\tilde{\mathbb{B}}) = (\mathbb{A}\otimes\mathbb{A})\boldsymbol{Var}^*_i(\tilde{\mathbb{B}}) + \boldsymbol{Var}^*(\tilde{\mathbb{A}})\mathbb{B}\boldsymbol{e}_i$$

We can gather all the columns of $\boldsymbol{Var}^*(\tilde{\mathbb{A}}$, and we get

$$\boldsymbol{Var}^*(\tilde{\mathbb{A}}\tilde{\mathbb{B}}) = (\mathbb{A}\otimes\mathbb{A})\boldsymbol{Var}^*(\tilde{\mathbb{B}}) + \boldsymbol{Var}^*(\tilde{\mathbb{A}})\mathbb{B} \tag{65}$$

We use Eq. **??** to compute $\boldsymbol{Var}^*(\tilde{\mathbb{M}}^t)$, by considering that $\tilde{\mathbb{M}}^t = \tilde{\mathbb{M}}\tilde{\mathbb{M}}^{t-1}$:

$$\boldsymbol{Var}^*(\tilde{\mathbb{M}}^t) = (\mathbb{M}\otimes\mathbb{M})\boldsymbol{Var}^*(\tilde{\mathbb{M}}^{t-1}) + \boldsymbol{Var}^*(\tilde{\mathbb{M}})\mathbb{M}^{t-1} \tag{66}$$

This also leads to a direct formulation:

$$\boldsymbol{Var^*}(\tilde{\mathbb{M}}^t) = \sum_{k=1}^{t}(\mathbb{M}\otimes\mathbb{M})^{k-1}(\boldsymbol{Var}^*(\tilde{\mathbb{M}})\mathbb{M}^{t-k})$$

Variance of $\tilde{\mathbb{D}}\mathbb{y}^{(t)}$

We apply Eq. 65 to $\tilde{\mathbb{D}}\mathbb{y}^{(t)} = \tilde{\mathbb{M}}^{t-1}\tilde{\mathbb{S}}\mathbb{J}^*$ directly and get the direct formulation:

$$\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{y}^{(t)}) = (\mathbb{M}\otimes\mathbb{M})^{t-1}\boldsymbol{Var}^*(\tilde{\mathbb{S}})\mathbb{J}^* + \boldsymbol{Var}^*(\tilde{\mathbb{M}}^{t-1})\mathbb{S}\mathbb{J}^* \quad \text{if } t>0 \text{ and } \mathbf{0} \text{ if } t=0 \tag{11a}$$

Note that by applying Eq. 65 to $\tilde{\mathbb{D}}\mathbb{y}^{(t)} = \tilde{\mathbb{M}}\tilde{\mathbb{D}}\mathbb{y}^{(t-1)}$, we get the following formula allowing to compute it recursively:

$$\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{y}^{(t)}) = \begin{cases} (\mathbb{M}\otimes\mathbb{M})\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{y}^{(t-1)}) + \boldsymbol{Var}^*(\tilde{\mathbb{M}})\mathbb{D}\mathbb{y}^{(t-1)} & \text{if } t>1 \\ \boldsymbol{Var}^*(\tilde{\mathbb{S}}\mathbb{J}^*) = & \text{if } t=1 \\ \mathbf{0} & \text{if } t=0 \end{cases}$$

7.7.4 Number of descendants via same litter sisters

process For same-litter sisters, writing it is $\tilde{\mathbb{D}}\mathbb{s}^{(t)}$ actually implies that the number of descendants stemming from same-litter sisters to the ancestor's daughter only depends on the class of the g_j-ancestor t time-steps ago. Whilst this is true for the number of descendants via younger and older sisters, since the reproductive success of the ancestor at other time-steps that $-t$ is independent from past or future reproduction, it is in general not true for the descendants via same-litter sisters which may depend on the particular class where the $g-1$ ancestor was produced in. We develop this general case in section xxxxsccs, and focus here on the three common cases (B-reproduction, P-reproduction and 1-reproduction), that we have highlighted in section dfsdfsf for which the number and class of same-litter sisters does not depend on the class of the g_j-ancestor, because (1), in the B-reproduction case, there is no same-litter sister, (2), in the P-reproduction case, each offspring in produced independently and (3), in the 1-reproduction case, the class of the $g-1$ ancestor at birth is fixed. Therefore, we have:

$$\tilde{\mathbb{D}}\mathbb{s}^{(t)} = \tilde{\mathbb{M}}^{t-1}\tilde{\mathbb{F}}^*\mathbb{J}^* \quad \text{if } t > 0 \text{ and } \mathbf{0} \text{ otherwise} \tag{67}$$

This process is very similar to $\tilde{\mathbb{D}}\mathbb{y}^{(t)}$, and with the approaches we get:

Expectation

$$\mathbb{D}\mathbb{s}^{(t)} = \mathbb{M}^{t-1}\mathbb{F}^*\mathbb{J}^* \quad \text{if } t > 0 \text{ and } \mathbf{0} \text{ otherwise} \tag{68}$$

PGF

$$\boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{s}^{(t)}} = (\boldsymbol{\varphi}_{\widetilde{\mathbb{F}^*}\mathbb{J}^*} \circ \bigcirc_{t-1} \boldsymbol{\varphi}_{\tilde{\mathbb{M}}}) \quad \text{if } t > 0 \text{ and } \mathbf{1} \text{ if } t = 0 \ , \tag{69}$$

Variance

$$\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{s}^{(t)}) = (\mathbb{M} \otimes \mathbb{M})^{t-1}\boldsymbol{Var}^*(\widetilde{\mathbb{F}^*})\mathbb{J}^* + \boldsymbol{Var}^*(\tilde{\mathbb{M}}^{t-1})\mathbb{F}^*\mathbb{J}^* \quad \text{if } t > 0 \text{ and } \mathbf{0} \text{ if } t = 0 \tag{11b}$$

7.7.5 Number of descendants via older sisters

Process

In order to compute the number of descendants via older sisters, one needs to go back further in time and track reproduction of the g-ancestor at multiple time-steps before it was it was last alive for certain (as it gave birth to the $g_j - 1$ ancestor if $g_j > 1$, now, as *ego* itself for $g_j = 1$). One can write that these descendants correspond to sum for all possible time-step α, before it was last alive for certain, of the descendants of the daughters it may have had then :

$$\forall t \geq 0, \quad \tilde{\mathbb{D}}\mathbb{o}^{(t)} = \sum_{\alpha>0} \tilde{\mathbb{M}}^{t-1+\alpha}\tilde{\mathbb{F}}\mathbb{J}^*\widetilde{\mathbf{P}_\mathbf{S}}^{\alpha} = \tilde{\mathbb{M}}^t\left(\sum_{\alpha>0}\tilde{\mathbb{M}}^{\alpha-1}\tilde{\mathbb{F}}\mathbb{J}^*\widetilde{\mathbf{P}_\mathbf{S}}^{\alpha}\right) = \tilde{\mathbb{M}}^t\tilde{\mathbb{D}}\mathbb{o}^{(0)}, \tag{70}$$

which converges almost surely to 0 (see 7.9). The "problem" with that sum, is that its different components do not have independent realisations. One can circumvent this problem in two ways, either by considering the distribution of all possible genealogical trajectories of the g_j-ancestor (as we did in the first process of the process building), or by considering the problem recursively. We go for the second option and write that the number of descendants via older sisters correspond to the number of descendants born from sisters potentially born the year before and to those born before that. This backward induction process is well defined and $\lim_{t\to\infty}\tilde{\mathbb{D}}\mathbb{o}^{(t)} = \mathbf{0}$(see 7.9):

$$\forall t \geq 0, \quad \tilde{\mathbb{D}}\mathbb{o}^{(t)} = \left(\tilde{\mathbb{M}}^t\tilde{\mathbb{F}}\mathbb{J}^* + \tilde{\mathbb{D}}\mathbb{o}^{(t+1)}\right)\widetilde{\mathbf{P}_\mathbf{S}} \text{ with } \lim_{t\to\infty}\tilde{\mathbb{D}}\mathbb{o}^{(t)} = \mathbf{0} \tag{5a}$$

Note that Eqs 70 and **??** imply that $\tilde{\mathbb{D}\mathbb{O}}^{(0)}$ is a solution of

$$\tilde{\mathbb{D}\mathbb{O}}^{(0)} = \left(\tilde{\mathbb{F}}\mathbb{J}^{*} + \tilde{\mathbb{M}}\mathbb{D}\mathbb{O}^{(0)}\right)\tilde{\mathbf{P}}_{\mathbf{S}} \tag{5b}$$

Expectation

We have directly from eq xxx that ,

$$\forall t \geq 0, \quad \mathbb{D}\mathbb{O}^{(t)} = \sum_{\alpha>0} \mathbb{M}^{t-1+\alpha}\mathbb{F}\mathbb{J}^{*}\mathbf{P}_{\mathbf{S}}{}^{\alpha} = \mathbb{M}^{t}\underbrace{\left(\sum_{\alpha>0} \mathbb{M}^{\alpha-1}\mathbb{F}\mathbb{J}^{*}\mathbf{P}_{\mathbf{S}}{}^{\alpha}\right)}_{\mathbb{D}\mathbb{O}^{(0)}}. \tag{71}$$

PGF

From Eq. 5a we have

$$\boldsymbol{\varphi}_{\tilde{\mathbb{D}\mathbb{O}}^{(t)}} = \boldsymbol{\varphi}_{\tilde{\mathbf{P}}_{\mathbf{S}}} \circ \boldsymbol{\varphi}_{\tilde{\mathbb{M}}^{t}\tilde{\mathbb{F}}\mathbb{J}^{*}+\tilde{\mathbb{D}\mathbb{O}}^{(t+1)}} = \boldsymbol{\varphi}_{\tilde{\mathbf{P}}_{\mathbf{S}}} \circ \left(\boldsymbol{\varphi}_{\tilde{\mathbb{M}}^{t}\tilde{\mathbb{F}}\mathbb{J}^{*}} \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{D}\mathbb{O}}^{(t+1)}}\right) = \boldsymbol{\varphi}_{\tilde{\mathbf{P}}_{\mathbf{S}}} \circ \left(\left(\boldsymbol{\varphi}_{\tilde{\mathbb{F}}\mathbb{J}^{*}} \circ \boldsymbol{\varphi}_{\tilde{\mathbb{M}}^{t}}\right) \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{D}\mathbb{O}}^{(t+1)}}\right) \tag{21a}$$

From eq. 5b, we also know that $\boldsymbol{\varphi}_{\tilde{\mathbb{D}\mathbb{O}}^{(0)}}$ is a solution of

$$\boldsymbol{\varphi}_{\tilde{\mathbb{D}\mathbb{O}}^{(0)}} = \boldsymbol{\varphi}_{\tilde{\mathbf{P}}_{\mathbf{S}}} \circ \boldsymbol{\varphi}_{\tilde{\mathbb{F}}\mathbb{J}^{*}+\tilde{\mathbb{M}}\mathbb{D}\mathbb{O}^{(0)}} = \boldsymbol{\varphi}_{\tilde{\mathbf{P}}_{\mathbf{S}}} \circ \left(\boldsymbol{\varphi}_{\tilde{\mathbb{F}}\mathbb{J}^{*}} \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{M}}\tilde{\mathbb{D}\mathbb{O}}^{(0)}}\right) = \boldsymbol{\varphi}_{\tilde{\mathbf{P}}_{\mathbf{S}}} \circ \left(\boldsymbol{\varphi}_{\tilde{\mathbb{F}}\mathbb{J}^{*}} \bullet \left(\boldsymbol{\varphi}_{\tilde{\mathbb{D}\mathbb{O}}^{(0)}} \circ \boldsymbol{\varphi}_{\tilde{\mathbb{M}}}\right)\right) \tag{8b}$$

Variance

In order to compute the variance of $\tilde{\mathbb{D}\mathbb{O}}^{(t)}$, we start from eq xxx , consider the process for one class only and decompose according to class of the "ancestor" via $\tilde{\mathbf{P}}_{\mathbf{S}}$

$$\boldsymbol{Var}(\tilde{\mathbb{D}\mathbb{O}}^{(t)}\boldsymbol{e}_{c_j}) = \boldsymbol{E}_{c_i}\boldsymbol{Var}\left((\tilde{\mathbb{M}}^{t}\tilde{\mathbb{F}}\mathbb{J}^{*} + \tilde{\mathbb{D}\mathbb{O}}^{(t+1)})\tilde{\mathbf{P}}_{\mathbf{S}}\boldsymbol{e}_{c_j}|\tilde{\mathbf{P}}_{\mathbf{S}}\boldsymbol{e}_{c_j} = \boldsymbol{e}_{c_i}\right) + \boldsymbol{Var}_{c_i}\boldsymbol{E}\left((\tilde{\mathbb{M}}^{t}\tilde{\mathbb{F}}\mathbb{J}^{*} + \tilde{\mathbb{D}\mathbb{O}}^{(t+1)})\tilde{\mathbf{P}}_{\mathbf{S}}\boldsymbol{e}_{c_j}|\tilde{\mathbf{P}}_{\mathbf{S}}\boldsymbol{e}_{c_j} = \boldsymbol{e}_{i}\right) \tag{72}$$

We have $\boldsymbol{Var}\left((\tilde{\mathbb{M}}^{t}\tilde{\mathbb{F}}\mathbb{J}^{*} + \tilde{\mathbb{D}\mathbb{O}}^{(t+1)})\tilde{\mathbf{P}}_{\mathbf{S}}\boldsymbol{e}_{c_j}|\tilde{\mathbf{P}}_{\mathbf{S}}\boldsymbol{e}_{c_j} = \boldsymbol{e}_{c_i}\right) = \boldsymbol{Var}\left((\tilde{\mathbb{M}}^{t}\tilde{\mathbb{F}}\mathbb{J}^{*} + \tilde{\mathbb{D}\mathbb{O}}^{(t+1)})\boldsymbol{e}_{i}\right) = \boldsymbol{Var}\left((\tilde{\mathbb{M}}^{t}\tilde{\mathbb{F}}\mathbb{J}^{*}\boldsymbol{e}_{i}\right) + \boldsymbol{Var}\left(\tilde{\mathbb{D}\mathbb{O}}^{(t+1)}\boldsymbol{e}_{i}\right)$
Therefore $\boldsymbol{E}_{c_i}\boldsymbol{Var}\left((\tilde{\mathbb{M}}^{t}\tilde{\mathbb{F}}\mathbb{J}^{*} + \tilde{\mathbb{D}\mathbb{O}}^{(t+1)})\tilde{\mathbf{P}}_{\mathbf{S}}\boldsymbol{e}_{c_j}|\tilde{\mathbf{P}}_{\mathbf{S}}\boldsymbol{e}_{c_j} = \boldsymbol{e}_{c_i}\right) = \sum_i P_S(c_i, c_j)(\boldsymbol{Var}\left((\tilde{\mathbb{M}}^{t}\tilde{\mathbb{F}}\mathbb{J}^{*}\boldsymbol{e}_{c_i}\right) + \boldsymbol{Var}\left(\tilde{\mathbb{D}\mathbb{O}}^{(t+1)}\boldsymbol{e}_{c_i}\right)) =$
$= (\boldsymbol{Var}^{*}(\tilde{\mathbb{M}}^{t}\tilde{\mathbb{F}}\mathbb{J}^{*}) + \boldsymbol{Var}^{*}(\tilde{\mathbb{D}\mathbb{O}}^{(t+1)}))\mathbf{P}_{\mathbf{S}}\boldsymbol{e}_{c_j}$ in vectorised form.
For the second component, we have $\boldsymbol{Var}_{c_i}\boldsymbol{E}(\tilde{\mathbb{M}}^{t}\tilde{\mathbb{F}}\mathbb{J}^{*} + \tilde{\mathbb{D}\mathbb{O}}^{(t+1)})\tilde{\mathbf{P}}_{\mathbf{S}}\boldsymbol{e}_{c_j}|\tilde{\mathbf{P}}_{\mathbf{S}}\boldsymbol{e}_{c_j} = \boldsymbol{e}_{c_i}) = \boldsymbol{Var}_{c_i}\boldsymbol{E}\left((\tilde{\mathbb{M}}^{t}\tilde{\mathbb{F}} + \tilde{\mathbb{D}\mathbb{O}}^{(t+1)})\boldsymbol{e}_{c_i}\right) =$
$\boldsymbol{Var}_{c_i}\left((\mathbb{M}^{t}(\mathbb{F}\mathbb{J}^{*} + \mathbb{M}\mathbb{D}\mathbb{O}^{(0)}))\boldsymbol{e}_{c_i}\right) = \boldsymbol{Var}\left((\mathbb{M}^{t}(\mathbb{F}\mathbb{J}^{*} + \mathbb{M}\mathbb{D}\mathbb{O}^{(0)}))\tilde{\mathbf{P}}_{\mathbf{S}}\boldsymbol{e}_{c_j}\right) = \boldsymbol{Var}^{*}_{c_j}\left((\mathbb{M}^{t}(\mathbb{F}\mathbb{J}^{*} + \mathbb{M}\mathbb{D}\mathbb{O}^{(0)}))\tilde{\mathbf{P}}_{\mathbf{S}}\right)$
$= \left(\left(\mathbb{M}^{t}(\mathbb{F}\mathbb{J}^{*} + \mathbb{M}\mathbb{D}\mathbb{O}^{(0)})\right) \otimes \left(\mathbb{M}^{t}(\mathbb{F}\mathbb{J}^{*} + \mathbb{M}\mathbb{D}\mathbb{O}^{(0)})\right)\right)\boldsymbol{Var}^{*}_{(c_j)}\left(\tilde{\mathbf{P}}_{\mathbf{S}}\right).$
So all in all $\boldsymbol{Var}^{*}_{c_j}(\tilde{\mathbb{D}\mathbb{O}}^{(t)}) = (\boldsymbol{Var}^{*}(\tilde{\mathbb{M}}^{t}\tilde{\mathbb{F}}\mathbb{J}^{*}) + \boldsymbol{Var}^{*}(\tilde{\mathbb{D}\mathbb{O}}^{(t+1)}))\mathbf{P}_{\mathbf{S}}\boldsymbol{e}_{c_j} + \left(\left(\mathbb{M}^{t}(\mathbb{F}\mathbb{J}^{*} + \mathbb{M}\mathbb{D}\mathbb{O}^{(0)})\right) \otimes \left(\mathbb{M}^{t}(\mathbb{F}\mathbb{J}^{*} + \mathbb{M}\mathbb{D}\mathbb{O}^{(0)})\right)\right)\boldsymbol{Var}^{*}_{(c_j)}\left(\tilde{\mathbf{P}}_{\mathbf{S}}\right)$
that we can write

$$\boldsymbol{Var}^{*}(\tilde{\mathbb{D}\mathbb{O}}^{(t)}) = (\boldsymbol{Var}^{*}(\tilde{\mathbb{M}}^{t}\tilde{\mathbb{F}})\mathbb{J}^{*} + \boldsymbol{Var}^{*}(\tilde{\mathbb{D}\mathbb{O}}^{(t+1)}))\mathbf{P}_{\mathbf{S}} + \left(\left(\mathbb{M}^{t}(\mathbb{F}\mathbb{J}^{*} + \mathbb{M}\mathbb{D}\mathbb{O}^{(0)})\right) \otimes \left(\mathbb{M}^{t}(\mathbb{F}\mathbb{J}^{*} + \mathbb{M}\mathbb{D}\mathbb{O}^{(0)})\right)\right)\boldsymbol{Var}^{*}\left(\tilde{\mathbf{P}}_{\mathbf{S}}\right) \tag{11c}$$

with $\lim_{t\to\infty}\boldsymbol{Var}^{*}(\tilde{\mathbb{D}\mathbb{O}}^{(t)}) = \mathbf{0}$ which is a backward induction in $\boldsymbol{Var}^{*}(\tilde{\mathbb{D}\mathbb{O}}^{(t)})$.

7.7.6 Summary

Process

From Eqs 4a,4c, 57, 61 , **??** and 70 we get the system

$$\tilde{\mathbb{K}}_j = \tilde{\mathbb{K}}_{c_j,g_j} = \tilde{\mathbb{D}}^{(t)}_{c_a} \text{ with probability } P_j(c_a,t) \tag{4a}$$

$$\tilde{\mathbb{D}}^{(t)} = \begin{cases} \underbrace{\textstyle\sum_{\alpha>0} \tilde{\mathbb{M}}^{\alpha-1}\tilde{\mathbb{F}}\mathbb{J}^*\tilde{\mathbf{P_S}}^{\alpha}}_{\tilde{\mathbb{D}}^{(0)}_{\mathbb{o}}} \text{if t=0} \\ \underbrace{\tilde{\mathbb{M}}^{t-1}\tilde{\mathbb{S}}\mathbb{J}^*}_{\tilde{\mathbb{D}}\mathbb{y}^{(t)}} + \underbrace{\tilde{\mathbb{M}}^{t-1}\tilde{\mathbb{F}^*}\mathbb{J}^*}_{\tilde{\mathbb{D}}\mathbb{s}^{(t)}} + \underbrace{\textstyle\sum_{\alpha>0} \tilde{\mathbb{M}}^{t-1+\alpha}\tilde{\mathbb{F}}\mathbb{J}^*\tilde{\mathbf{P_S}}^{\alpha}}_{\tilde{\mathbb{D}}^{(t)}_{\mathbb{o}}} \text{ if t>0} \end{cases} \tag{4b}$$

$$P_{c_j,g_j}(c_a,t) = P_j(c_a,t) = \begin{cases} \boldsymbol{e}^T_{c_a}\mathbb{J}^{*T}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}\boldsymbol{e}_{(c_j,g_j)} & \text{if } t>0 \\ \boldsymbol{e}^T_{c_a}\mathbb{J}^{*T}\boldsymbol{e}_{(c_j,g_j)} & \text{if } t=0 \end{cases}, \tag{4c}$$

which we can simplify as

$$\tilde{\mathbb{K}}_{c_j,g_j} = \begin{cases} \underbrace{(\textstyle\sum_{\alpha>0} \tilde{\mathbb{M}}^{\alpha}\tilde{\mathbb{F}}\mathbb{J}^*\tilde{\mathbf{P_S}}^{\alpha}}_{\tilde{\mathbb{D}}^{(0)}_{\mathbb{o}}})\boldsymbol{e}_{c_a} \text{ with probability } \boldsymbol{e}^T_{\boldsymbol{c_a}}\mathbb{J}^{*T}\boldsymbol{e}_{\boldsymbol{(c_j,g_j)}} \\ (\underbrace{\tilde{\mathbb{M}}^{t-1}\tilde{\mathbb{S}}\mathbb{J}^*}_{\tilde{\mathbb{D}}\mathbb{y}^{(t)}} + \underbrace{\tilde{\mathbb{M}}^{t-1}\tilde{\mathbb{F}^*}\mathbb{J}^*}_{\tilde{\mathbb{D}}\mathbb{s}^{(t)}} + \underbrace{\textstyle\sum_{\alpha>0} \tilde{\mathbb{M}}^{t-1+\alpha}\tilde{\mathbb{F}}\mathbb{J}^*\tilde{\mathbf{P_S}}^{\alpha}}_{\tilde{\mathbb{D}}^{(t)}_{\mathbb{o}}})\boldsymbol{e}_{c_a} \text{ with probability } \boldsymbol{e}^T_{\boldsymbol{c_a}}\mathbb{J}^{*T}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}\boldsymbol{e}_{\boldsymbol{(c_j,g_j)}} \text{ (only on } t>0) \end{cases},$$

with $\tilde{\mathbb{D}}\mathbb{o}^{(t)}$ such that $\tilde{\mathbb{D}}\mathbb{o}^{(t)} = \tilde{\mathbb{M}}^t\tilde{\mathbb{D}}\mathbb{o}^{(0)}$, with the following backward induction $\tilde{\mathbb{D}}\mathbb{o}^{(t)} = \left(\tilde{\mathbb{M}}^t\tilde{\mathbb{F}}\mathbb{J}^* + \tilde{\mathbb{D}}\mathbb{o}^{(t+1)}\right)\tilde{\mathbf{P_S}}$ with $\lim_{t\to\infty} P(\tilde{\mathbb{D}}\mathbb{o}^{(t)} = \mathbf{0}) = 1$ and solution of $\tilde{\mathbb{D}}\mathbb{o}^{(0)} = \left(\tilde{\mathbb{F}}\mathbb{J}^* + \tilde{\mathbb{M}}\mathbb{D}\mathbb{o}^{(0)}\right)\tilde{\mathbf{P_S}}$.

Expectations

From Eqs 54, 58, 62, 68 and 71, we can write $\mathbb{K}$ as the number of kin for $t = 0$, which corresponds to $\mathbb{D}\mathbb{o}^{(0)}$, plus the sum on all $t > 0$ of $\mathbb{D}\mathbb{y}^{(t)} + \mathbb{D}\mathbb{s}^{(t)} + \mathbb{D}\mathbb{o}^{(t)}$: $\mathbb{K} = \underbrace{(\sum_{\alpha>0}\mathbb{M}^{\alpha-1}\mathbb{F}\mathbb{J}^*\mathbf{P_S}^{\alpha})}_{\mathbb{D}\mathbb{o}^{(0)}}\mathbb{J}^{*T} + \sum_{t>0}\Big(\mathbb{M}^{t-1}\mathbb{S}\mathbb{J}^* + \mathbb{M}^{t-1}\mathbb{F}^*\mathbb{J}^* + \mathbb{M}^t\underbrace{(\sum_{\alpha>0}\mathbb{M}^{\alpha-1}\mathbb{F}\mathbb{J}^*\mathbf{P_S}^{\alpha})}_{\mathbb{D}\mathbb{o}^{(0)}}\Big)\mathbb{J}^{*T}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}$ since $\sum_{c_a}\boldsymbol{e}_{c_a}\boldsymbol{e}^T_{c_a} = \mathbf{I}_s$ We regroup with first all kin issued by older sisters and then the kin issued by younger and same-litter sisters:

$$\mathbb{K} = \underbrace{(\sum_{\alpha>0}\mathbb{M}^{\alpha-1}\mathbb{F}\mathbb{J}^*\mathbf{P_S}^{\alpha})\mathbb{J}^{*T} + \sum_{t>0}\Big(\mathbb{M}^t(\sum_{\alpha>0}\mathbb{M}^{\alpha-1}\mathbb{F}\mathbb{J}^*\mathbf{P_S}^{\alpha})\Big)\mathbb{J}^{*T}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}}_{\mathbb{K}_{\mathbb{o}}} + \underbrace{\sum_{t>0}\Big(\mathbb{M}^{t-1}\mathbb{S} + \mathbb{M}^{t-1}\mathbb{F}^*\Big)(\mathbb{J}^*\mathbb{J}^{*T})\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}}_{\mathbb{K}_{\mathbb{y}} + \mathbb{K}_{\mathbb{s}}}$$

We focus first on $\mathbb{K}_{\mathbb{y}} + \mathbb{K}_{\mathbb{s}}$, where via an index change,and from $\mathbb{J} = (\mathbb{J}^*\mathbb{J}^{*T})$ we get

$$\mathbb{K}_{\mathbb{y}} + \mathbb{K}_{\mathbb{s}} = \sum_{t'=t-1=0}^{\infty}\mathbb{M}^{t'}(\mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}} + \mathbb{F}^*\mathbb{J}\mathbb{P}_{\mathbb{F}})\mathbb{P}^{t'}$$

First note that $\mathbb{J}^*\mathbf{P_S}^{\alpha}\mathbb{J}^{*T} = \mathbb{J}\mathbb{P_S}^{\alpha} = \mathbb{P_S}^{\alpha}\mathbb{J}$. Now consider the first element of $\mathbb{K}_{\mathbb{o}}$, to which we apply the same change of index, it is $\sum_{\beta=\alpha-1=0}^{\infty}\mathbb{M}^{\beta}\mathbb{F}\mathbb{J}\mathbb{P_S}\mathbb{P_S}^{\beta}$. We proceed similarly with the second element, where the index change consist in using $\gamma = \alpha + t - 1$, where $1 \le \gamma \le \infty$ and then $1 \le \alpha \le \gamma$: $\sum_{t>0}\Big(\mathbb{M}^t(\sum_{\alpha>0}\mathbb{M}^{\alpha-1}\mathbb{F}\mathbb{P_S}^{\alpha})\Big)\mathbb{J}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1} = \sum_{\gamma=1}^{\infty}\sum_{\alpha=1}^{\infty}\mathbb{M}^{\gamma}\mathbb{F}\mathbb{J}\mathbb{P_S}^{\alpha}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{\gamma-\alpha} = \sum_{\gamma=1}^{\infty}\mathbb{M}^{\gamma}\mathbb{F}\mathbb{J}\mathbb{P_S}\sum_{\alpha=1}^{\infty}\mathbb{P_S}^{\alpha-1}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{\gamma-\alpha}$. We can sum these two expressions:

$$\mathbb{K}_{\mathbb{o}} = \sum_{\beta=0}^{\infty}\mathbb{M}^{\beta}\mathbb{F}\mathbb{J}\mathbb{P_S}\mathbb{P_S}^{\beta} + \sum_{\gamma=1}^{\infty}\mathbb{M}^{\gamma}\mathbb{F}\mathbb{J}\mathbb{P_S}\sum_{\alpha=1}^{\gamma}\mathbb{P_S}^{\alpha-1}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{\gamma-\alpha}$$

In the first element we can isolate the case $\beta = 0$ and then group the terms:

$$\mathbb{K}_{\mathbb{O}} = \mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \sum_{\gamma=1}^{\infty} \mathbb{M}^{\gamma}\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} \left({\mathbb{P}_{\mathbb{S}}}^{\gamma} + \sum_{\alpha=1}^{\gamma} {\mathbb{P}_{\mathbb{S}}}^{\alpha-1}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{\gamma-\alpha} \right)$$

Let us now consider the development of $\mathbb{P}^{\gamma} = (\mathbb{P}_{\mathbb{S}}+\mathbb{P}_{\mathbb{F}})^{\gamma}$. We can write that $\mathbb{P}^{\gamma} = \sum_{\boldsymbol{n}\in[0,1]^{\gamma}} ({\mathbb{P}_{\mathbb{S}}}^{1-n_1}{\mathbb{P}_{\mathbb{F}}}^{n_1}{\mathbb{P}_{\mathbb{S}}}^{1-n_2}{\mathbb{P}_{\mathbb{F}}}^{n_2}...{\mathbb{P}_{\mathbb{S}}}^{1-n_{\gamma}}{\mathbb{P}_{\mathbb{F}}}^{n_{\gamma}})$. Let us now order these terms according to u the position of the first 1 in $\boldsymbol{n}$, with the convention that $u = \gamma + 1$ if $\boldsymbol{n} = \boldsymbol{0}$. We can then re-write $\mathbb{P}^{\gamma}$ by segregating the case $u = \gamma + 1$ (then there is only one product and its ${\mathbb{P}_{\mathbb{S}}}^{\gamma}$) and the other cases: $\mathbb{P}^{\gamma} = {\mathbb{P}_{\mathbb{S}}}^{\gamma} + \sum_{u=1}^{\gamma} {\mathbb{P}_{\mathbb{S}}}^{u-1}\mathbb{P}_{\mathbb{F}} \sum_{\boldsymbol{m}\in[0,1]^{\gamma-u}} ({\mathbb{P}_{\mathbb{S}}}^{1-m_1}{\mathbb{P}_{\mathbb{F}}}^{m_1}{\mathbb{P}_{\mathbb{S}}}^{1-m_2}{\mathbb{P}_{\mathbb{F}}}^{m_2}...{\mathbb{P}_{\mathbb{S}}}^{1-m_{\gamma-u}}{\mathbb{P}_{\mathbb{F}}}^{m_{\gamma-u}})$ But we have established already that $\sum_{\boldsymbol{m}\in[0,1]^{\gamma-u}} ({\mathbb{P}_{\mathbb{S}}}^{1-m_1}{\mathbb{P}_{\mathbb{F}}}^{m_1}{\mathbb{P}_{\mathbb{S}}}^{1-m_2}{\mathbb{P}_{\mathbb{F}}}^{m_2}...{\mathbb{P}_{\mathbb{S}}}^{1-m_{\gamma-u}}{\mathbb{P}_{\mathbb{F}}}^{m_{\gamma-u}}) = \mathbb{P}^{\gamma-u}$ and therefore $\mathbb{P}^{\gamma} = {\mathbb{P}_{\mathbb{S}}}^{\gamma} + \sum_{u=1}^{\gamma} {\mathbb{P}_{\mathbb{S}}}^{u-1}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{\gamma-u}$ which is the term in parenthesis in the expression above, and therefore

$$\mathbb{K}_{\mathbb{O}} = \mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \sum_{t=1}^{\infty} \mathbb{M}^{t}\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}}\mathbb{P}^{t} = \sum_{t=0}^{\infty} \mathbb{M}^{t}\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}}\mathbb{P}^{t}$$

This yields finally,

$$\mathbb{K} = \sum_{t=0}^{\infty} \mathbb{M}^{t} \left(\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}} + \mathbb{F}^{*}\mathbb{J}\mathbb{P}_{\mathbb{F}} \right) \mathbb{P}^{t} \tag{6}$$

Direct formulation of expectation

One can reach Eq. 6 directly: as described in Eq 4c, the probability for an individual *ego* in class c_j that its g_j-ancestor was in class c_a, $t+1$ time-steps from now is are to be read in the $c_a, (c_j, g_j)$ element of $\mathbb{J}\mathbb{P}^{t+1}$. This probability can be split $\mathbb{J}\mathbb{P}^{t+1} = \mathbb{J}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t} + \mathbb{J}\mathbb{P}_{\mathbb{S}}\mathbb{P}^{t}$ according to whether the ancestor just gave birth to the $g_j - 1$ ancestor ($\mathbb{J}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t}$) or if this occurs later ($\mathbb{J}\mathbb{P}_{\mathbb{S}}\mathbb{P}^{t}$). One can account for the descendants of the ancestor via younger and same-litter sisters by summing, for all times t at which the ancestor gave birth to the $g_j - 1$ ancestor, the descendants produced, via $\mathbb{M}^{t}(\mathbb{S} + \mathbb{F}^{*})\mathbb{J}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t}$. One can account for the descendants of the ancestor via older sisters by considering, by summing, for all time-steps t before the birth of the $g_j - 1$ ancestor, the descendants of offspring produced then, via $\mathbb{M}^{t}\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}}\mathbb{P}^{t}$. The sum of these two components yields Eq .... We cannot use such a reasoning for the general description of the process and the computation of its variance and *pdf*, since the ancestors of the descendants in the latter sum are not independent from each other. A more intuitive, but less rigorous, way to fathom Eq **??**, is to consider naively that one can find the descendants of the ancestor of *ego* by considering $\sum_{t=0}^{\infty} \mathbb{M}^{t+1}\mathbb{J}\mathbb{P}^{t+1}$. By decomposing $\mathbb{M}^{t+1}\mathbb{J}\mathbb{P}^{t+1} = \mathbb{M}^{t}\mathbb{M}\mathbb{J}\mathbb{P}\mathbb{P}^{t} = \mathbb{M}^{t}\left(\mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}} + \mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{F}} + \mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{S}}\right)\mathbb{P}^{t}$, in which we recognise the three components of the correct formula in the case of Poisson reproduction, plus a term $\mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{S}}$ that indeed needs to be eliminated, since it corresponds to an ancestor which was alive $t-1$ time-steps before now, and "only" survived during the previous time-step. Its descendants are therefore already counted in $\mathbb{M}^{t}\mathbb{J}\mathbb{P}^{t}$.

PGF

We have, from Eqs. 7a, 59, 63 and 69

$$\varphi_{\tilde{\mathbb{K}}_j} = \sum_{c_a, t} P_j(c_a, t)\varphi_{\tilde{\mathbb{D}}^{(t)}_{c_a}} \tag{7a}$$

which can be written as

$$\boldsymbol{\varphi}_{\tilde{\mathbb{K}}} = \boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(0)}}\mathbb{J}^{*} + \sum_{t>0} \boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(t)}}\mathbb{J}^{*}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1} \tag{55}$$

with

$$\begin{cases} \boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(0)}} = \boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(0)}} \\ \boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(t)}} = \underbrace{(\boldsymbol{\varphi}_{\tilde{\mathbb{S}}\mathbb{J}^*} \circ \bigcirc_{t-1} \boldsymbol{\varphi}_{\tilde{\mathbb{M}}})}_{\varphi_{\tilde{\mathbb{D}}\mathbb{y}^{(t)}}} \bullet \underbrace{(\boldsymbol{\varphi}_{\tilde{\mathbb{F}^*}\mathbb{J}^*} \circ \bigcirc_{t-1} \boldsymbol{\varphi}_{\tilde{\mathbb{M}}})}_{\boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{s}^{(t)}}} \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}} \text{ if } t > 0, \end{cases} \tag{7b}$$

where $\bullet$ is the termwise product of vectors and $\bigcirc_k$ denotes k compositions of the same function (if $k = 0$, it is the identity). Note that $\varphi_{\tilde{\mathbb{D}}^{(t)}_{c_a}}(\boldsymbol{x})$ is the c_a element of $\boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(t)}}$, which contains only s elements (contrary to $\varphi_{\tilde{\mathbb{K}}}$ which contains sg_m). For $\varphi_{\tilde{\mathbb{D}}^{(t)}}$, we have

$$\boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}} = \boldsymbol{\varphi}_{\tilde{\mathbf{P}_\mathbf{S}}} \circ ((\boldsymbol{\varphi}_{\tilde{\mathbb{F}}\mathbb{J}^*} \circ \boldsymbol{\varphi}_{\tilde{\mathbb{M}}^t}) \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(t+1)}}) \text{ with } \lim_{t\to\infty} \boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}} = \mathbf{1} \tag{21a}$$

$$\boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(0)}} = \boldsymbol{\varphi}_{\tilde{\mathbf{P}_\mathbf{S}}} \circ (\boldsymbol{\varphi}_{\tilde{\mathbb{F}}\mathbb{J}^*} \bullet (\boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(0)}} \circ \boldsymbol{\varphi}_{\tilde{\mathbb{M}}})) \tag{8b}$$

As mentioned earlier, the theoretical approach (Eq 8b) will be investigated in a follow up paper. In the implementation part of this article (sec dfsdf) and its illustration, we will use Eq 21a

From Eq 55, we get :

$$\boldsymbol{\varphi}_{\tilde{\mathbb{K}}} = \boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(0)}} \mathbb{J}^{*T} + \sum_{t>0} \underbrace{(\boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{y}^{(t)}} \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{y}^{(t)}} \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}})}_{\varphi_{\tilde{\mathbb{D}}^{(t)}}} \mathbb{J}^{*T} \mathbb{P}_\mathbb{F} \mathbb{P}^{t-1} \tag{73}$$

which we can write

$$\varphi_{\tilde{\mathbb{K}}} = \varphi_{\tilde{\mathbb{D}}\mathbb{o}^{(0)}} \mathbb{J}^{*T} + \sum_{t>0} \left((\boldsymbol{\varphi}_{\tilde{\mathbb{S}}\mathbb{J}^*} \circ \bigcirc_{t-1} \boldsymbol{\varphi}_{\tilde{\mathbb{M}}}) \bullet (\boldsymbol{\varphi}_{\tilde{\mathbb{F}^*}\mathbb{J}^*} \circ \bigcirc_{t-1} \boldsymbol{\varphi}_{\tilde{\mathbb{M}}}) \bullet \varphi_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}} \right) \mathbb{J}^{*T} \mathbb{P}_\mathbb{F} \mathbb{P}^{t-1} \tag{74}$$

Variance

from the above, we have

$$\boldsymbol{Var}^*_j(\tilde{\mathbb{K}}) = \underbrace{\sum_{c_a,t} P_j(c_a,t) \boldsymbol{Var}^*_{(c_a)}(\tilde{\mathbb{D}}^{(t)})}_{\boldsymbol{Var}_{des}(\tilde{\mathbb{K}}_j)} + \underbrace{\left(\sum_{c_a,t} P_j(c_a,t) \mathbb{D}^{(t)}_{c_a} \mathbb{D}^{(t)T}_{c_a} \right) - \mathbb{K}_j \mathbb{K}_j^T}_{\boldsymbol{Var}_{anc}(\tilde{\mathbb{K}}_j)} \tag{9}$$

which can be written in tensor notation as

$$\boldsymbol{Var}^*(\tilde{\mathbb{K}}) = \underbrace{\boldsymbol{Var}^*(\tilde{\mathbb{D}}^{(0)})\mathbb{J}^{*T} + \sum_{t>0} \boldsymbol{Var}^*(\tilde{\mathbb{D}}^{(t)})\mathbb{J}^{*T}\mathbb{P}_\mathbb{F}\mathbb{P}^{t-1}}_{\boldsymbol{Var}^*_{des}(\tilde{\mathbb{K}})} + \underbrace{(\mathbb{D}^{(0)} \bigotimes \mathbb{D}^{(0)})\mathbb{J}_2 + \sum_{t>0} (\mathbb{D}^{(t)} \bigotimes \mathbb{D}^{(t)})\mathbb{J}_2\mathbb{P}_\mathbb{F}\mathbb{P}^{t-1} - (\mathbb{K} \bigotimes \mathbb{K})\mathbb{J}_3}_{\boldsymbol{Var}^*_{anc}(\tilde{\mathbb{K}})} \tag{56}$$

with

$$\mathbf{Var}^*(\tilde{\mathbb{D}}^{(t)}) = \mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{y}^{(t)}) + \mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{s}^{(t)}) + \mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) \tag{10}$$

where

$$\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{y}^{(t)}) = (\mathbb{M} \otimes \mathbb{M})^{t-1} \boldsymbol{Var}^*(\tilde{\mathbb{S}})\mathbb{J}^* + \boldsymbol{Var}^*(\tilde{\mathbb{M}}^{t-1})\mathbb{S}\mathbb{J}^* \quad \text{if } t > 0 \text{ and } \mathbf{0} \text{ if } t = 0 \tag{11a}$$

and

$$\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{s}^{(t)}) = (\mathbb{M} \otimes \mathbb{M})^{t-1} \boldsymbol{Var}^*(\tilde{\mathbb{F}^*})\mathbb{J}^* + \boldsymbol{Var}^*(\tilde{\mathbb{M}}^{t-1})\mathbb{F}^*\mathbb{J}^* \quad \text{if } t > 0 \text{ and } \mathbf{0} \text{ if } t = 0 \tag{11b}$$

can be obtained from

$$\boldsymbol{Var}^*(\tilde{\mathbb{M}}^t) = (\mathbb{M} \otimes \mathbb{M}) \boldsymbol{Var}^*(\tilde{\mathbb{M}}^{t-1}) + \boldsymbol{Var}^*(\tilde{\mathbb{M}})\mathbb{M}^{t-1} \tag{66}$$

and

$$\begin{cases} \boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) = (\boldsymbol{Var}^*(\tilde{\mathbb{M}}^t\tilde{\mathbb{F}})\mathbb{J}^* + \boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t+1)}))\mathbf{P}_\mathbf{S} + \left(\left(\mathbb{M}^t(\mathbb{F}\mathbb{J}^* + \mathbb{M}\mathbb{D}\mathbb{o}^{(0)})\right) \otimes \left(\mathbb{M}^t(\mathbb{F}\mathbb{J}^* + \mathbb{M}\mathbb{D}\mathbb{o}^{(0)})\right)\right) \boldsymbol{Var}^*\left(\tilde{\mathbf{P}_\mathbf{S}}\right) \\ \lim_{t\to\infty} \boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) = \mathbf{0} \end{cases} \tag{11c}$$

Decomposition of the Variance according to projective and genealogical stochasticity

We can decompose eq 11c further according to whether the variance stems from stochasticity in the forward projectors, we call it the projective component of individual stochasticity, or simply projective stochasticity or from stochasticity in the backward projectors, which we call genealogical stochasticity:

$$\begin{cases} \boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) = \boldsymbol{Var}^*_{(ps)}(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) + \boldsymbol{Var}^*_{(gs)}(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) \text{ with ,} \\ \boldsymbol{Var}^*_{(ps)}(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) = (\boldsymbol{Var}^*(\tilde{\mathbb{M}}^t\tilde{\mathbb{F}})\mathbb{J}^* + \boldsymbol{Var}^*_{(ps)}(\tilde{\mathbb{D}}\mathbb{o}^{(t+1)}))\mathbf{P_S} \\ \boldsymbol{Var}^*_{(gs)}(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) = \boldsymbol{Var}^*_{(gs)}(\tilde{\mathbb{D}}\mathbb{o}^{(t+1)})\mathbf{P_S} + \left(\left(\mathbb{M}^t(\mathbb{F}\mathbb{J}^* + \mathbb{M}\mathbb{D}\mathbb{o}^{(0)})\right) \otimes \left(\mathbb{M}^t(\mathbb{F}\mathbb{J}^* + \mathbb{M}\mathbb{D}\mathbb{o}^{(0)})\right)\right) \boldsymbol{Var}^*\left(\tilde{\mathbf{P_S}}\right) \end{cases} \tag{12}$$

This allows to write

$$\begin{cases} \boldsymbol{Var}^*_j(\tilde{\mathbb{K}}) = \boldsymbol{Var}^*_{(ps)j}(\tilde{\mathbb{K}}) + \boldsymbol{Var}^*_{(gs)j}(\tilde{\mathbb{K}}), \text{ with} \\ \boldsymbol{Var}^*_{(ps)j}(\tilde{\mathbb{K}}) = \sum_{c_a,t} P_j(c_a,t)\left(\boldsymbol{Var}^*_{(ps)c_a}(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) + \boldsymbol{Var}^*_{(c_a)}(\tilde{\mathbb{D}}\mathbb{y}^{(t)}) + \boldsymbol{Var}^*_{(c_a)}(\tilde{\mathbb{D}}\mathbb{s}^{(t)})\right) \\ \boldsymbol{Var}^*_{(gs)j}(\tilde{\mathbb{K}}) = \sum_{c_a,t} P_j(c_a,t)\boldsymbol{Var}^*_{(gs)c_a}(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) + \underbrace{\sum_{c_a,t} P_j(c_a,t)(\mathbb{D}^{(t)}_{c_a,1})(\mathbb{D}^{(t)}_{c_a,1})^T - \mathbb{K}_j\mathbb{K}_j^T}_{\boldsymbol{Var}_{anc}(\tilde{\mathbb{K}}_j)} \end{cases} \tag{13}$$

In this decomposition, $\boldsymbol{Var}_{ps}(\tilde{\mathbb{K}}_j)$ corresponds to the variance in the number of kin caused by "projective stochasticity", that is by stochasticity in fertility (both in the number of offspring produced and in the state that they are produced in) and survival (both stochasticity in survival per se, and in the transitions towards future states). $\boldsymbol{Var}_{ps}(\tilde{\mathbb{K}}_j)$ corresponds to what population ecologists call individual stochasticity, which aggregates at the population level as demographic stochasticity and which affects the annual growth rate, as measured by demographic variance, itself affecting (negatively) the long term growth rate. It is of note that even in a model where individual stochasticity is zero, that is, all individuals have the same exact life trajectories (age at death, fertility schedule, net reproductive rate etc...), the variance in the number of kin can be non-zero, because of $\boldsymbol{Var}_{gs}(\tilde{\mathbb{K}}_j)$ which embeds "genealogical stochasticity", that is stochasticity in the genealogy of individuals, which causes two individual in the same state at the same time-step to have mothers, grand-mother, etc. of different ages and different states.

7.7.7 Embedded maximum age

Consider the case where there is a maximum age embedded in the model, that is there exist ω whereby $\mathbf{S}^\omega = \mathbf{0}$ (but $\mathbf{S}^{\omega-1} \neq \mathbf{0}$ and provide a bottom-up method (that is, a pseudo-code) for the computation of the desired moments and distributions. In that case $\tilde{\mathbf{S}}^\omega = \mathbf{0}$ and ${\mathbf{P_S}}^\omega = \mathbf{0}$ as well. This is also true in the generation structured model: $\tilde{\mathbb{S}}^\omega = \tilde{\mathbb{P}_{\mathbb{S}}}^\omega = \mathbf{0}$. To the contrary, $\tilde{\mathbf{F}}$ and $\tilde{\mathbb{F}}$ do not behave similarly, and contrary to the building block, we have for the generation structured model $\tilde{\mathbb{F}}^{g_m} = \tilde{\mathbb{P}_{\mathbb{F}}}^{g_m} = \mathbf{0}$.

Any product of $\tilde{\mathbb{F}}$ and $\tilde{\mathbb{S}}$ such as $\tilde{\mathbb{S}}\tilde{\mathbb{S}}\tilde{\mathbb{F}}\tilde{\mathbb{F}}\tilde{\mathbb{F}}\tilde{\mathbb{S}}\tilde{\mathbb{F}}$, of any size, that contains more than g_m $\tilde{\mathbb{F}}$s is worth $\mathbf{0}$ since $\tilde{\mathbb{F}}$ increase the generation number by 1 and $\tilde{\mathbb{S}}$ preserves it. Therefore, as it contains either a run of $\tilde{\mathbb{S}}$s longer than $\omega - 1$ or more than $g_m - 1$ $\tilde{\mathbb{S}}$s, we have $\tilde{\mathbf{M}}^{g_m\omega} = \mathbf{0}$ and similarly $\tilde{\mathbf{P}}^{g_m\omega} = \mathbf{0}$. Each element of $(\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}} + \mathbb{F}^*\mathbb{J}\mathbb{P}_{\mathbb{F}})$ contains either one $\mathbb{F}$ or one $\mathbb{P}_{\mathbb{F}}$, and therefore

$$\mathbb{K} = \sum_{t=0}^{\omega(g_m-1)-1} \mathbb{M}^t \left(\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}} + \mathbb{F}^*\mathbb{J}\mathbb{P}_{\mathbb{F}}\right)\mathbb{P}^t \tag{17}$$

In terms of the intermediary components, we have, for the same reasons, that $\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1} = \mathbf{0}$ for $t \geq \omega(g_m-1)+1$ and therefore

$$\tilde{\mathbb{K}} = \tilde{\mathbb{D}}^{(0)}\mathbb{J}^* + \sum_{t=1}^{\omega.(g_m-1)} \tilde{\mathbb{D}}^{(t)}\mathbb{J}^*\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}. \tag{75}$$

This implies that there is no need to consider $\tilde{\mathbb{D}}\mathbb{y}^{(t)}$,$\tilde{\mathbb{D}}\mathbb{s}^{(t)}$ and $\tilde{\mathbb{D}}\mathbb{o}^{(t)}$ for $t > \omega(g_m-1)$ For $\tilde{\mathbb{D}}\mathbb{o}^{(t)}$ specifically, we have

$$\tilde{\mathbb{D}}\mathbb{o}^{(0)} = \sum_{\alpha=1}^{\omega-1} \tilde{\mathbb{M}}^{\alpha-1}\tilde{\mathbb{F}}\mathbb{J}^*\tilde{\mathbf{P}_{\mathbf{S}}}^{\alpha}. \tag{76}$$

Because of the $\tilde{\mathbb{F}}$ in their formulation we have

$$\tilde{\mathbb{D}}\mathbb{s}^{(\omega(g_m-1)+1)} = \tilde{\mathbb{D}}\mathbb{o}^{(\omega(g_m-1))} = \mathbf{0}; \tag{77}$$

note however that we have, in general, $\tilde{\mathbb{D}}\mathbb{y}^{(\omega(g_m-1)+1)} \neq \mathbf{0}$. Indeed whilst in a population where one can survive and reproduce until age 20, the earliest in time my grandmother could have given birth to my mother is 40 years ago (if i am 20 and my mother gave birth to me when she was 20), my grandmother can have grandchildren alive up to 60 years after her own birth. The difference of 1 betwen $\tilde{\mathbb{D}}\mathbb{s}^{(\omega(g_m-1)+1)}$ and $\tilde{\mathbb{D}}\mathbb{o}^{(\omega(g_m-1))} = \mathbf{0}$ and between the indices in eq .. ... above, is caused by the change of index related to Doo that we see in section sdfsdgfds.

With the same reasoning we get $\boldsymbol{Var}_j^*(\tilde{\mathbb{K}}) = \boldsymbol{Var}_{des}(\tilde{\mathbb{K}}_j) + \boldsymbol{Var}_{anc}(\tilde{\mathbb{K}}_j)$ with :

$$\boldsymbol{Var}_{des}(\tilde{\mathbb{K}}_j) = \sum_{t=0}^{\omega.(g_m-1)} \sum_{c_a} P_j(c_a,t)\boldsymbol{Var}_{c_a}^*(\tilde{\mathbb{D}}^{(t)}) \tag{78a}$$

$$\boldsymbol{Var}_{anc}(\tilde{\mathbb{K}}_j) = \sum_{t=0}^{\omega.(g_m-1)} \sum_{c_a} P_j(c_a,t)\mathbb{D}_{c_a,1}^{(t)}{\mathbb{D}_{c_a,1}^{(t)}}^T - \mathbb{K}_j\mathbb{K}_j^T \tag{78b}$$

Eq 18a implies to know $\boldsymbol{Var}^*(\tilde{\mathbb{D}}^{(t)}) = \mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{y}^{(t)}) + \mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{s}^{(t)}) + \mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)})$ for all ts up to $\omega.(g_m-1)$ (and for all classes, but only generation 1 is required). The computation of $\mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{y}^{(t)})$ and $\mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{s}^{(t)})$ is obtained directly from eqs11a and 11b. Because of Eq. 77,we now have a starting point for the backward induction calculation of $\mathbf{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)})$:

$$\begin{cases} \boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(\omega.(g_m-1))}) = \mathbf{0} & \text{,and } \forall 0 \leq t < \omega.(g_{max}-1) \\ \boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) = (\boldsymbol{Var}^*(\tilde{\mathbb{M}}^t\tilde{\mathbb{F}}) + \boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t+1)}))\mathbb{P}_{\mathbb{S}} + \left(\left(\mathbb{M}^t(\mathbb{F} + \mathbb{M}\mathbb{D}\mathbb{o}^{(0)})\right) \otimes \left(\mathbb{M}^t(\mathbb{F} + \mathbb{M}\mathbb{D}\mathbb{o}^{(0)})\right)\right)\boldsymbol{Var}^*\left(\tilde{\mathbb{P}_{\mathbb{S}}}\right) \end{cases} \tag{19}$$

For the *pgf*, we get similarly

$$\varphi_{\tilde{\mathbb{K}}_j} = \sum_{t=0}^{\omega.(g_{max}-1)} \sum_{c_a} P_j(c_a,t)\varphi_{\tilde{\mathbb{D}}_{c_a}^{(t)}}, \qquad \text{with} \tag{79a}$$

$$\boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(t)}} = \underbrace{(\boldsymbol{\varphi}_{\tilde{\mathbb{S}}} \circ \bigcirc_{t-1} \boldsymbol{\varphi}_{\tilde{\mathbb{M}}})}_{\varphi_{\tilde{\mathbb{D}}\mathbb{y}^{(t)}}} \bullet \underbrace{(\boldsymbol{\varphi}_{\tilde{\mathbb{F}^*}} \circ \bigcirc_{t-1} \boldsymbol{\varphi}_{\tilde{\mathbb{M}}})}_{\boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{s}^{(t)}}} \bullet \boldsymbol{\varphi}_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}} \tag{79b}$$

where the various $\boldsymbol{\varphi}_{\tilde{\mathbb{D}}^{(t)}}$ of Eq 20b need only generating for $0 \leq t \leq \omega.(g_{max}-1)$. The *pgf* of $\tilde{\mathbb{D}}\mathbb{o}^{(t)}$ can be computed by :

$$\varphi_{\tilde{\mathbb{D}}\mathbb{o}^{(\omega.(g_{max}-1))}} = \mathbf{1} \text{ ,and } \forall 0 \le t < \omega.(g_{max}-1) \tag{80a}$$

$$\varphi_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}} = \varphi_{\tilde{\mathbb{P}_{\mathbb{S}}}} \circ \left(\left(\varphi_{\tilde{\mathbb{F}}} \circ \varphi_{\tilde{\mathbb{M}}^t} \right) \bullet \varphi_{\tilde{\mathbb{D}}\mathbb{o}^{(t+1)}} \right) \tag{80b}$$

To summarise, we get from Eq.55, the following system of equations:

$$\begin{cases} \varphi_{\tilde{\mathbb{K}}} = \varphi_{\tilde{\mathbb{D}}^{(0)}} \mathbb{J}_5 + \sum_{t=1}^{\omega.(g_{max}-1)-1} \varphi_{\tilde{\mathbb{D}}^{(t)}} \mathbb{J}_5 \mathbb{P}_{\mathbb{F}} \mathbb{P}^{t-1} \\ \varphi_{\tilde{\mathbb{D}}^{(t)}} = \underbrace{\left(\varphi_{\tilde{\mathbb{S}}} \circ \bigcirc_{t-1} \varphi_{\tilde{\mathbb{M}}} \right)}_{\varphi_{\tilde{\mathbb{D}}\mathbb{y}^{(t)}}} \bullet \underbrace{\left(\varphi_{\tilde{\mathbb{F}^*}} \circ \bigcirc_{t-1} \varphi_{\tilde{\mathbb{M}}} \right)}_{\varphi_{\tilde{\mathbb{D}}\mathbb{s}^{(t)}}} \bullet \varphi_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}} \\ \varphi_{\tilde{\mathbb{D}}\mathbb{o}^{(\omega.(g_{max}-1))}} = \mathbf{1} \text{ ,and } \forall 0 \le t < \omega.(g_{max}-1) \\ \varphi_{\tilde{\mathbb{D}}\mathbb{o}^{(t)}} = \varphi_{\tilde{\mathbb{P}_{\mathbb{S}}}} \circ \left(\left(\varphi_{\tilde{\mathbb{F}}} \circ \varphi_{\tilde{\mathbb{M}}^t} \right) \bullet \varphi_{\tilde{\mathbb{D}}\mathbb{o}^{(t+1)}} \right) \end{cases}$$

7.7.8 Expectation in the general case

Since $\sum_{c_b} \boldsymbol{e}_{c_a}^T \mathbb{P}_{\mathbb{F}} \boldsymbol{e}_{(c_b,2)} \boldsymbol{e}_{(c_b,2)}^T \mathbb{P}^{t-1} = \boldsymbol{e}_{c_a}^T \mathbb{P}_{\mathbb{F}} \mathbb{P}^{t-1}$ we have $\mathbb{K} = \sum_{t=0}^{\infty} \mathbb{M}^t \left(\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}} \right) \mathbb{P}^t + \sum_{c_a,c_b,t>0} (\mathbb{M}^{t-1}\mathbb{F}^{(c_b)}) \boldsymbol{e}_{c_a}^T \mathbb{P}_{\mathbb{F}} \boldsymbol{e}_{(c_b,2)} \boldsymbol{e}_{(c_b,2)}^T \mathbb{P}^{t-1}$. We can rewrite the last term as $\sum_{t>0} \mathbb{M}^{t-1} \sum_{c_a,c_b} (\mathbb{F}^{(c_b)}) \boldsymbol{e}_{c_a}^T \mathbb{P}_{\mathbb{F}} \boldsymbol{e}_{(c_b,2)} \boldsymbol{e}_{(c_b,2)}^T \mathbb{P}^{t-1}$.

Let us focus now on $\sum_{c_a,c_b} (\mathbb{F}^{(c_b)}) \boldsymbol{e}_{c_a}^T \mathbb{P}_{\mathbb{F}} \boldsymbol{e}_{(c_b,2)} \boldsymbol{e}_{(c_b,2)}^T = \sum_{c_b} (\mathbb{F}^{(c_b)}) \mathbb{J}\mathbb{P}_{\mathbb{F}} \boldsymbol{e}_{(c_b,2)} \boldsymbol{e}_{(c_b,2)}^T$. This is the product of two block matrices that are zero apart from the second block of the diagonal. Let us call $\mathbb{Z} = \mathbf{J2} \otimes \mathbf{Z} = \sum_{c_b} (\mathbb{F}^{(c_b)}) \mathbb{J}\mathbb{P}_{\mathbb{F}} \boldsymbol{e}_{(c_b,2)} \boldsymbol{e}_{(c_b,2)}^T$
Then $\mathbf{Z}_{c_i,c_j} = \sum_{c_k} \mathbf{F}_{c_i,c_k}^{(c_j)} \mathbf{P}_{\mathbf{F}c_k,c_j}$ an expectation we call the same-litter sister matrix. Therefore

$$\mathbb{K} = \sum_{t=0}^{\infty} \mathbb{M}^t \left(\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}} + \mathbb{Z} \right) \mathbb{P}^t \tag{7.7.8}$$

noting that in the three common cases, we have $\mathbb{Z} = \mathbb{F}^* \mathbb{J}\mathbb{P}_{\mathbb{F}}$

7.8 SM: Implementation of computation of variance-covariance of number of kin

pseudocode

- Compute the variance of $\tilde{\mathbb{M}}^t$, iteratively, for $0 \le t \le \omega(g_m-1)-1$ from $\boldsymbol{Var}^*(\tilde{\mathbb{M}}^t) = (\mathbb{M} \otimes \mathbb{M}) \boldsymbol{Var}^*(\tilde{\mathbb{M}}^{t-1}) + \boldsymbol{Var}^*(\tilde{\mathbb{M}}) \mathbb{M}^{t-1}$ (starting from $\boldsymbol{Var}^*(\tilde{\mathbb{M}}^0) = \mathbf{0}$), from Eq.66. Store these matrices as sparse matrices.
- From this, compute the variance of the number of descendants via younger and same-litter sisters, for all $0 \le t \le \omega(g_m-1)-1$, via $\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{y}^{(t)}) = (\mathbb{M} \otimes \mathbb{M})^{t-1} \boldsymbol{Var}^*(\tilde{\mathbb{S}}) \mathbb{J}^* + \boldsymbol{Var}^*(\tilde{\mathbb{M}}^{t-1}) \mathbb{S}\mathbb{J}^*$ and $\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{s}^{(t)}) = (\mathbb{M} \otimes \mathbb{M})^{t-1} \boldsymbol{Var}^*(\tilde{\mathbb{F}^*}) \mathbb{J}^* + \boldsymbol{Var}^*(\tilde{\mathbb{M}}^{t-1}) \mathbb{F}^* \mathbb{J}^*$ with $\boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{y}^{(0)}) = \boldsymbol{Var}^*(\tilde{\mathbb{D}}\mathbb{s}^{(t)}) = \mathbf{0}$ (Eqs.11a and 11b).
- Compute $\mathbb{D}\mathbb{o}^{(0)} = \sum_{\alpha=1}^{\omega} \tilde{\mathbb{M}}^{\alpha-1} \mathbb{F}\mathbb{J}^* \tilde{\mathbf{P}_{\mathbf{S}}}^{\alpha}$ (Eq.4b) by calculating the summands iteratively.
- Compute and store $\mathbb{D}^{(t)} = \mathbb{M}^{t-1} (\mathbb{M}\mathbb{D}\mathbb{o}^{(0)} + \mathbb{S} + \mathbb{F}^*)$ for all $0 \le t \le \omega(g_m-1)$, with $\mathbb{D}^{(0)} = \mathbb{D}\mathbb{o}^{(0)}$.
- From $\boldsymbol{Var}_{(ps)}^*(\tilde{\mathbb{D}}\mathbb{o}^{(\omega(g_m-1))}) = \boldsymbol{Var}_{(gs)}^*(\tilde{\mathbb{D}}\mathbb{o}^{(\omega(g_m-1))}) = \mathbf{0}$ compute (Eq.12)
  - $\boldsymbol{Var}_{(ps)}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) = (\boldsymbol{Var}^*(\tilde{\mathbb{M}}^t \tilde{\mathbb{F}}) \mathbb{J}^* + \boldsymbol{Var}_{(ps)}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t+1)})) \mathbf{P_S}$ and
  - $\boldsymbol{Var}_{(gs)}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) = (\boldsymbol{Var}_{(gs)}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t+1)}) \mathbf{P_S} + \left(\left(\mathbb{M}^t (\mathbb{F}\mathbb{J}^* + \mathbb{M}\mathbb{D}\mathbb{o}^{(0)}) \right) \otimes \left(\mathbb{M}^t (\mathbb{F}\mathbb{J}^* + \mathbb{M}\mathbb{D}\mathbb{o}^{(0)}) \right) \right) \boldsymbol{Var}^* \left(\tilde{\mathbf{P}_{\mathbf{S}}} \right)$

  as backward iterations. This gives $\boldsymbol{Var}_{(gs)}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)}) + \boldsymbol{Var}_{(ps)}^*(\tilde{\mathbb{D}}\mathbb{o}^{(t)})$ for all $0 \le t \le \omega(g_m-1)$

- compute
  - $\boldsymbol{Var}^*_{ps}(\tilde{\mathbb{K}}) = \boldsymbol{Var}^*_{ps}(\tilde{\mathbb{D}}^{(0)})\mathbb{J}_4 + \sum_{t=1}^{\omega(g_m-1)} \boldsymbol{Var}^*_{ps}(\tilde{\mathbb{D}}^{(t)})\mathbb{J}_4\mathbb{P}_{\mathbb{F}}\mathbb{P}^t$
  - $\boldsymbol{Var}^*_{gs1}(\tilde{\mathbb{K}}) = \boldsymbol{Var}^*_{gs}(\tilde{\mathbb{D}}^{(0)})\mathbb{J}_4 + \sum_{t=1}^{\omega(g_m-1)} \boldsymbol{Var}^*_{gs}(\tilde{\mathbb{D}}^{(t)})\mathbb{J}_4\mathbb{P}_{\mathbb{F}}\mathbb{P}^t$
  - $\boldsymbol{Var}^*_{gs2}(\tilde{\mathbb{K}}) = (\mathbb{D}^{(0)} \otimes \mathbb{D}^{(0)})\mathbb{J}_6 + \sum_{t=1}^{\omega(g_m-1)} (\mathbb{D}^{(t)} \otimes \mathbb{D}^{(t)})\mathbb{J}_6\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1} - (\mathbb{K} \otimes \mathbb{K})\mathbb{J}_7$

  From this one gets $\boldsymbol{Var}^*_{gs}(\tilde{\mathbb{K}}) = \boldsymbol{Var}^*_{gs1}(\tilde{\mathbb{K}}) + \boldsymbol{Var}^*_{gs2}(\tilde{\mathbb{K}})$ the component of $\boldsymbol{Var}^*(\tilde{\mathbb{K}})$ caused by "genealogical stochasticity" and $\boldsymbol{Var}^*_{gs}(\tilde{\mathbb{K}})$ the component caused by "projective stochasticity", and finally $\boldsymbol{Var}^*(\tilde{\mathbb{K}}) = \boldsymbol{Var}^*_{ps}(\tilde{\mathbb{K}}) + \boldsymbol{Var}^*_{gs}(\tilde{\mathbb{K}})$

- If one is not interested in this decomposition, one can replace the last two steps by
  - $\boldsymbol{Var}^*(\tilde{\mathbb{D}\mathbb{o}}^{(t)}) = (\boldsymbol{Var}^*(\tilde{\mathbb{M}}^t\tilde{\mathbb{F}})\mathbb{J}^* + \boldsymbol{Var}^*(\tilde{\mathbb{D}\mathbb{o}}^{(t+1)}))\mathbf{P_S} + \left(\left(\mathbb{M}^t(\mathbb{F}\mathbb{J}^* + \mathbb{M}\mathbb{D}\mathbb{o}^{(0)})\right) \otimes \left(\mathbb{M}^t(\mathbb{F}\mathbb{J}^* + \mathbb{M}\mathbb{D}\mathbb{o}^{(0)})\right)\right) \boldsymbol{Var}^*\left(\tilde{\mathbf{P_S}}\right)$
  - $\boldsymbol{Var}^*_{des}(\tilde{\mathbb{K}}) = \boldsymbol{Var}^*(\tilde{\mathbb{D}}^{(0)})\mathbb{J}_4 + \sum_{t=1}^{\omega(g_m-1)} \boldsymbol{Var}^*(\tilde{\mathbb{D}}^{(t)})\mathbb{J}_4\mathbb{P}_{\mathbb{F}}\mathbb{P}^t$
  - $\boldsymbol{Var}^*_{anc}(\tilde{\mathbb{K}}) = (\mathbb{D}^{(0)} \otimes \mathbb{D}^{(0)})\mathbb{J}_6 + \sum_{t=1}^{\omega(g_m-1)} (\mathbb{D}^{(t)} \otimes \mathbb{D}^{(t)})\mathbb{J}_6\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1} - (\mathbb{K} \otimes \mathbb{K})\mathbb{J}_7$
  - $\boldsymbol{Var}^*(\tilde{\mathbb{K}}) = \boldsymbol{Var}^*_{anc}(\tilde{\mathbb{K}}) + \boldsymbol{Var}^*_{des}(\tilde{\mathbb{K}})$

7.9 SM: Convergence and Definition of processes

From its shape we see that $\tilde{\mathbb{S}}$ has the same behaviour that $\tilde{\mathbf{S}}$, and $\tilde{\mathbb{P}_{\mathbb{S}}}$ that of $\tilde{\mathbf{P_S}}$. Indeed we have $\mathbb{S}^k = Jx \otimes \mathbb{S}^k$ and ${\mathbb{P}_{\mathbb{S}}}^k = Jx \otimes {\mathbb{P}_{\mathbb{S}}}^k$. This is not the case for $\tilde{\mathbb{F}}$ (and $\tilde{\mathbb{P}_{\mathbb{F}}}$) and $\tilde{\mathbf{F}}$ (and $\tilde{\mathbf{P_F}}$), since contrary to their class-structure equivalents, the generation structured matrices are such that $\mathbb{F}^{g_m} = {\mathbb{P}_{\mathbb{F}}}^{g_m} = \mathbf{0}$.
We know that $\mathbb{S}^k$ and ${\mathbb{P}_{\mathbb{S}}}^k$ converge exponentially to $\mathbf{0}$ as $k \to \infty$, as $\sum_{k=0}^{\infty} \mathbf{S}^k = (\mathbf{I}-\mathbf{S})^{-1}$ and $\sum_{k=0}^{\infty} {\mathbf{P_S}}^k = (\mathbf{I}-\mathbf{P_S})^{-1}$.
Consider the quantity $\sum_{k=0}^{\infty} \mathbf{P}^k = \sum_{k=0}^{\infty}(\mathbf{P_S}+\mathbf{P_F})^k = \sum_{k=0}^{\infty} \sum_{\boldsymbol{v}\in[0,1]^k} \prod_{i=1}^k {\mathbf{P_S}}^{v_i} {\mathbf{P_F}}^{1-v_i} = \sum_{k=0}^{\infty} \sum_{n=0}^{k} \sum_{\boldsymbol{v}\in[0,1]^k \text{with } n \text{ zeros}} \prod_{i=1}^k {\mathbf{P_S}}^{v_i} {\mathbf{P_F}}^{1-v_i}$ $\sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \sum_{\boldsymbol{v}\in[0,1]^k \text{with } n \text{ zeros}} \prod_{i=1}^k {\mathbf{P_S}}^{v_i} {\mathbf{P_F}}^{1-v_i} = \sum_{n=0}^{\infty} \sum_{j_1,j_2,\dots,j_n\geq 0} {\mathbf{P_S}}^{j_1}\mathbf{P_F}{\mathbf{P_S}}^{j_2}\mathbf{P_F}\cdots{\mathbf{P_S}}^{j_n}\mathbf{P_F} = \sum_{n=0}^{\infty} \prod_{n \text{ times}} \sum_{j=0}^{\infty} {\mathbf{P_S}}^{j}\mathbf{P_F} = \sum_{n=0}^{\infty} \prod_{n \text{ times}} (MalbuithbfI - \mathbf{P_S})^{-1}\mathbf{P_F} = \sum_{n=0}^{\infty} \prod_{n \text{ times}} \mathbf{N_g}\mathbf{P_F} = \sum_{n=0}^{\infty} \prod_{n \text{ times}} \mathbf{G} = \sum_{n=0}^{\infty} \mathbf{G}^n$ (see xczsczc) which does not converge.
However, when turning to generation models, we have also $\sum_{k=0}^{\infty} \mathbb{P}^k = \sum_{n=0}^{\infty}((\mathbb{I}-\mathbb{P}_{\mathbb{S}})^{-1}\mathbb{P}_{\mathbb{F}})^n$ from the same calculations, but we also have (for the same reasons than ${\mathbb{P}_{\mathbb{F}}}^{g_m} = \mathbf{0}$, $((\mathbb{I}-\mathbb{P}_{\mathbb{S}})^{-1}\mathbb{P}_{\mathbb{F}})^{g_m} = \mathbf{0}$ and therefore

$$\sum_{k=0}^{\infty} \mathbb{P}^k = \sum_{n=0}^{g_m-1} ((\mathbb{I}-\mathbb{P}_{\mathbb{S}})^{-1}\mathbb{P}_{\mathbb{F}})^n \tag{81}$$

Similarly,

$$\sum_{k=0}^{\infty} \mathbb{M}^k = \sum_{n=0}^{g_m-1} ((\mathbb{I}-\mathbb{S})^{-1}\mathbb{F})^n \tag{82}$$

This implies that both $\mathbb{M}^k$ and $\mathbb{P}^k$ converge exponentially towards $\mathbf{0}$, and that $\tilde{\mathbb{M}}^k$ and $\tilde{\imath}^k$ converge almost surely towards $\mathbf{0}$. As a consequence $\forall c_j, g_j, c_i \lim_{t\to\infty} P_{c_j}(t, c_i) = \lim_{t\to\infty} \boldsymbol{e}_{c_i,1}\mathbb{P}_{\mathbb{F}}\mathbb{P}^{t-1}\boldsymbol{e}_{c_j,g_j} = 0$. As another consequence, $\tilde{\mathbb{D}}_{\mathbb{y}}^{(t)} = \tilde{\mathbb{M}}^{(t-1)}\tilde{\mathbb{S}}$ and $\tilde{\mathbb{D}}_{\mathbb{s}}^{(t)} = \tilde{\mathbb{M}}^{(t-1)}\tilde{\mathbb{F}^*}$ converge almost surely towards $\mathbf{0}$. This implies also that $\tilde{\mathbb{D}\mathbb{o}}^{(0)} = \sum_{\alpha>0} \tilde{\mathbb{M}}^{\alpha-1}\tilde{\mathbb{F}}{\tilde{\mathbb{P}_{\mathbb{S}}}}^{\alpha}$ is well defined and that $\tilde{\mathbb{D}\mathbb{o}}^{(t)} = \tilde{\mathbb{M}}^t\tilde{\mathbb{D}\mathbb{o}}^{(0)}$ converge exponentially towards $\mathbf{0}$, and that we can write $\forall t \geq 0, \quad \tilde{\mathbb{D}\mathbb{o}}^{(t)} = \left(\tilde{\mathbb{M}}^t\tilde{\mathbb{F}} + \tilde{\mathbb{D}\mathbb{o}}^{(t+1)}\right)\tilde{\mathbb{P}_{\mathbb{S}}}$ as $\lim_{t\to\infty} P(\tilde{\mathbb{D}\mathbb{o}}^{(t)} = \mathbf{0}) = 1$

7.10 SM: Symmetrical unstructured kinship

intermediary result: $\mathbb{K}_w$ is symetrical

We first show here that $\mathbb{K}_w = \mathbb{K}(\mathbf{I}_{g_m} \otimes \boldsymbol{diag}(\boldsymbol{w}))$ is symmetrical ($\mathbb{K}_w$ can also be noted $\mathbb{K} \bullet (\mathbf{1}_{g_m,g_m} \otimes \boldsymbol{w}^T)$). This tensor matrix has for element in position $(c_i, g_i), (c_j, g_j)$: $\mathbb{K}_{w\,(c_i,g_i),(c_j,g_j)} = w_{c_j} \mathbb{K}_{(c_i,g_i),(c_j,g_j)}$.

Let us note $\mathbb{K}_w^t = \mathbb{K}^t(\mathbf{I}_{g_m} \otimes \boldsymbol{diag}(\boldsymbol{w}))$, with $\mathbb{K}^t = \mathbb{M}^t\,(\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}} + \mathbb{F}^*\mathbb{J}\mathbb{P}_{\mathbb{F}})\,\mathbb{P}^t$.

Consider that $\mathbb{K}_w^t$ is symmetrical, that is : $\forall c_i, g_i, c_j, g_j, {\mathbb{K}_w^{(t)}}_{(c_i,g_i),(c_j,g_j)} = {\mathbb{K}_w^{(t)}}_{(c_j,g_j),(c_i,g_i)}$, that is

$$\forall c_i, g_i, c_j, g_j, \qquad w_{c_j} \mathbb{K}^{(t)}_{(c_i,g_i),(c_j,g_j)} = w_{c_i} \mathbb{K}^{(t)}_{(c_j,g_j),(c_i,g_i)} \tag{83}$$

Let us now focus on $\mathbb{K}_w^{(t+1)}$. Remembering that $\mathbb{K}^{t+1} = \mathbb{M}\mathbb{K}^t\mathbb{P}$, we have ${\mathbb{K}_w^{(t+1)}}_{(c_i,g_i),(c_j,g_j)} = w_{c_j} \mathbb{K}^{(t+1)}_{(c_i,g_i),(c_j,g_j)} = w_{c_j} \{\mathbb{M}\mathbb{K}^t\mathbb{P}\}_{(c_i,g_i),(c_j,g_j)} = \sum_{c_k,g_k,c_l,g_l} w_{c_j} \mathbb{M}_{(c_i,g_i),(c_k,g_k)} \mathbb{K}^t_{(c_k,g_k),(c_l,g_l)} \mathbb{P}_{(c_l,g_l),(c_j,g_j)}$.

$\mathbb{M}_{(c_i,g_i),(c_k,g_k)}$ is only non-zero for $g_k = g_i$ or $g_k = g_i - 1$. In the first case, the entry is part of a $\mathbf{S}$ block, and it is therefore equal to $\mathbf{S}_{c_i,c_k}$. In the second case, the entry is part of a $\mathbf{F}$ block, and it is therefore equal to $\mathbf{F}_{c_i,c_k}$. Similarly, $\mathbb{P}_{(c_l,g_l),(c_j,g_j)}$ is only non-zero for $g_l = g_j$ or $g_l = g_j - 1$. In the first case, the entry is part of a $\mathbf{P}_{\mathbf{S}}$ block, and it is therefore equal to ${\mathbf{P}_{\mathbf{S}}}_{c_l,c_j}$. In the second case, the entry is part of a $\mathbf{F}$ block, and it is therefore equal to ${\mathbf{P}_{\mathbf{F}}}_{c_l,c_j}$. Therefore ${\mathbb{K}_w^{(t+1)}}_{(c_i,g_i),(c_j,g_j)} = \mathbb{E}1_{(c_i,g_i),(c_j,g_j)} + \mathbb{E}2_{(c_i,g_i),(c_j,g_j)} + \mathbb{E}3_{(c_i,g_i),(c_j,g_j)} + \mathbb{E}4_{(c_i,g_i),(c_j,g_j)}$

with $\mathbb{E}1_{(c_i,g_i),(c_j,g_j)} = \sum_{c_k,g_k,c_l,g_l} w_{c_j} \mathbf{S}_{c_i,c_k} \mathbb{K}^t_{(c_k,g_i),(c_l,g_j)} {\mathbf{P}_{\mathbf{S}}}_{c_l,c_j}$

$\mathbb{E}2_{(c_i,g_i),(c_j,g_j)} = \sum_{c_k,g_k,c_l,g_l} w_{c_j} \mathbf{S}_{c_i,c_k} \mathbb{K}^t_{(c_k,g_i),(c_l,g_j-1)} {\mathbf{P}_{\mathbf{F}}}_{c_l,c_j}$

$\mathbb{E}3_{(c_i,g_i),(c_j,g_j)} = \sum_{c_k,g_k,c_l,g_l} w_{c_j} \mathbf{F}_{c_i,c_k} \mathbb{K}^t_{(c_k,g_i-1),(c_l,g_j)} {\mathbf{P}_{\mathbf{S}}}_{c_l,c_j}$

$\mathbb{E}4_{(c_i,g_i),(c_j,g_j)} = \sum_{c_k,g_k,c_l,g_l} w_{c_j} \mathbf{F}_{c_i,c_k} \mathbb{K}^t_{(c_k,g_i-1),(c_l,g_j-1)} {\mathbf{P}_{\mathbf{F}}}_{c_l,c_j})$.

From their formulation, we have ${\mathbf{P}_{\mathbf{S}}}_{c_l,c_j} = \frac{\mathbf{S}_{c_j,c_l}.w_{c_l}}{\lambda.w_{c_j}}$ and similarly ${\mathbf{P}_{\mathbf{F}}}_{c_l,c_j} = \frac{\mathbf{F}_{c_j,c_l}.w_{c_l}}{\lambda.w_{c_j}}$ and equivalently $\mathbf{S}_{c_i,c_k} = \frac{\lambda.{\mathbf{P}_{\mathbf{S}}}_{c_k,c_i}.w_{c_i}}{w_{c_k}}$ and $\mathbf{F}_{c_i,c_k} = \frac{\lambda.{\mathbf{P}_{\mathbf{F}}}_{c_k,c_i}.w_{c_i}}{w_{c_k}}$.

Replacing in the expression above, where the w_{c_j} is distributed on the summands, we have, for the first term $w_{c_j} \mathbf{S}_{c_i,c_k} \mathbb{K}^t_{(c_k,g_i),(c_l,g_j)} {\mathbf{P}_{\mathbf{S}}}_{c_l,c_j} = w_{c_j} \frac{\lambda.{\mathbf{P}_{\mathbf{S}}}_{c_k,c_i}.w_{c_i}}{w_{c_k}} \mathbb{K}^t_{(c_k,g_i),(c_l,g_j)} \frac{\mathbf{S}_{c_j,c_l}.w_{c_l}}{\lambda.w_{c_j}} = \frac{w_{c_i}}{w_{c_k}} \mathbf{S}_{c_j,c_l} w_{c_l} \mathbb{K}^t_{(c_k,g_i),(c_l,g_j)} {\mathbf{P}_{\mathbf{S}}}_{c_k,c_i}$.

that is, $w_{c_j} \mathbf{S}_{c_i,c_k} \mathbb{K}^t_{(c_k,g_i),(c_l,g_j)} {\mathbf{P}_{\mathbf{S}}}_{c_l,c_j} = w_{c_i} \mathbf{S}_{c_j,c_l} \mathbb{K}^{(t)}_{(c_l,g_j),(c_k,g_i)} {\mathbf{P}_{\mathbf{S}}}_{c_k,c_i}$.

This means, from switching indices c_l and c_k $\mathbb{E}1_{(c_i,g_i),(c_j,g_j)} = \sum_{c_k,g_k,c_l,g_l} w_{c_i} \mathbf{S}_{c_j,c_k} \mathbb{K}^{(t)}_{(c_k,g_j),(c_l,g_i)} {\mathbf{P}_{\mathbf{S}}}_{c_l,c_i} = \mathbb{E}1_{(c_j,g_j),(c_i,g_i)}$.

Similarly for the other components we have $w_{c_j} \mathbf{S}_{c_i,c_k} \mathbb{K}^t_{(c_k,g_i),(c_l,g_j-1)} {\mathbf{P}_{\mathbf{F}}}_{c_l,c_j} = w_{c_i} \mathbf{F}_{c_j,c_l} \mathbb{K}^{(t)}_{(c_l,g_j-1),(c_k,g_i)} {\mathbf{P}_{\mathbf{S}}}_{c_k,c_i}$.

$w_{c_j} \mathbf{F}_{c_i,c_k} \mathbb{K}^t_{(c_k,g_i-1),(c_l,g_j)} {\mathbf{P}_{\mathbf{S}}}_{c_l,c_j} = w_{c_i} \mathbf{S}_{c_j,c_l} \mathbb{K}^{(t)}_{(c_l,g_j),(c_k,g_i-1)} {\mathbf{P}_{\mathbf{F}}}_{c_k,c_i}$.

$w_{c_j} \mathbf{F}_{c_i,c_k} \mathbb{K}^t_{(c_k,g_i-1),(c_l,g_j-1)} {\mathbf{P}_{\mathbf{F}}}_{c_l,c_j} = w_{c_i} \mathbf{F}_{c_j,c_l} \mathbb{K}^{(t)}_{(c_l,g_j-1),(c_k,g_i-1)} {\mathbf{P}_{\mathbf{F}}}_{c_k,c_i}$.

This implies $\mathbb{E}2_{(c_i,g_i),(c_j,g_j)} = \mathbb{E}3_{(c_j,g_j),(c_i,g_i)}$; $\mathbb{E}3_{(c_i,g_i),(c_j,g_j)} = \mathbb{E}2_{(c_j,g_j),(c_i,g_i)}$ and $\mathbb{E}4_{(c_i,g_i),(c_j,g_j)} = \mathbb{E}4_{(c_j,g_j),(c_i,g_i)}$.

Therefore ${\mathbb{K}_w^{(t+1)}}_{(c_i,g_i),(c_j,g_j)} = \mathbb{E}1_{(c_i,g_i),(c_j,g_j)} + \mathbb{E}2_{(c_i,g_i),(c_j,g_j)} + \mathbb{E}3_{(c_i,g_i),(c_j,g_j)} + \mathbb{E}4_{(c_i,g_i),(c_j,g_j)} = \mathbb{E}1_{(c_j,g_j),(c_i,g_i)} + \mathbb{E}3_{(c_j,g_j),(c_i,g_i)} + \mathbb{E}2_{(c_j,g_j),(c_i,g_i)} + \mathbb{E}4_{(c_j,g_j),(c_i,g_i)} = {\mathbb{K}_w^{(t+1)}}_{(c_j,g_j),(c_i,g_i)}$.

and therefore if $\mathbb{K}_w^t$ is symmetrical, $\mathbb{K}_w^{t+1}$ is also symmetrical.

With the exact same calculation approach, we can show that $\mathbb{K}_w^0$, such that $\mathbb{K}^0_{w\,(c_i,g_i),(c_j,g_j)} = w_{c_j} \{\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}} + \mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}} + \mathbb{Z})\}_{(c_i,g_i),(c_j,g_j)}$ is symmetrical too in the three common cases highlighted in this article. $\mathbb{K}_w^0$ is made of three blocks only: one in position (1,2), corresponding to $\mathbb{S}\mathbb{J}\mathbb{P}_{\mathbb{F}}$ and equal to $\mathbf{S}\mathbf{P}_{\mathbf{F}}$, one in the opposite block (2,1) corresponding to $\mathbb{F}\mathbb{J}\mathbb{P}_{\mathbb{S}}$ and equal to $\mathbf{F}\mathbf{P}_{\mathbf{S}}$. We have $w_j \mathbf{S}\mathbf{P}_{\mathbf{F}}(i,j) = \sum_k w_j \mathbf{S}(i,k)\mathbf{P}_{\mathbf{F}}(k,i) = \sum_k w_i \mathbf{F}(j,k)\mathbf{P}_{\mathbf{S}}(k,i) = w_i \mathbf{F}\mathbf{P}_{\mathbf{S}}(j,i)$.

The third block corresponds to $\mathbf{Z}$ in position (2,2). The symmetry is easy to prove for the three common cases we have highlighted: If reproduction is Bernoulli $\mathbf{Z} = \nvdash$ and therefore $\{w_j \mathbf{Z}(i,j)\}_{i,j}$ is symmetrical as it is worth zero. If there is only one class of offspring, then $\mathbf{Z}$ is zero but for $\mathbf{Z}(1,1)$ and therefore $\{w_j \mathbf{Z}(i,j)\}_{i,j}$ is

symmetrical, as it is made of one element only and that it is on the diagonal. If reproduction is Poisson, then $\mathbf{Z} = \mathbf{F}\mathbf{P_F}$. Then $w_j \mathbf{F}\mathbf{P_F}(i,j) = \sum_k w_j \mathbf{F}(i,k)\mathbf{P_F}(k,i) = \sum_k w_i \mathbf{F}(j,k)\mathbf{P_F}(k,i) = w_i \mathbf{F}\mathbf{P_S}(j,i)$.
In the general case, we have from Eq. 39 $Z_{i,j} = \sum_k \left(\frac{Cov(\tilde{F}(i,k),\tilde{F}(j,k))}{F_{j,k}} + F_{i,k} - \delta(i,j) \right) P_F(k,j)$ As we have done a few operations ago, we replace $\mathbf{P_F}_{k,j}$ by $\frac{\mathbf{F}_{j,k} w_k}{\lambda w_j}$ in $Z_{i,j} w_j$. For $i \neq j$ (we do not care about what happens on the diagonal), we have $Z_{i,j} w_j = \sum_k w_j \left(\frac{Cov(\tilde{F}(i,k),\tilde{F}(j,k))}{F_{j,k}} + F_{i,k} \right) \frac{\mathbf{F}_{j,k} w_k}{\lambda w_j} = \frac{1}{\lambda} \sum_k \left(Cov(\tilde{F}(i,k), \tilde{F}(j,k)) + F_{i,k}\mathbf{F}_{j,k} \right) w_k$ which is symetrical.

And therefore $\mathbb{K}^0_w$ is symmetrical.
Therefore, by induction, for all ts $\mathbb{K}^t_w$ is symmetrical.
We have $\mathbb{K} = \sum_t \mathbb{K}^{(t)}$ and therefore $\mathbb{K}_w = \mathbb{K} \bullet (\mathbf{1}_{g_m,g_m} \otimes \boldsymbol{w}^T) = \sum_t \mathbb{K}^{(t)} \bullet (\mathbf{1}_{g_m,g_m} \otimes \boldsymbol{w}^T) = \sum_t \mathbb{K}^{(t)} \bullet (\mathbf{1}_{g_m,g_m} \otimes \boldsymbol{w}^T) = \sum_t \mathbb{K}^{(t)}_w$ is the sum of symmetrical matrices and is itself symmetrical: $\mathbb{K}_w = \mathbb{K}^T_w$.

$\mathbb{K}_u$ is symmetrical

$\mathbb{K}_w$ corresponds to $\mathbb{K}$ but with every entry multiplied by the corresponding w_{c_j}. This means that $\mathbb{K}_u = \mathbb{J}_y \mathbb{K} \mathbb{J}_w$ can also be written: $\mathbb{K}_u = \mathbb{J}_y \mathbb{K}_w \mathbb{J}^T_y$. From this we have $\mathbb{K}^T_u = \mathbb{J}_y \mathbb{K}^T_w \mathbb{J}^T_y = \mathbb{J}_y \mathbb{K}_w \mathbb{J}^T_y = \mathbb{K}_u$, which completes the proof.

7.11 SM: Complete set

For a given state j of *ego*, $\{P_j(c_a,t)\}_{c_a,t}$ describes a complete set. Indeed $\sum_{c_a,t} P_j(c_a,t) = \sum_{c_a,t>0} P_j(c_a,t) + \sum_{c_a} P_j(c_a,0) = \sum_{c_a,t>0} \boldsymbol{e}^T_{\mathbf{(1,c_a)}} \mathbb{P}_\mathbb{F} \mathbb{P}^{t-1} \boldsymbol{e_j} + \sum_{c_a} \boldsymbol{e}^T_{\mathbf{(1,c_a)}} \boldsymbol{e_j} = \sum_{c_a} \left(\boldsymbol{e}^T_{\mathbf{(1,c_a)}} \left(\mathbb{I} + \mathbb{P}_\mathbb{F} \sum_{t' \geq 0} \mathbb{P}^{t'} \right) \boldsymbol{e_j} \right)$. Let's denote $\mathbb{G} = \mathbb{I} + \mathbb{P}_\mathbb{F} \sum_{t' \geq 0} \mathbb{P}^{t'}$, we have $\sum_{c_a,t} P_j(c_a,t) = \sum_{c_a} \boldsymbol{e}^T_{\mathbf{(1,c_a)}} \mathbb{G} \boldsymbol{e_j}$ which corresponds to the sum of the first s elements of the jth column of $\mathbb{G}$

$\mathbb{G} = \mathbb{I} + \mathbb{P}_\mathbb{F} \sum_{t' \geq 0} \mathbb{P}^{t'} = \mathbb{I} + \mathbb{P}_\mathbb{F}(\mathbb{I} - \mathbb{P})^{-1}$. Therefore $\mathbb{G}(\mathbb{I} - \mathbb{P}) = \mathbb{I} - \mathbb{P} + \mathbb{P}_\mathbb{F} = \mathbb{I} - \mathbb{P}_\mathbb{S}$ and thus $\mathbb{G} = (\mathbb{I} - \mathbb{P}_\mathbb{S})(\mathbb{I} - \mathbb{P})^{-1} = \left((\mathbb{I} - \mathbb{P})(\mathbb{I} - \mathbb{P}_\mathbb{S})^{-1} \right)^{-1} = \left((\mathbb{I} - \mathbb{P}_\mathbb{S} - \mathbb{P}_\mathbb{F})(\mathbb{I} - \mathbb{P}_\mathbb{S})^{-1} \right)^{-1} = \left((\mathbb{I} - \mathbb{P}_\mathbb{S})(\mathbb{I} - \mathbb{P}_\mathbb{S})^{-1} - \mathbb{P}_\mathbb{F}(\mathbb{I} - \mathbb{P}_\mathbb{S})^{-1} \right)^{-1} = \left(\mathbb{I} - \mathbb{P}_\mathbb{F}(\mathbb{I} - \mathbb{P}_\mathbb{S})^{-1} \right)^{-1} = \sum_{k \geq 0} \left(\mathbb{P}_\mathbb{F}(\mathbb{I} - \mathbb{P}_\mathbb{S})^{-1} \right)^k$.

Now generation structured matrix, $\mathbb{P}_\mathbb{F}(\mathbb{I} - \mathbb{P}_\mathbb{S})^{-1}$ can be written $\mathbb{P}_\mathbb{F}(\mathbb{I} - \mathbb{P}_\mathbb{S})^{-1} = \mathbf{J_1}^T \otimes \left(\mathbf{P_F}(\mathbf{I} - \mathbf{P_S})^{-1} \right)$ in which we recognise in $\mathbf{P_F}(\mathbf{I} - \mathbf{P_S})^{-1}$ the "previous-generation matrix", which is the transition matrix of the genealogical Markov chain associated with the Euler-Lokta matrix $(\lambda\mathbf{I} - \mathbf{S})^{-1}\mathbf{F}$ (see SM5 in (Coste et al., 2021)). This property can be extended to generation structured model: $\mathbb{P}_\mathbb{F}(\mathbb{I} - \mathbb{P}_\mathbb{S})^{-1}$ is the "previous-generation matrix structured by generation", which is the transition matrix of the genealogical Markov chain associated with the "Euler-Lokta generation" matrix $(\lambda\mathbb{I} - \mathbb{S})^{-1}\mathbb{F}$ by noting that $(\lambda\mathbb{I} - \mathbb{S})^{-1}\mathbb{F} = \mathbf{J_1} \otimes (\lambda\mathbf{I} - \mathbf{S})^{-1}\mathbf{F}$. Using the same consideration is in section xxxx, where we compute $\mathbb{P}_\mathbb{F}$ from $\mathbb{P}\star_\mathbb{F}$ the transition matrix of genealogical markov chain assocated with $(\lambda\mathbb{I} - \mathbb{S})^{-1}\mathbb{F}$, corresponds to the one associated with the relative generation process $(\lambda\mathbb{I} - \mathbb{S}\star)^{-1}\mathbb{F}\star$ bar for the bottom-left block which is zero and therefore it is $\mathbf{J_1}^T \otimes \left(\mathbf{P_F}(\mathbf{I} - \mathbf{P_S})^{-1} \right)$.

Now since $\mathbb{G} = \sum_{k \geq 0} \left(\mathbb{P}_\mathbb{F}(\mathbb{I} - \mathbb{P}_\mathbb{S})^{-1} \right)^k$ and we know that $\mathbb{P}_\mathbb{F}(\mathbb{I} - \mathbb{P}_\mathbb{S})^{-1}$ is, like $\mathbb{P}_\mathbb{F}$, a block over-diagonal matrix, with all blocks on the over diagonal equal to $\mathbf{P_F}(\mathbf{I} - \mathbf{P_S})^{-1}$, we can compute any block of $\mathbb{G}$: $\mathbb{G}(g_i, g_j) = \left(\mathbf{P_F}(\mathbf{I} - \mathbf{P_S})^{-1} \right)^{g_j - g_i}$ for $g_j \geq g_i$ and 0 for $g_j < g_i$. We are interested in the first s elements of the jth column of $\mathbb{G}$, that is to the blocks on the top row: $\mathbb{G}(1, g_j) = \left(\mathbf{P_F}(\mathbf{I} - \mathbf{P_S})^{-1}\right)^{g_j - 1}$ and therefore

$$\sum_{c_a, t} P_{c_j, g_j}(c_a, t) = \sum_{c_i=1}^{s} \left(\left(\mathbf{P_F}(\mathbf{I} - \mathbf{P_S})^{-1}\right)^{g_j-1}\right)_{c_i, c_j} = 1, \tag{84}$$

since for any column stochastic such as $\mathbf{P_F}(\mathbf{I} - \mathbf{P_S})^{-1}$, its powers $\left(\mathbf{P_F}(\mathbf{I} - \mathbf{P_S})^{-1}\right)^k$ are also colun stochastic, and therefore their columns sum to 1.

7.12 SM: Programming softwares

While the computation of expectation and variance of the number of kin, can be easily done with any classical object-oriented program, it is made faster when the program focuses on matrices, such as Matlab or Octave. To the contrary, the generation of *pgf*s, and the extraction of information from them, such as probability distributions, is easier and faster performed by functional programs (such as Lisp) even though most programs can handle functions, especially when the support of $\tilde{\mathbf{F}}$ is finite (that is, when there is n_m, $\forall c_i, c_j$ $\sum_{n_m} P(F_{c_j, c_i} = n) = 1$) which implies that the support of $\tilde{\mathbf{K}}$ is also finite. If the support of $\tilde{\mathbf{F}}$ is not (for instance if it is a Poisson), one may need the help of a formal computation program (such as Mathematica) to get exact results or one can opt to truncate the distribution to the right for approximations.

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