Perceptual axioms are irreconcilable with Euclidean geometry

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Abstract

In systems of logic (as in mathematics) axioms are statements whose truths are self-evident but cannot be proven; they are the foundation from which further propositional truths are derived. Previous attempts, led by David Hilbert, to show that all of mathematics can be built into an axiomatic system that is complete and consistent failed when Kurt Gödel proved that there will always be statements which are known to be true but can never be proven within the logical system. But Gödel and his followers took no account of brain mechanisms that generate and mediate logic. Here we show that in the case of so-called "optical illusions" there exists a significant and irreconcilable difference between their visual perception and their description according to Euclidean geometry. In psychophysical experiments, when participants are asked to adjust, from an initial randomised state, the perceptual geometric axioms to conform to the Euclidean description, the two never match. These results provide evidence that perceptual axioms, or statements known to be true perceptually, cannot be described mathematically. Thus the logic of the visual perceptual system is irreconcilable with the cognitive (mathematical) system and cannot be updated even when knowledge of the difference between the two is available. Hence no one brain reality is more "objective" than any other.

Keywords: perception, illusions, axioms, geometry, cognition, priors

INTRODUCTION

A set of axioms from which geometric truths arise are those of Euclidean geometry, as laid out in Euclid's Elements (Casey, 1885). That constituted the first systematic approach to geometry and was thought to be the only system leading to accurate and correct geometrical knowledge of physical space until the 19th century, when new geometries were developed (Gray & Ferreirós, 2022).

Statements of mathematical truths are often considered superior to knowledge acquired by different means, such as perceptual processes; the latter, apparently departing from rigid mathematical precision, are therefore commonly regarded as "illusions". It was posited recently (Filippov & Zeki, 2022) that there exist alternative truthful axiomatic statements about the organization of our world; these are determined by the organization of the visual brain and its logic, which differs from that dictating mathematical axioms; it is irreconcilable with the mathematical system though

no less real and compelling. We refer to this system as the perceptual axiomatic system (Filippov & Zeki, 2022). Comparison of perceptual axioms with geometric axioms leads us to conclude that axiomatic truths in systems of logic are dictated largely, if not exclusively, by the brain's different logical systems; each of these adheres to its own governing logic, which may differ significantly from alternative logical systems, thus leading to contradictory truths. Neither system is more objective than the other, because they are both objective, though in different and irreconcilable ways. By axiomatising the statements of perceptual truths, then, we are setting a precedent by which axioms outside the (cognitive) logical deductive system of the brain might challenge the idea that conventional (mathematical and geometric) domains of knowledge are complete.

In this work, we consider certain unique configurations of two-dimensional geometries, so-called "optical illusions", where a mathematical description of the geometric components is irreconcilable with the perceptual appearance of those same components. When viewing these geometries, the perceptual truth differs from the mathematical truth in a way that is persistent and self-evident. Even when one is informed of the mathematical truth, perception is highly resistant to change.

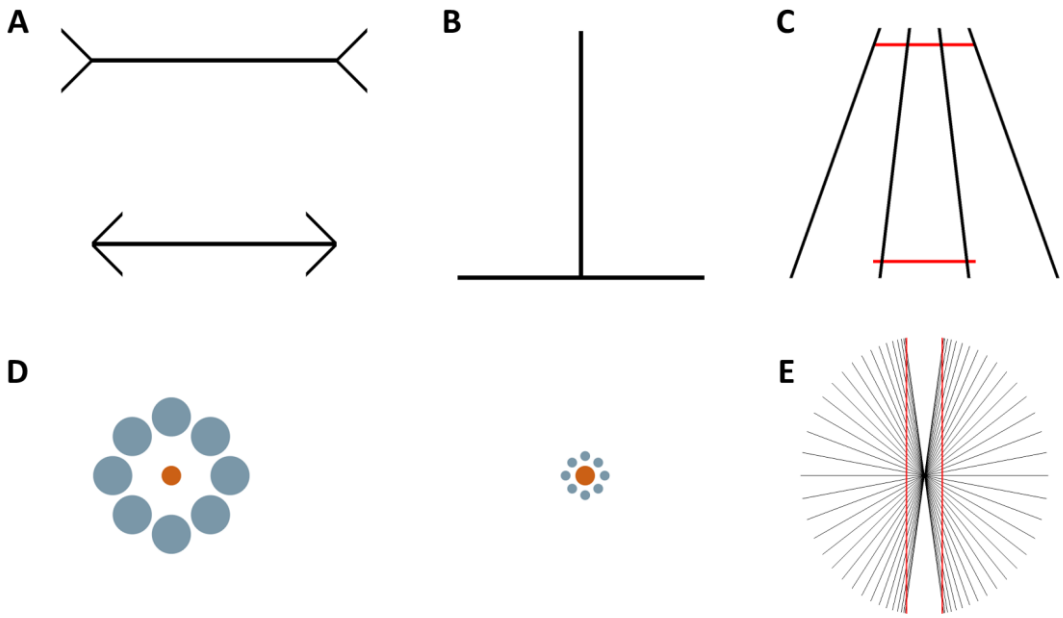

Figure 1. Illusions used in this experiment.
A. The Müller-Lyer illusion. B**.** The vertical-horizontal illusion. C**.** The Ponzo illusion. **D.** The Ebbinghaus illusion. **E.** The Hering illusion.

METHODS

Participants

We recruited 28 participants (18 female, 10 male, ages 25.7 ± 5.6) through an online recruitment agency (https://www.callforparticipants.com/) and through internal faculty bulletins at University College London (UCL). Our pool included participants from nine different ethnic backgrounds and all had normal or corrected-to-normal vision; all provided informed consent and the experiment was approved by the UCL Research Ethics Committee.

Design of stimuli

The illusions used were programmed in Matlab (The Mathworks) using PsychoPhysics Toolbox Version 3 (Brainard, 1997) such that one or more components of each illusion could be adjusted by a fixed increment (or decrement). Each illusion had three starting positions: mathematically truthful (equality), and two increments smaller or two increments larger than the mathematical equality point.

When the starting positions were mathematically equal, line lengths in the Müller-Lyer and vertical-horizontal illusions covered 15.9 visual degrees, and the Ponzo illusion 9.2 degrees. The increment was set at 0.2 visual degrees on each side of the bottom line for the Müller-Lyer , the Ponzo, and the vertical-horizontal illusions (Figure 1a-c). For the Ebbinghaus illusion, the diameter of the central circles was one visual degree when mathematically equal. The increment of the left inner circle corresponded to a change in diameter of 0.2 visual degrees.

Participants were able to alter the Hering illusion by increasing the curvature of the two lines inwards (toward the central point). This was achieved by modelling the lines with a quadratic formula, $x = ay^2$, and where changing the value of a led to a change in the line's curvature: $a = 0$ corresponded to a straight line, and $a = a_{max}$ corresponded to the configuration of maximum allowed curvature. The increment corresponded to $a = 1$; the maximum increment for this illusion was set to $a = 30$, which corresponded to a range of 1.3 visual degrees from the starting point (straight line) to the peak of the curve.

All stimuli were displayed on a monitor with a resolution of 1920×1080 (W×H) pixels against a plain white background and subtended a maximum of 16° at the eye, from a viewing distance of 60 cm. Pilot data confirmed that the adjustment effected, as a fraction of the absolute size of the initial configuration, was scale-independent (see supplementary material).

Procedure

The experiment began with a set of preliminary questions where participants indicated whether they perceived the effect of each separate illusion; this amounted to asking whether the two respective lines or circles appeared equal and/or parallel when the shapes are mathematically parallel or equal in length. Participants were told to base their judgements solely on visual perception, rather than prior knowledge or what they thought was veridical. Those illusions for which the participant reported perceiving a geometric asymmetry were then displayed in the task. If a participant indicated not perceiving the illusion, the illusion was not displayed further.

Each illusion was presented randomly five times for 10 s at one of the three initial positions (mathematical truth, two increments smaller than the mathematical truth, or two increments larger), amounting to fifteen presentations in total. During the presentation time, participants had the opportunity to alter the illusion; they were asked to keep adjusting the size of the bottom part of the Müller-Lyer illusion, the Ponzo illusion, and the vertical-horizontal illusion until they appeared equal in length. For the Ebbinghaus illusion, participants were asked to match the size of the two inner disks by adjusting the size of the left inner disk. Adjusting the Hering illusion entailed altering the curvature of the two vertical lines from a state where they appeared to be convex to a state where they appeared to be parallel. Left hand key presses on the keyboard made the lines more concave while while right hand keypresses made them more convex. If participants were satisfied with the adjustment before 10 s elapsed, they moved on to the next trial by pressing a key. A fixation cross was shown for half a second in the interval between rounds, and participants were given a break halfway through the set of trials.

Statistics

To model the distribution of these adjustments over the whole population, we first removed points that were more than three scaled median absolute deviations (MAD) from the median as these were considered outliers. Single instances of negative adjustments were also removed as these were attributed to accidental entries. We then used maximum likelihood estimation to fit both a normal distribution and a log-normal distribution to the adjustments made to each illusion for the normal, or all adjustments greater than zero for the log-normal distribution.

Table 1. Sample size and coefficient of determination values for each illusion.

Illusion	Sample size (n)	Normal coefficient of determination (R^2)	Log-normal coefficient of determination (R^2)
Müller-Lyer	314	0.52	0.73
Vertical-horizontal	375	0.96	0.68
Ponzo	371	0.88	0.73
Ebbinghaus	399	0.80	0.94
Hering	239	0.90	0.54

To test goodness of fit, 50 points were interpolated over the histograms using bin heights as anchor points. These points were then compared with values of best-fit normal and log-normal distributions at the location of those points. For each fitted distribution the coefficient of determination, R^2 is shown in Table 1. Log-normal distributions were selected over normal distributions on the occasions when they produced a better fit according to this metric. Residuals were inspected for non-random features.

It was noted that the shape of the distribution of responses, i.e. the presentation of the histogram, depended to a large degree on the choice of bin size for some illusions. This effect was particularly pronounced for the Müller-Lyer and the Hering illusion, and less so for the vertical-horizontal, Ponzo, and Ebbinghaus illusions.

RESULTS

The optical "illusions" (perceptual axioms)

We selected five geometric optical illusions for their simplicity and ease of (incremental) adjustment. These were the Müller-Lyer, the vertical-horizontal, the Ponzo, the Hering, and the Ebbinghaus illusions (Figure 1). In the Müller-Lyer and Ponzo illusions the lower of two horizontal lines appears shorter than the higher one, whereas in the vertical-horizontal illusion the vertical line appears longer than the horizontal one, when the pairs of lines are mathematically equal. In the Hering illusion the two vertical lines appear convexly curved when they are mathematically parallel. In the Ebbinghaus illusion the central disk surrounded by smaller disks appears larger than the one surrounded by larger disks, when they are mathematically identical.

Participants' adjustments

Participants viewed the above five geometric optical "illusions" and were asked to adjust their components from an initial configuration to a final one in which the features appeared equal (see Methods). We began each participant's testing session by showing them the various stimuli and asking whether they perceived the illusions. Table 2 shows that most participants did, but not all illusions were equally effective, although the magnitude of the modification required to bring the perceptual reality to match the mathematical description (e.g., to make the two lines of the Ponzo illusion appear equal in length) was significant for all illusions.

Although the magnitude of change varied between participants for any given illusion, this change was relatively stable within-participants (Table 2, Figure 2). This indicates that, although no general fixed modification can bring the perceptual and mathematical equalities to match for all participants, the difference between the two is more stable within each individual than across the whole population.

Table 2. Susceptibility to visual illusions and magnitude of the illusory effect.

Illusion	Proportion of participants who perceived the illusion	Description of modification required to match the mathematical and perceptual equalities	Magnitude of modification	Average individual standard deviation
Müller-Lyer	75%	increase length of lower line	16.1%	4.8%
Vertical-horizontal	93%	increase length of lower line	16.6%	6.1%
Ponzo	93%	increase length of lower line	48.6%	7.6%
Ebbinghaus	96%	increase diameter of left disk	37.4%	10.6%
Hering	57%	increase curvature of vertical lines	28.4%	11.3%

Most participants (N = 28) were susceptible to the illusory effect of the chosen stimuli, but the stimuli were not all equally effective and required different amounts of modification to transition from mathematical equality to perceptual equality.

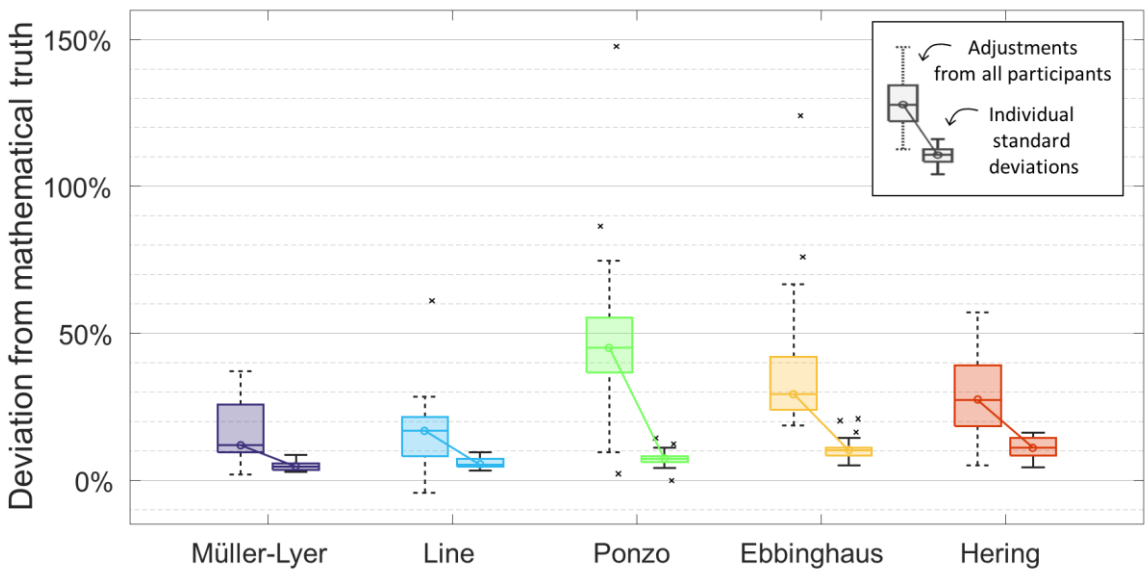

Figure 2. Percentage change from mathematical equality per illusion and standard deviation within participant.
For each illusion, the boxplot on the left shows the distribution of the participants' mean modification in percentages of the original reference size. The boxplot of the same colour on the right shows the distribution of participants' standard deviations. The left boxplot therefore represents the variability of mean adjustments at the population level, while the right boxplot represents their variability at the individual level. Boxes cover points between the 25th and 75th percentile and central lines and circles indicate the median point of each dataset. The whiskers extend to 1.5x the interquartile range (IQR) from the top/bottom of each box.

Modelling

We denote ψ_i as the percept experienced by a random participant on a random trial from the "illusions", where $i = 1,2,3 \ldots$ is an index that indicates each "illusion" in the order listed above (Figures 2 and 3). We fit probability density functions to the distribution of all responses across all participants for each illusion, and evaluated their goodness of fit by calculating the coefficient of determination, R^2 (see Methods). We chose a normal or a log-normal probability density function to model participants' responses, depending on which had a higher R^2 value. We found that the vertical-horizontal, Ponzo, and Hering illusions adjustments are best modelled by normal distributions:

$$\psi_i \sim N(\mu_i, \sigma_i^2) \qquad (Eq. 1)$$

and the Müller-Lyer and Ebbinghaus illusions adjustments by log-normal distributions:

$$\log(\psi_i) \sim N(\mu_i, \sigma_i^2) \qquad (Eq. 2)$$

In equations 1 and 2, $N(\mu, \sigma^2)$ is a normal distribution with mean μ and standard deviation σ, where numerical values for the pairs μ_i, σ_i are listed below:

$$\mu_i, \sigma_i = \begin{cases} 2.46, & 0.78 & 1 \\ 13.20, & 9.09 & 2 \\ 40.30, & 18.05 & i = 3 \\ 3.46, & 0.50 & 4 \\ 28.11, & 17.28 & 5 \end{cases}$$

We defined f_i as the corresponding probability density functions for either the normal or log-normal distributions, which allowed us to arrive at a quantified estimate for the probability of a participant perceiving that illusion within a certain range. The probability that the perceptual axiom lies between a and b percentage deviations from the mathematical (Euclidean) truth is given by:

$$P(b < \psi_i < a) = \int_b^a f_i(x)\mathrm{d}x \qquad (Eq. 3)$$

Note that for the log-normal probability density functions, μ_i and σ_i parametrise the normal distribution of $log(\psi_i)$, not the distribution of ψ_i itself.

We also constructed an equation for the average perceptual axiom, $\overline{\psi}_i$, across all potential participants, including those who do not perceive an illusory effect; this combines information on the distribution of adjustments made and the susceptibility of participants to each illusion (for the vertical-horizontal, and Ebbinghaus illusions),

$$\overline{\psi}_i = m_i + s_i\mu_i \qquad (Eq. 4)$$

where $\overline{\psi}_i$ is the average perceptual axiom over all potential participants for illusion i, m_i is the mathematical truth, μ_i is the average adjustment taken from f_i, and s_i is the percentage of participants who reported seeing the illusion, as given below.

$$s_i = \begin{cases} 75.1 & 1 \\ 92.9 & 2 \\ 92.9 & i = 3 \\ 96.4 & 4 \\ 57.1 & 5 \end{cases}$$

This allows us to formulate perceptual axioms, which most consider to be subjective, in mathematical terms; these perceptual axioms differ from geometric axioms but are no less truthful.

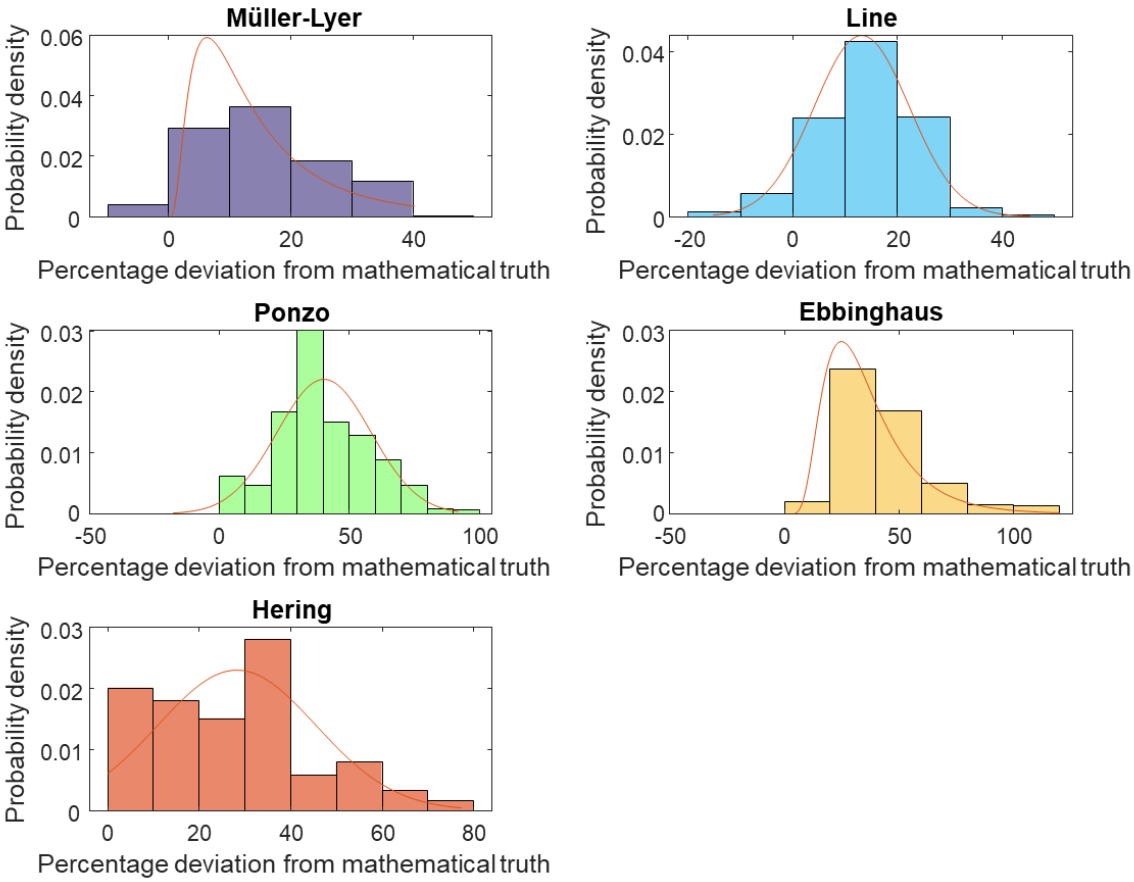

Figure 3. Distribution of adjustments across the population and the probability density functions fitted to them.
A normal distribution was fitted to the adjustments of the vertical-horizontal, Ponzo, and Hering illusions, and a log-normal distribution was fitted to the adjustments of the Müller-Lyer and Ebbinghaus illusions after statistical testing. Here bin width is set at 10% for the Müller-Lyer, vertical-horizontal, Ponzo, and Ebbinghaus illusions, and at 20% for the Ebbinghaus illusion. This is because the adjustment increment (in percentage size) is larger for the Ebbinghaus illusion than the others.

DISCUSSION

In this study we sought to demonstrate the irreconcilable nature of mathematical truths and perceptual ones. We presented participants with a variety of "illusions", which can be precisely defined mathematically, and asked them to adjust them until the relevant properties of each appeared equal in size, which entailed a deviation from their mathematically defined properties. The majority of participants perceived these "illusions" and that their perception deviated from mathematical equality. This indicates that mathematics does not accurately describe perceptual reality.

The magnitude by which individuals modify these illusions to reach perceptual equality is variable, but the amount of change was fairly consistent for each participant,

indicating that perceptual truths are stable at the individual level. We refer to these perceptual truths as *perceptual axioms* (Filippov & Zeki, 2022). It is presumed that these perceptual axioms are strong in that they rely on priors which are resistant to updating, even when cognitive knowledge of the mathematical reality is acquired.

The Ebbinghaus illusion

We will use the Ebbinghaus illusion as an example with which to draw on the axioms of Euclidean geometry and present their irreconcilability with the perceptual axiom. Euclid's eighth axiom states that "magnitudes that can be made to coincide are equal" (Casey, 1885). Digitally and mentally the two inner disks in the Ebbinghaus illusion might be exactly superposed and be defined as equal according to this axiom. However, 96% of participants perceived inequality in this same state, and made an average adjustment of 37.4% of the initial disk diameter to reach a state in which the two disks are *perceived* as equal in size. These participants' perceptual axioms, determined by their unique perceptual systems, were incompatible with the Euclidean axiom. The same is true for the other "illusions".

Incompleteness in mathematics

An early demonstration of incompleteness in mathematics came with the discovery of the sphere. Curved surfaces proved irreconcilable with certain axioms and propositions of Euclidean geometry, necessitating the formulation of a novel system under different axioms capable of dealing with curved surfaces, namely that of Riemannian geometry (Lee, 1997).

Later, David Hilbert and the formalist school of mathematics sought for a mathematical axiomatic system that was complete and consistent (Hilbert, 1930), one from which any statement formulated in the language of that system might be proven true or disproven. They also searched for an underlying set of axioms which were themselves proved to be consistent. In the first of his so-called Incompleteness Theorems, Kurt Gödel (1931) showed that within any system of first-order arithmetic logic there exist statements which are known to be syntactically true but can never be proven as such within that system. He also showed, in his second Incompleteness Theorem, that the consistency of the axioms of a logical system can never be proven within that system (Raatikainen, 2022). It is obvious that Gödel did not consider the possibility, and nor have others, of alternative brain logical systems. Following on from Gödel, our results further expose incompleteness in reliance on any one logical system to obtain knowledge. The perceptual logical system has often been labelled "illusory" because it departs from what has been considered to be the "objective" and therefore truthful system. We are

not unaware of the further implications of this for knowledge derived from other brain logical systems.

Beyond the "ocular deception"

By labelling the stimuli used in this experiment as "illusions" psychologists and psychophysicists took a firm position – that these displays departed from the "objective" reality. Yet, the only reality that we are capable of experiencing is that which is generated by, and compliant to, brain mechanisms. The cognitively (mathematically) "objective" axioms of Euclidean geometry hold true within their system, but every effort made on a perceptual basis to make, for example, the two central disks in the Ebbinghaus illusion appear perceptually equal will entail a departure from the objective mathematical condition. The same is true of other "illusions."

Various pieces from the literature have utilised ideas from Bayesian probability as a conceptual basis for perception (Brown & Friston, 2012; Geisler & Kersten, 2002; Weiss et al., 2002; Yang et al., 2021; Zeki & Chén, 2020). Given that multiple environmental configurations can map onto the same sense data, this framework states that the brain must make inferences about the environment; these inferences are influenced by priors, which can be innate or learned. We state that the brain's perceptual reality, which is exposed by optical "illusions", relies on strong priors and is resistant to updating, even when cognitive knowledge of the mathematical truth is acquired.

The irreconcilable nature of geometric mathematical and perceptual axioms was implicitly recognised during the architectural and perceptual experiments of the great Greek architects (Ictinus and Callicrates) who designed the Parthenon. They deliberately modified the geometry of the massive Parthenon columns and spaced them more widely at the ends to counteract the visual effect of perceiving the columns as appearing unequal in size and in spacing when viewed from a distance (Notomi, 1999). The geometric modification was undertaken "To counteract the ocular deception by an adjustment of proportions" (Vitruvius, 20 B.C.E.). Here, Vitruvius, while implicitly acknowledging that perception trumps geometry, refers to the perceptual reality as a "deception". In our experiments we proceeded in exactly the opposite direction – we counteracted the "geometric deception" by adjusting it to fit the "ocular reality" or, in our terms, to fit the perceptual axioms. By adjusting the presiding intuition around the truth of perceptual axioms such that it is recognised that perceptual truths are entirely compliant to brain mechanisms that generate logic and experience, we move beyond the stigma associated with so-called "optical illusions." Even those mathematical truths considered real in a Platonic sense – that they are abstract forms that exist independently of human action – are still generated by brain mechanisms.

CONCLUSION

The results presented here show that in the case of certain geometric pictures there exists a significant and irreconcilable difference between the perception of equality and its mathematical description. These pictures are commonly referred to as "optical illusions," and are examples of cases in which perceptual truths are self-evident but cannot be proven by mathematics. Here we argue that these perceptual truths qualify as axioms and that the term "optical illusion" is a misnomer because it emphasises primarily the cognitive mathematical description of these pictures when in fact perception cannot be updated once this knowledge is acquired. Both perceptual and mathematical knowledge follow their own cogent logic, and in certain cases these systems are irreconcilable and reliant upon differing brain mechanisms. If mathematics aspires to a complete description of reality such perceptual axioms must be imported into its canon.

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AUTHOR CONTRIBUTIONS

S.Z. designed and oversaw the experiment; S.E.R. and A.B. programmed the stimuli used; Z.F.H. modified the code and ran the experiment; S.E.R., A.B., and Z.F.H. performed statistical analyses; Z.F.H. and S.E.R. wrote the manuscript; all authors contributed to manuscript revisions.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon request.

CODE AVAILABILITY

The code used in this study is available from the corresponding author upon request.

COMPETING INTERESTS

The authors declare no competing interests.

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