LOCALIZATION OF SPATIALLY EXTENDED BRAIN SOURCES BY FLEXIBLE ALTERNATING PROJECTION (FLEX-AP)

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ABSTRACT

Magnetoencephalography (MEG) and electroencephalography (EEG) are widely employed techniques for the *in-vivo* measurement of neural activity with exceptional temporal resolution. Modeling the neural sources underlying these signals is of high interest for both neuroscience research and pathology. The method of Alternating Projections (AP) was recently shown to outperform the well-established recursively applied and projected multiple signal classification (RAP-MUSIC) algorithm. In this work, we further enhanced AP to allow for source extent estimation, a novel approach termed flexible extent AP (FLEX-AP). We found that FLEX-AP achieves significantly lower errors for spatially coherent sources compared to AP, RAP-MUSIC, and the corresponding extension, FLEX-RAP-MUSIC. We also found an advantage for discrete dipoles under forward modeling errors encountered in real-world scenarios. Together, our results indicate that the FLEX-AP method can unify dipole fitting and distributed source imaging into a single algorithm with promising accuracy.

***Index Terms*—** MEG, EEG, source localization, least-squares, alternating projection, RAP-MUSIC, FLEX

1. INTRODUCTION

MEG and EEG (M/EEG) are crucial techniques for recording neural activity with high temporal precision. Despite their importance, a significant challenge in M/EEG is their low spatial resolution, complicating neural source localization due to an ill-posed inverse problem [1, 2].

Traditional localization approaches often rely on the least squares criterion over a source space, such as the minimum norm estimate (MNE) [3]. While widely used, MNE-based methods often struggle with identifying sparse neural sources. In contrast, dipole scanning techniques follow an alternate inversion strategy. Rather than estimating a distributed current density across the entire expansive source space, these methods focus on identifying a confined set of discrete sources. Typically, a localizer function is derived, and dipoles are shortlisted as candidates based on specific threshold criteria. A prominent technique in the M/EEG field is the recursively applied and projected Multiple Signal Classification (RAP-MUSIC, [4, 5]). In this iterative approach, individual dipoles are successively added to the set of active sources, and their signals are systematically removed from the measurements through an out-projection operation until a predefined criterion is met. A natural extension of this approach, the recent introduction of FLEX-RAP-MUSIC, allowed for the estimation of the spatial extent of neural sources [6]. The results showed that extent estimation is possible by smoothing the forward matrix with increasing order, thereby reducing localization errors significantly. A conceptually similar approach to RAP-MUSIC is Alternating Projections (AP) [7]. It differs from RAP-MUSIC in two key properties: (1) Unlike RAP-MUSIC, AP does not rely on the signal subspace but instead directly utilizes the data covariance. This property enhances the robustness of AP in localizing highly correlated sources, and (2) following the initial search, AP further refines the position of the discovered dipole through additional iterations. Despite being a widely adopted approach in sensor array signal processing, its advantage over many iterative scanning techniques was shown only recently [8]. Notably, AP exhibited significantly lower localization error compared to RAP-MUSIC in scenarios featuring highly correlated sources and low signal-to-noise ratios (SNR). This makes AP a viable choice in many real-world settings.

In this study, we combine the advantages of the AP approach and the FLEX framework to allow for the estimation of source extent (FLEX-AP). We compare FLEX-AP with other approaches and evaluate its efficacy under various conditions based on a wide variety of neural source simulations.

2. METHODS

2.1. Problem formulation

M/EEG recordings $\boldsymbol{y}(t) \in \mathbb{R}^q$, measured in q sensors at time t, can be modelled as the product of the forward matrix $\mathbf{G} \in \mathbb{R}^{q \times p}$ and the current vector $\boldsymbol{j}(t) \in \mathbb{R}^p$, where p is the number of candidate dipoles spanning the source space:

$$\boldsymbol{y}(t) = \mathbf{G}\boldsymbol{j}(t) + \boldsymbol{n}(t), \tag{1}$$

with $\boldsymbol{n}(t) \in \mathbb{R}^q$ an additive noise term. The forward matrix[1]

[1]The forward matrix is also referred to as *lead field* or *gain matrix*.

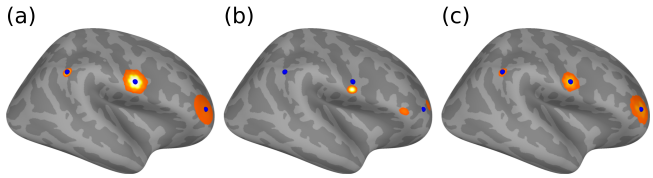

Fig. 1: **Illustration of Flexible Extent Estimation.** (a) Simulation of a discrete source (left) and two increasingly extended sources (middle & right). (b) AP inverse solution results in biased (middle source) or multiple sources assigned to a single large extent source (right source). (c) FLEX-AP inverse solution. Blue spheres in (b) and (c) mark the center of the ground truth locations depicted in (a).

describes the mapping of the sources of neural activity in the brain to the measurements recorded by the sensors, and takes into account the geometry of the head, the sensor positions, and the physical properties of the tissues between the sources and sensors. Note that, for simplicity, we focus here in the case of *fixed-orientated* dipoles, which are aligned with the aggregate direction of the dendrites of pyramidal cells, the primary generators of M/EEG signals [9, 10].

2.2. FLEX-AP

We describe here the calculation of the FLEX-AP inverse solution, which encompasses the standard AP approach as a special case with a maximum smoothness order of 0. In this work, we assumed that the number of active sources Q is known. To transition from a discrete dipoles to an extended source, we represent the mapping of a local patch of *coherent* brain activity to the sensors through an equivalent local smoothing operation on the forward matrix. Smoothness is achieved using a heat diffusion model, which simulates the concurrent activation of dipoles with diminishing amplitude as we move away from the center of the patch.

Mathematically, we first define the adjacency matrix $\mathbf{A} \in \mathbb{R}^{p\times p}$ of the cortical mesh (triangulated surface) that spans the source space. The degree matrix $\mathbf{D}$ is then defined as the diagonal matrix with elements $\mathbf{D}_{ii} = \sum_j \mathbf{A}_{ij}$. Using the Laplacian matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$ the diffusion operator $\mathbf{K}$ is computed as:

$$\mathbf{K} \quad = \quad \mathbf{A}(\mathbf{I} - \alpha\mathbf{L}) \tag{2}$$

where $0 < \alpha < 1$ is a free diffusion parameter, and $\mathbf{I} \in \mathbb{R}^{p\times p}$ is the identity matrix. With the diffusion operator $\mathbf{K}$, we can generate a series of increasingly smoothed forward matrices to simulate sources of gradually greater extent, up to a maximum smoothness order of k:

$$\mathbf{G}_n = \mathbf{G}_0\mathbf{K}^n \;\forall\; 0 < n \leq k, \tag{3}$$

where $\mathbf{G}_0$ is the initial ("unsmoothed") forward matrix and n reflects the order of the source extent, indicating the inclusion of neighbours up to the n-th order from the center of the patch.

Given this set of forward matrices, we then proceed with an exhaustive grid search over all candidate dipole locations p and source extents k to maximize the AP localizer function (equation 21 in [8]), yielding the initial (0-th) estimate of the first source:

$$\hat{p}_1^0, \hat{n}_1^0 = \arg\max_{p_1,n_1} \frac{g^T_{(p_1,n_1)}\mathbf{C}g_{(p_1,n_1)}}{g^T_{(p_1,n_1)}g_{(p_1,n_1)}}, \tag{4}$$

where $\mathbf{C} = \mathbf{Y}\mathbf{Y}^T \in \mathbb{R}^{q\times q}$ is the data covariance matrix and $g_{(p_1,n_1)}$ is the p_1-th column of the forward matrix with the extent n_1 ($\mathbf{G}_{n_1}$). After detecting the initial source location and extent, we recursively add one source at a time, solving for the initial (0-th) estimate of the q-th source as:

$$\mathbf{G}_q^0 = [g_{(\hat{p}_1^0,\hat{n}_1^0)}, ..., g_{(\hat{p}_{q-1}^0,\hat{n}_{q-1}^0)}] \tag{5a}$$

$$\mathbf{P}_q^0 = \mathbf{I} - \mathbf{\Pi}_{\mathbf{G}_q^0} \tag{5b}$$

$$\hat{p}_q^0, \hat{n}_q^0 = \arg\max_{p_q,n_q} \frac{g^T_{(p_q,n_q)}\mathbf{P}_q^0\mathbf{C}\mathbf{P}_q^0 g_{(p_q,n_q)}}{g^T_{(p_q,n_q)}\mathbf{P}_q^0 g_{(p_q,n_q)}}, \tag{5c}$$

where $\mathbf{\Pi}_{\mathbf{G}_q^{0\top}} = \mathbf{G}_q^0(\mathbf{G}_q^{0\top}\mathbf{G}_q^0)^{-1}\mathbf{G}_q^{0\top}$ is the projection operator within the column space of $\mathbf{G}$. Sources are detected recursively, $q = 2, ..., Q$, until all sources Q are found. $\mathbf{G}_q^0$ is the set of forward matrix columns corresponding to the previously selected source candidates. $\mathbf{P}$ denotes the projection matrix which projects out these components as part of the localizer function. This ensures that the next source candidate (1) aligns with the data covariance matrix, after (2) the previously detected sources are projected out.

Once all the Q sources are found, the AP algorithm begins the refinement process of the location and extent of each sources over several iterations j:

$$\mathbf{G}_q^j = [g_{(\hat{p}_1^j,\hat{n}_1^j)}, ..., g_{(\hat{p}_{q-1}^j,\hat{n}_{q-1}^j)}, g_{(\hat{p}_{q+1}^{j-1},\hat{n}_{q+1}^{j-1})}, ..., g_{(\hat{p}_Q^{j-1},\hat{n}_Q^{j-1})}] \tag{6a}$$

$$\mathbf{P}_q^j = \mathbf{I} - \mathbf{\Pi}_{\mathbf{G}_q^j} \tag{6b}$$

$$\hat{p}_q^j, \hat{n}_q^j = \arg\max_{p_q,n_q} \frac{g^T_{(p_q,n_q)}\mathbf{P}_q^j\mathbf{C}\mathbf{P}_q^j g_{(p_q,n_q)}}{g^T_{(p_q,n_q)}\mathbf{P}_q^j g_{(p_q,n_q)}}, \tag{6c}$$

where $q = 1, ..., Q$ refines all sources. $\mathbf{G}_q^j$ comprises the set of all but one forward matrix column, corresponding to the online estimate of the source candidates. The one not contained in the set is removed to find a potentially better candidate in the process. $\mathbf{P}_q^j$ therefore out-projects all but one forward matrix column, and the optimal new candidate is found through exhaustive search over all dipole locations and extents just as in Eq. (5). An illustration of the concept and impact of source extent estimation is shown in Fig. 1. The maximum number of iterations was set to 6 as was found to

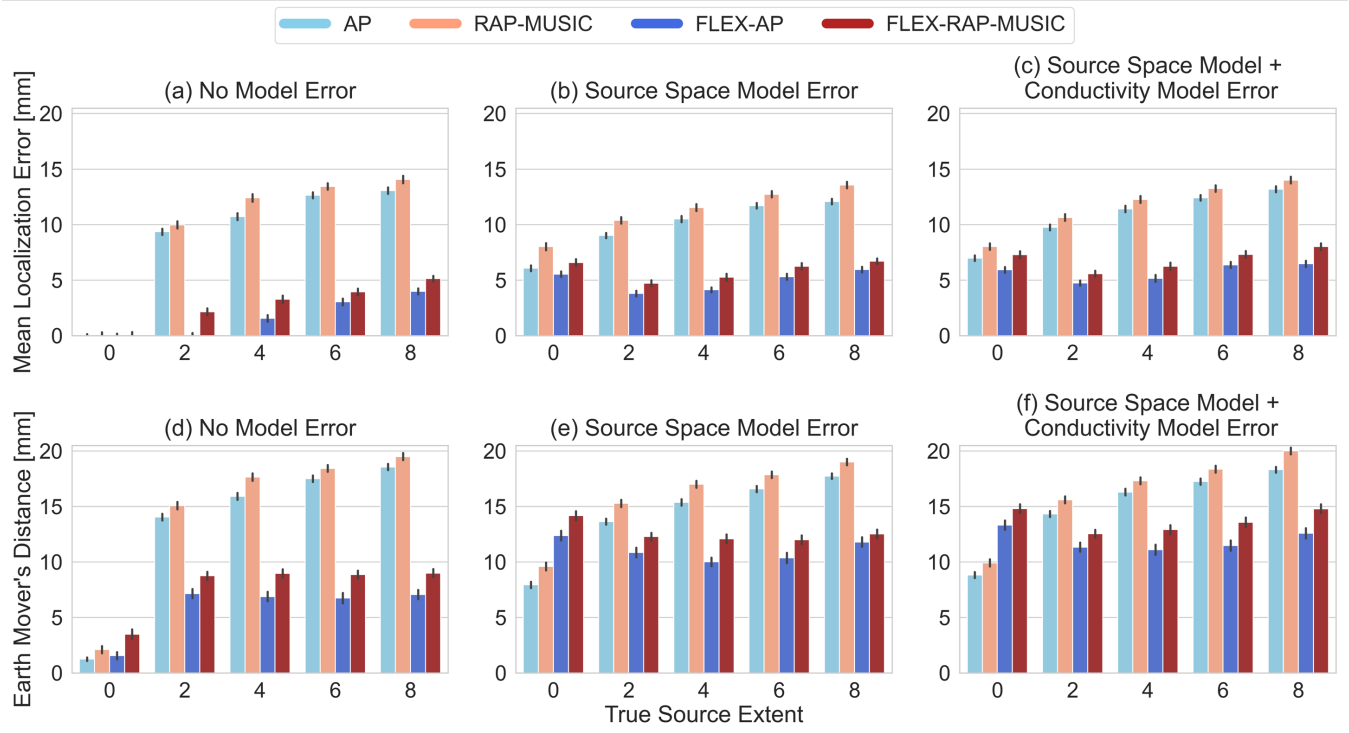

Fig. 2: **Localization Performance for Different Source Extents:** (a)-(c) Mean Localization Error; and (d)-(f) Earth Mover's Distance. The left column represents inversion with the same forward model as used for data generation. The center column shows inversion on a different source space grid and the right column shows inversion on a different grid and a mismatch of skull-to-brain conductivity ratio between simulation and inversion. Each bar shows the median error over 500 Monte Carlo simulations. The error bars indicate standard errors.

be optimal by [8], although convergence was achieved earlier in many cases.

The computational complexity of the FLEX-AP algorithm can be broken down to its four different stages, where (1) the covariance computation follows $\mathcal{O}(q^3)$, (2) the search for the initial source follows $\mathcal{O}(k \times p \times q^2)$, (3) the first round of iterations follows $\mathcal{O}(k \times Q \times q^2)$ and (4) the refinement stage follows $\mathcal{O}(k \times Q \times q^3)$. The dominant computational cost is tied to operations on the number of channels (q), but the most extensive interactions in terms of data size involve the number of dipoles ($p \gg q$). Given the provided complexities and constraints, the algorithm is designed to handle larger number of dipoles effectively while the maximum number of smoothness orders k, and the number of sources Q and the maximum number of refinements k have only a low impact on computational complexity. In comparing the FLEX-AP to the standard AP, both algorithms exhibit the same computational complexity for stage (1) and (4). The search for the initial set of sources Q (stage 3 & 4) scale linearly with the maximum smoothness order k.

2.3. Evaluation on Simulated EEG Data

For the evaluation we used a 128-channel EEG montage based on the international 1020 system. The *fsaverage* template T1 image provided by the freesurfer software suite [11, 12] was used to model the cortical source space. To prevent committing the "inverse crime," a diverse set of forward models was employed. In the base forward model we used $p = 5124$ dipoles sampled along the cortical surface with fixed orientations. For the first alternative forward model we have changed this source space into a finer mesh composed of $p = 8196$ dipoles. The second alternative forward model used the same source space with an additional change of the skull-to-brain conductivity ratio from $1:80$ to $1:50$.

Every condition included 500 Monte Carlo simulations. Each simulation was composed of two randomly located sources with either no extent (discrete source) or with an extent of a given neighborhood order (extended source). A discrete source was turned into an extended source of order n by applying the diffusion operator $\mathbf{K}^n$ to the source vector $\mathbf{J}$ as described in Eq. (2). The activity of all members of an extended source was coherent and the time course followed a random sequence with a frequency spectrum following $\frac{1}{f}$. After that, the time courses of both sources was mixed such

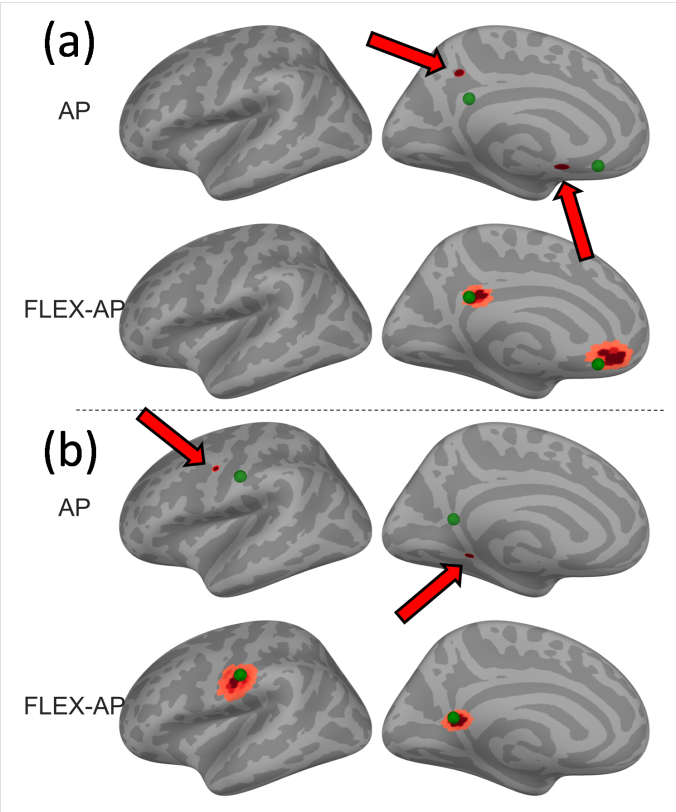

Fig. 3: **Inverse Solutions of Discrete Sources with Source Space Modelling Error.** Two samples of two simulated discrete sources are depicted (a & b). The ground truth locations are indicated by green spheres, whereas red indicates the inverse solution. Note that the AP solutions are displaced, while the sources found with FLEX-AP show the correct center of activity.

that they obeyed a medium inter-source correlation of $\rho = 0.5$ (cf. [13]). Independent and identically distributed (IID) noise was added after projecting the source vectors J through the forward matrix to yield an SNR of $0dB$.

We evaluated the FLEX-AP method by comparing it to (non-FLEX) AP, RAP-MUSIC and FLEX-RAP-MUSIC. Note, that the smoothing of forward matrices in the FLEX-RAP-MUSIC algorithm as described in [6] was corrected by the heat diffusion model. Therefore, both FLEX-AP and FLEX-RAP-MUSIC share the same set of smoothed forward models (Eq. (2)). For the evaluation we utilized two metrics to quantify the accuracy of the inverse solutions: (1) Mean Localization Errors (MLE) were calculated as proposed by [14]. For extended sources we identified the maxima of the source vector prior to the MLE calculation. (2) The Earth Mover's Distance (EMD) was calculated to also account for the error in extent estimation [15]. We used the python package *POT* by [16] to compute EMDs.

3. PERFORMANCE EVALUATION

FLEX-AP was developed to localize not only discrete but also extended sources. Figure 2 shows the MLE and EMD for all methods with sources of increasing source extent. A neighborhood order of 0 indicates discrete sources, while an order of 8 suggests a large extended source of significant size.

First, we like to highlight that our results replicate the seminal paper on using AP for the M/EEG inverse problem [8]. In our experiments, AP yielded lower localization errors compared to RAP-MUSIC in every comparison, both in terms of MLE and EMD.

In the absence of forward modelling error (Fig. 2 (a & d)), we find that AP had only a small advantage over the FLEX-AP algorithm when the true source extent was 0. With increased source extents we find that both FLEX-AP and FLEX-RAP-MUSIC achieve significantly lower errors compared to the non-FLEX variants, both in terms of MLE and EMD. This indicates that the spatial extent estimation was successful. Analogously to the comparison between AP and RAP-MUSIC, FLEX-AP achieves lower errors in compared to FLEX-RAP-MUSIC in every scenario shown.

Interestingly, we find that the FLEX methods produce lower MLE for discrete source in the presence of source modelling error (Fig. 2 (b & c)). Figure 3 illustrates this behavior with two examples from our simulations. The green spheres indicate ground truth locations of single two single dipoles per example (a & b) and red patches are the inverse solution produced by AP and FLEX-AP. Note how the extended sources found by FLEX-AP align closely with the ground truth foci, whereas the discrete sources found by AP are displaced.

4. CONCLUSIONS

We presented an extension of the AP algorithm to allow for flexible extent estimation of neural sources. We show that extent estimation reduces overall localization errors and that the AP framework is superior to the more widely adopted RAP-MUSIC algorithm. We further show that extent estimation can even reduce localization errors for discrete sources in the presence of source modelling errors, a scenario that is to be expected in real data. Future research should aim to validate these findings on real data and could expand on finding spatially coherent sources with non-coherent time-series. Furthermore, the diffusion parameter α (cf. Eq. 2) makes strong assumptions about the distribution of currents within an extended source. Future work should investigate the influence of its misestimation and possibly include new sets of forward matrices smoothed with different degrees of diffusion.

5. DATA & CODE AVAILABILITY

All code used to produce the results of this work is open source and can be found at `https://github.com/`

`LukeTheHecker/ap_paper_analyses`. A python library that implements FLEX-AP alongside other inverse solutions can be found at:
`https://github.com/LukeTheHecker/invert`.

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