Probabilistic projections of distributions of kin over the life course

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Contents

1 Introduction **4**

1.1 Relevant mathematical frameworks 6

1.2 Algebraic notation 9

2 Model ingredients and assumptions **11**

2.1 Demographic dynamics 12

2.2 Defining kin of Focal: her q-th ancestor and their g-th descendant 13

2.3 The probable ages of reproduction 13

2.4 Projecting survivorship of a probability distribution: The matrix U 14

2.5 Projecting reproduction of a probability mass function: The matrix F . . . 16

2.6 Total kin versus age-distributed kin 18

3 Towards the kin-formulae **19**

3.1 Kin which descend through older sisters of Focal's $(q-1)$-th ancestor . . . 19

3.2 Kin which descend through younger sisters of Focal's $(q-1)$-th ancestor . . 20

3.3 Descendants of Focal 22

3.4 Ancestors of Focal 22

4 Application **23**

4.1 Unconditional distributions of kin 24

4.1.1 Sisters 24

4.1.2 Ancestors and descendants 27

4.2 Conditional distributions of kin 31

5 Discussion **38**

6 Acknowledgements **43**

References **44**

Appendix **49**

A Proof of Theorem 1 **49**

B Illustration of the formulae in relation to Caswell's approach. 50

B.1 Older sisters (m) or $(g = 1, q = 1)$-kin 50

B.2 Nieces from older sisters (p) or $(g = 2, q = 1)$-kin 51

B.3 Aunts older than mother (r) or $(g = 1, q = 2)$-kin 52

B.4 Cousins from aunts older than mother (t) or $(g = 2, q = 2)$-kin 52

B.5 Younger sisters (n) or $(q = 1, g = 1)$-kin 53

B.6 Nieces from younger sisters (q) or $(q = 1, g = 2)$-kin 53

B.7 Aunts younger than Focal's mother (s) or $(q = 2, g = 1)$-kin 54

B.8 Cousins from aunts younger than Focal's mother (v) or $(q = 2, g = 1)$-kin . 54

B.9 Daughters (a) or $(g = 1, q = 0)$-kin 54

B.10 Grand daughters (b) or $(g = 2, q = 0)$-kin 55

B.11 Mother (d) or $(g = 0, q = 1)$-kin 56

B.12 Grandmother (g) or $(g = 0, q = 2)$-kin 56

Abstract

BACKGROUND

Mathematical kinship demography is an expanding area of research. Recent papers have explored the expected number of kin a typical individual should experience. Despite the uncertainty of the future number and distributions of kin, just one paper investigates it.

OBJECTIVE

To develop a new method for obtaining the probability that a typical population member experiences one or more of some kin at any age through the life course.

METHODS

We use combinatorics and matrix algebra to construct and project a discrete probability distribution of kin. Our model requires as inputs, age-specific mortality and fertility.

CONCLUSIONS

We derive probabilities of kin-number for fixed age of kin and over all possible ages of kin. We derive expected numbers and variance of kin. We demonstrate how kinship structures are conditional on familial events.

CONTRIBUTION

The paper presents the first analytic approach allowing the projection of a full probability distribution of the number of kin of arbitrary type that a population member has over the life course.

1. Introduction

Kinship relationships are a fundamental part of the social structures of human societies (Alburez-Gutierrez et al., 2022). Kinship ties have impacts on physical and mental wellbeing (Vlachantoni et al., 2024), social and economic inequalities

between and within generations (Verdery and Margolis, 2017; Margolis et al., 2024), and social and fiscal policy (Pittavino, Arpino, and Pirani, 2025). However, direct data on kinship relationships are rarely collected, and generally only available for limited samples or in countries with high quality demographic registers (Kolk, 2017). Studies that quantify the development and evolution of kinship are therefore extremely valuable.

In particular, understanding inequalities resulting from kinship relationships requires quantification of the diversity of kinship structures within the population. In addition, the future numbers and structures of kin remain uncertain. Micro-simulation provides for the study of these kinds of structures, offering rich and detailed output about family composition both inside and outside the household (e.g. Wachter, Hammel, and Laslett, 1978; Hammel, 2005; Margolis and Verdery, 2019). However, micro-simulation is computationally expensive and technically challenging to set up, which can be a barrier to deployment (notwithstanding creditable recent work in opening up access to these tools, such as Thiele et al., 2023). Analytical models of kinship relationships are therefore highly desirable for their much greater computational efficiency.

Recent years have witnessed a resurgence in the mathematical modelling of kinship (Caswell, 2019, 2020, 2022, 2024; Caswell and Song, 2021; Coste et al., 2021), allowing richer insight into future kinship structure (Alburez-Gutierrez, Williams, and Caswell, 2023). However, all but one of the recent models developed since 2019 have been deterministic (see Caswell, 2024): they produce expected numbers of kin distributed by age or *stage* (or *state*) and sex for a representative population member. The stages can refer to countries or regions of residence to account for migration, although this dimension remains out of scope of the present study, which focuses on the natural population dynamics.

Population reproduction in the demographic context is driven by birth and

death processes, both of which are random. Each (female) individual in a population can give birth to a positive integer (or zero) number of offspring, with variation in reproductive profiles between them. Each individual in a population dies with an age-specific probability, again, with variation between them. Demographers usually refer to these probabilistic events as "demographic stochasticity" (Keyfitz and Caswell, 1997). As population dynamics are stochastic processes, so are kinship dynamics.

This research makes a contribution through complementing and extending recent advances mathematical kinship demography to account for the above-mentioned stochasticity. Our paper is structured as follows: Section 1.1 provides a brief review of the relevant literature and existing mathematical frameworks for analysing kinship, while clarifying what sets this work apart. Section 1.2 outlines notation used. In Section 2 we introduce our assumptions and pre-requisites for formally constructing the model: Section 2.1 outlines the general demographic assumptions and Sections 2.2-2.6 lay the mathematical foundations for probabilistically projecting the numbers of different kin. Section 3 provides our novel derivations, organised into analytically distinct regimes. In Section 4 we apply the framework, using UK age-specific mortality and fertility data sourced from the Human Mortality Database (2024), and Human Fertility Collection (2024). We illustrate the analysis by showing selected model output, and explore its implications. We conclude in Section 5 by considering the prospects and limitations of this theoretical development with respect to the leading frameworks on kinship, and by discussing our contribution to the formal demography of kinship.

1.1 Relevant mathematical frameworks

There is a rich history of mathematical modelling of kinship which have considered demographic stochasticity. Already in the 1980s, the research by Waugh (1981)

and Joffe and Waugh (1982) considered randomness in kinship networks through applying branching processes with non-overlapping generations, namely the so-called Galton-Watson process, primarily focusing on biology and population ecology. Advancing on such methods Jagers (1982) and Jagers and Nerman (1984) included age structures (and thereby allowing for overlapping generations) by utilising so-called "Crump-Mode-Jagers" (CMJ) branching process (Crump and Mode, 1968). Although very relevant to the field of kinship research, due to their complexity, these papers proved for the most part too technically complicated to allow their wider adoption, and as such, never became mainstream frameworks.

In a recent remarkable paper, Caswell (2024) acknowledged that as an individual's kin are essentially a population of their own, they will be subject to the same variability in the births and deaths as any other population. The author's proposed model accounts for demographic stochasticity through projecting the mean and variance of an unknown kin-distribution, using a multi-type Galton-Watson process (see also Pollard (1966)). From these quantities (the mean and variance) the author subsequently assumes appropriate statistical distributions to represent kin, and then fits prediction intervals thereby estimating the uncertainty associated with kin distributions.

While the proposed framework is more accessible to demographers, ecologists, mathematicians, and perhaps the occasional physicist, the technical modeling approach can be challenged on theoretical grounds. For instance, although the distribution of kin will always be discrete with a non-zero support on the non-negative integers, an appropriate choice might require specific knowledge. Caswell (2024) states that Poisson distributions should apply when the mean is close to the variance and cites previous work which adheres to this assumption (Caswell, Margolis, and Verdery, 2023). Moreover, previous research has shown that using a Galton-Watson process with age-structure coerced as "type" can

under-estimate variation in population sizes (Mode, Jacobson, and Pickens, 1987).

An alternative approach to Caswell (2024) would be to apply similar methods to Tuljapurkar et al. (2020) who directly calculate the so-called lifetime reproductive success (i.e., a distribution for the number of offspring an individual has over their life course). The benefit of this approach, emphasised by Tuljapurkar et al. (2020), is that, by constructing a probability distribution for the number of offspring one can account for variability in reproduction between individuals. Another obvious benefit is that higher-order moments (such as the skewness) become readily available.

In this research we develop a methodology for projecting the kin-number of a typical female population member in a one-sex time-invariant age-structured demography (see the illustration in Figure 1 for kin-types). As in the leading mathematical models of kin (Caswell, 2019, 2020; Caswell and Song, 2021; Caswell, 2022) we refer to this individual as 'Focal'. We want to know Focal's kin-network at each age of Focal's life. In the same vein as Caswell (2024), we seek to account for demographic stochasticity in the kinship network. However, instead of projecting the mean and variance in kin-number, we are motivated by the methods of Tuljapurkar et al. (2020), and as such seek to project age-specific probabilities that Focal experiences some integer number of kin. In particular, at each age in Focal's life, she will experience a non-negative integer number of some kin-type of a given age. Each instance of a given number of kin occurs with a given (possibly zero) probability. For example, regarding older sisters

(i.e., $\mathbf{m}$ in Figure 1), we seek a function:

$$\mathbf{m}_s(y) = \begin{pmatrix} m_s^{(y)}(0) \\ m_s^{(y)}(1) \\ m_s^{(y)}(2) \\ \vdots \\ m_s^{(y)}(Q) \end{pmatrix} \tag{1}$$

which associates with each possible number of older sisters of age s of Focal age y, a probability of occurrence. That is, we seek a probability mass function (pmf) describing the distribution of the possible number of sisters. The first entry of the pmf, $m_s^y(0)$, provides the probability that Focal does not have an older sister aged s when she is of age y – either because Focal's mother does not produce one or the sister does not survive to age s. The second entry of the pmf, $m_s^y(1)$, provides the probability that Focal has exactly one older sister. The third entry of the pmf, $m_s^y(2)$, provides the probability that Focal has exactly two older sisters, and so on. A biologically reasonable upper bound Q must be imposed to limit maximum lifetime kin-number, for which, in the above example of sisters, corresponds to the last ($(Q+1)$-th) entry $m_s^{(y)}(Q)$; the probability that Focal has Q older sisters of age s when she is aged y.

1.2 Algebraic notation

As the framework presented in this article includes mathematic derivations, some of which may be non-obvious, we start by defining the main algebraic object of interest and summarising the notation used throughout this paper. Thus, stochastic matrices are denoted using blackboard bold – A, other matrices are denoted boldface upper-case $\mathbf{A}$, vectors lower-case boldface $\mathbf{a}$, and (discrete) probability distribution functions are boldface Greek symbols, e.g., $\boldsymbol{\psi}$. We also

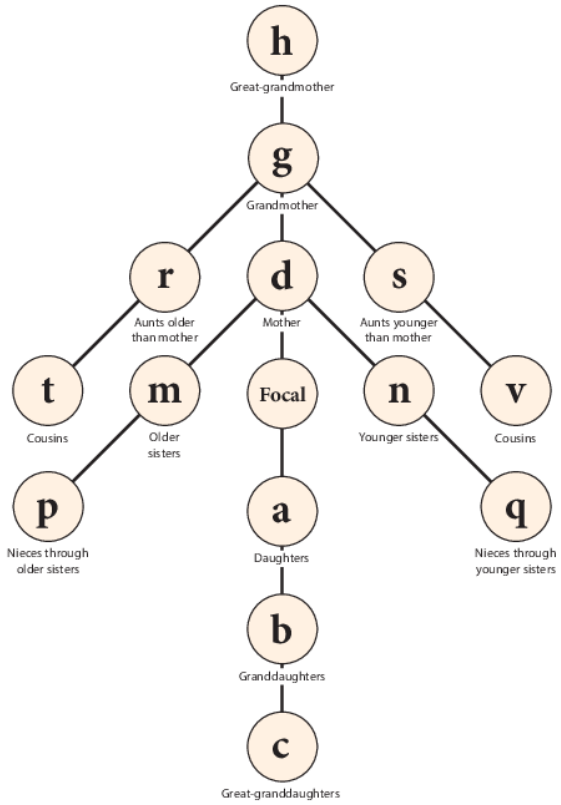

Figure 1: The diagram and notation of kin used in the formal models of Caswell (see e.g. Caswell, 2019, passim).

use the unit vector with i-th entry one by $\mathbf{e}_i$. When possible matrix entries will be given by lower-case letters, e.g., $a_{i,j}$ the i,j entry of $\mathbf{A}$, but when notation is a pain – for instance when the matrix is a function of parameters, $\mathbf{A}(x)$ – we may use $[\mathbf{A}(x)]_{i,j}$. The transpose of a matrix is denoted by $\dagger$, the Kronecker product of matrices by $\otimes$, the vec-operator vec(.), the Hadamard product $\circ$. The one-norm is given by $||.||_1$ and its extension to the (operator) column-max norm is $|||.|||_1$. An ordered (left) product of matrices $\mathbf{A}_i, i = 1, m$ is written:

$$\prod_{i=1}^{m} \mathbf{A}_i = \mathbf{A}_m \dots \mathbf{A}_1 \tag{2}$$

A discrete convolution of two functions f and g defined on the integers is given by $(f * g)[n] = \sum_{i=-\infty}^{\infty} f[i]g[n-i]$. For two distributions $\boldsymbol{\psi}_1 \in Z^n$ and $\boldsymbol{\psi}_2 \in Z^n$ we write the discrete convolution as $\boldsymbol{\psi}_1 * \boldsymbol{\psi}_2$ with m-th entry defined through $(\psi_1 * \psi_2)[m] = \sum_{i=0}^{m} \psi_1(i)\psi_2(m-i)$. Over independent distributions $\boldsymbol{\psi}_j, \dots, \boldsymbol{\psi}_n$ we write

$$|_{s=i}^{n} \boldsymbol{\psi}_s = \boldsymbol{\psi}_i * \boldsymbol{\psi}_{i+1} * \cdots * \boldsymbol{\psi}_{n-1} * \boldsymbol{\psi}_n \tag{3}$$

with the convention that $|_{i=1}^{0} = 1$, and to represent the $n-$th convolution power of a distribution, we write

$$\boldsymbol{\psi}^{*n} = \overbrace{\boldsymbol{\psi} * \cdots * \boldsymbol{\psi}}^{n}. \tag{4}$$

2. Model ingredients and assumptions

In this section we outline our assumptions and formally construct a model. We seek to propose a novel model for deriving probability distributions of Focal's kin, as shown in Figure 1. As described by Eq (1) in the Introduction, we construct $\mathbf{k}_s(y)$ to be a $(Q+1)$-dimensional kin-number pmf with $j+1$-th

entry giving the probability that Focal has j kin of age s when she is age y (for $j = 0, 1, 2, \ldots, Q$). For consistency henceforth, we refer to the random variable j as "kin-number-class".

2.1 Demographic dynamics

Consider a female age-structured population of ages $i = 1, 2, \ldots, n$. Suppose that at each age i, an individual gives birth to a random number of offspring $F_i \in \{0, 1, \ldots\}$ with probabilities $\{\psi_i(0), \psi_i(1), \ldots\}$, and encoded by the vector $\boldsymbol{\psi}_i = (\psi_i(0), \psi_i(1), \ldots)^\dagger$. That is, for $F_i = 2$ a woman of age i has 2 offspring with probability $\psi_i(2)$. Suppose that at each age i, an individual's survival to age $i+1$ is defined by a random variable, $U_i \in \{1, 0\}$ (where 1 represents survival and 0 death) with respective probabilities $\{u_i, 1 - u_i\}$.

Using the expected fertility rate $f_i = E[F_i] = \sum_j j\psi_i(j)$ and observing the expectation of a survival is given through a Bernoulli variable $u_i = E[U_i]$, we obtain the standard Leslie matrix (Allen, 2010):

$$\mathbf{A} = \begin{pmatrix} f_1 & f_2 & f_3 & \ldots & f_n \\ u_1 & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \ldots & \ldots & u_{n-1} & u_n \end{pmatrix}. \tag{5}$$

An age-structured population vector $\mathbf{x}$ is projected from one time to the next according to $\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t$. Given some initial age-structure $\mathbf{x}_0$ we see $\mathbf{x}_t = \mathbf{A}^t\mathbf{x}_0$. The large-time behaviour of $\mathbf{A}^T\mathbf{x}_0$ ($T >> 0$) leads to demographic stability; the relative sizes of each age-class become constant and with structure $\mathbf{w}$ where $\mathbf{A}\mathbf{w} = \lambda\mathbf{w}$ (with $||\mathbf{w}||_1 = 1$). Here, λ is the population growth rate and the stable reproductive values in the population are given by $\mathbf{v}$ where $\mathbf{A}^\dagger\mathbf{v} = \lambda\mathbf{v}$ (we appropriately normalise $||\mathbf{v}^\dagger\mathbf{w}||_1 = 1$).

2.2 Defining kin of Focal: her q-th ancestor and their g-th descendant

Following framework of Pullum (1982) we define Focal and kin through a common ancestor. Let q be the number of generations separating Focal and ancestor. Let g be the number of generations separating kin from the ancestor. We define b_i as the age at which Focal's i-th generation ancestor gives birth to Focal's $(i-1)$-th generation ancestor (so e.g., b_1 is the age at which mother had Focal; b_2 is the age at which Focal's grandmother had Focal's mother). By definition set $b_0 := 0 \equiv y$ to reflect Focal's current age, and define $\mathbf{b} = (b_1, \ldots, b_q)^\dagger$ with $\beta_q = y + \sum_{i=1}^{q} b_i$ representing the age of Focal's q-th ancestor when Focal is y. Note that β_q may be biologically unrealistic (i.e., $\beta_i > n$); we simply use this variable to ground time within a time-invariant framework.

We define s_i as the current age of Focal's kin, related to Focal as the $i-$th generation descendant from Focal's q-th ancestor. For instance if $q = 2$; then s_1 is the age of Focal's aunt when Focal is age y; and s_2 is age of Focal's cousin when Focal is age y. Notice that if $g = 0$ then s_0 is simply the age of a direct ancestor of Focal. Some of the s_i may be biologically unrealistic (i.e., $s_i > n$), but again these values are required to root time. What is biologically reasonable and relevant is the age at which the $(i-1)$-th generation descendant produces the i-th generation descendant; $\Delta_i = s_{i-1} - s_i$. For efficiency purposes in the formulae that follow we define $\underline{n}$ as the minimum fertile age and $\bar{n}$ the maximum.

We write the probability distribution for Focal's g, q kin which are of age s_g when Focal is age y by $\mathbf{k}_{s_g}^{g,q}(y)$. We write the probability distribution of kin over all possible ages when Focal is y by $\mathbf{k}_{\Sigma}^{g,q}(y)$.

2.3 The probable ages of reproduction

The first step in inferring the kinship network of Focal is to probabilistically predict the age of her q-th ancestor. Let ρ_x be the probability that if we

randomly sample a newborn from the population, her mother is of age x. It is well known that under demographic stability, the stable population age-structure is proportional to $\lambda^{-i}u_1 \dots u_{i-1}$ (e.g., Goldman (1978)). Hence, $\rho_x = u_1 \dots u_{x-1} f_x \lambda^{-x}/\mathcal{N}$ where $\mathcal{N}$ is a normalisation constant, which from the Euler-Lotka equation: $\mathcal{N} = \sum_{i=1}^{n} \left(\prod_{j=0}^{i-1} u_j\right) f_i \lambda^{-i} = 1$, we see $w_1 u_1 \dots u_{x-1} = \lambda^{x-1} w_x$ and $\rho_x = f_x w_x/(\lambda w_1)$. Note that, drawing from an alternative derivation (e.g., Demetrius (1975); Tuljapurkar (1993)), we see that ρ_x recovers the $(1, x)-$th entry of the transition matrix of the so-called genealogical Markov chain associated with the population.

2.4 Projecting survivorship of a probability distribution: The matrix U

Here we seek to project a pmf of kin, $\mathbf{k}_{s'}(.)$, of age s' (independent of Focal's age), conditional on survival, to a another pmf of the kin $\mathbf{k}_s(.)$ at age s (where $s' < s$). To do so, we must conserve probability in the sense that any death of kin simply flows from one entry in the pmf to another entry. Our approach is to construct a matrix $U(s', s) \in R^{(Q+1)\times(Q+1)}$, the role of which is to project kin of age s' within the $j = 0, \dots, Q$ kin-number-classes to age s.

We formally construct the matrix $U(s', s)$ through the following steps. First, note that each kin independently experiences the same probability $1 - u_{s'} \dots u_s$ of death between age s' and age s. Within each of the $j = 1, \dots, Q$ kin-number-classes, the probability that all j survive from s' to s is given by $u_{s'}^j \dots u_s^j$. These probabilities leave the pmf of kin unchanged (no kin are lost), and thus we set the diagonal:

$$\left[U(s', s)\right]_{i,i} = \begin{cases} 1, \quad i = 1 \\ u_{s'}^{i-1} \dots u_s^{i-1}, \quad 1 < i \leq Q \end{cases} \qquad \text{(diagonal)} \tag{6}$$

The $(1,1)$ entry acts on the $j=0$ kin-number-class ensuring the probability of no kin of age s' contributes a one-to-one correspondence to the probability of no kin by age s.

Off-diagonal entries of $U(s',s)$ encode all other ways in which other kin can die between age s' and age s. The probability that one (or more) kin die between s' and s contributes to the probability that Focal has one (or more) less kin of age s. In general, the probability that out of j kin, some l die between ages s' and s, is given by

$$\mathcal{D}(j,l,s',s) = \binom{j}{l}(1-u_{s'}\cdots u_s)^l(u_{s'}\cdots u_s)^{j-l}. \tag{7}$$

Applying Eq (7), we define the top-row of $U(s',s)$ through probabilities that deaths from the $j=1,2,\ldots,Q$ kin-number-classes result in Focal having zero kin of age s

$$\big[U(s',s)\big]_{1,i} = \mathcal{D}(i-1,i-1,s',s), \;\; i>1 \quad \text{(top row)}, \tag{8}$$

while the remaining (l,j) entries of $U(s',s)$ for $j>l>1$ are defined through

$$\big[U(s',s)\big]_{j-l+1,j} = \mathcal{D}(j-1,l-1,s',s) \quad \text{(upper triangle)}, \tag{9}$$

and give probabilities that out of j kin at age s' some $j-l$ die by age s. Note that lower triangle is zero since survival cannot act to increase kin number.

As an example, suppose that Focal's mother gives birth to (prospective) sisters of Focal when she is aged i. Then, $\mathcal{D}(3,2,1,10)\psi_i(3) = \binom{3}{2}(1-u_1\cdots u_{10})^2(u_1\cdots u_{10})^1\psi_i(3)$ gives the probability that Focal's mother had 3 offspring, and exactly 2 of them die before reaching age 10. This quantity contributes to the probability that Focal has a $j=1$ kin-number of sisters. More generally, conditional on sur-

vival, $U(0,10)\boldsymbol{\psi}_i$ projects the pmf of a mother's newborns up to age 10, while $U(s',s)\mathbf{k}_{s'}$ projects a kin pmf from age s' up to age s.

Note that the matrix is column-stochastic, and can be checked by observing that entries of the $j > 1$ columns obey the binomial expansion:

$$\begin{aligned} 1 &= \left[u_{s'}\dots u_s - (1-u_{s'}\dots u_s)\right]^j = \sum_{i=0}^{j} \binom{j}{l} (1-u_{s'}\dots u_s)^{j-l}(u_{s'}\dots u_s)^l \\ &= \underbrace{(1-u_{s'}\dots u_s)^j}_{\left[U(s's)\right]_{1,j}} + \underbrace{\binom{j}{1}(1-u_{s'}\dots u_s)^{j-1}u_{s'}\dots u_s}_{\left[U(s's)\right]_{2,j}} + \dots + \underbrace{u_{s'}^j\dots u_s^j}_{\left[U(s's)\right]_{j,j}} \end{aligned} \tag{10}$$

To summarise: the matrix U projects kin-number pmfs of age s' to age s. As the matrix is column-stochastic it conserves probability: the total probability of death within any kin-number-class j', is simply transferred to another kin-number-class $j'' \leq j'$. We adopt the convention that $U(s_1,s_2) = \mathbf{I},\ \forall s_2 < s_1$.

2.5 Projecting reproduction of a probability mass function: The matrix F

Here we seek to create a pmf describing the number of newborns through the age-specific reproduction of one of Focal's kin. While we are dealing with the reproduction of a direct ancestor of Focal, say of age b_i, because Focal has only one of these kin, we know that its reproduction is given through $\boldsymbol{\psi}_{b_i}$. However, if we are dealing with the reproduction of kin who are defined through a pmf (e.g., sisters), we have to create a matrix function $F(s)$ (a function of kin's age s) which projects the pmf of kin onto a pmf of their offspring.

Because reproduction is independent, the probability that j kin produce l

offspring is the $l-$th entry of the $j-$th power convolution of ψ_s:

$$\begin{aligned}\text{Prob}\{j \text{ kin of age } s \text{ produce } l \text{ offspring }\} &= \sum_{l_1=0}^{l} \psi_s(l_1) \sum_{l_2=0}^{l-l_1} \psi_s(l_2) \times \cdots \times \\ &\quad \sum_{l_j=0}^{l-\cdots-l_{j-1}} \psi_s(l_j)\psi_s(l-l_1-\cdots-l_{j-1}-l_j) \\ &= \psi_s^{*j}(l)\end{aligned} \tag{11}$$

Using this information, one can create a column-stochastic matrix which projects the number-distribution of producer kin of age s onto offspring kin:

$$F(s) = \begin{pmatrix} 1 & \psi_s(0) & \psi_s^{*2}[0] & \dots & \psi_s^{*Q}[0] \\ 0 & \psi_s(1) & \psi_s^{*2}[1] & \dots & \psi_s^{*Q}[1] \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \psi_s(Q) & \psi_s^{*2}[Q] & \dots & \psi_s^{*Q}[Q] \end{pmatrix} \in R^{(Q+1)\times(Q+1)}. \tag{12}$$

Through $F(s)\mathbf{k}_s$ we produce a pmf of offspring number encapsulating the reproduction of an arbitrary kin-type $\mathbf{k}_s$ of age s. The first entry of $F(s)\mathbf{k}_s$ for instance is $k_s(0) + \psi_s(0)k_s(1) + \psi_s^{*2}[0]k_s(2) + \cdots + \psi_s^{*Q}[0]k_s(Q)$, which one interprets as the probability that there are no producer kin (who do not reproduce), plus the probability that there is one kin who does not have offspring, plus the probability that there are two kin and neither have offspring, and so on, up to the probability that there are Q kin and none of them have offspring. The second entry is $\psi_s(1)k_s(1) + \psi_s^{*2}[1]k_s(2) + \cdots + \psi_s^{*Q}[1]k_s(Q)$ which reads: the probability that there is one kin who has one offspring, plus the probability that there are two kin and between them produce one offspring (irrespective of ordering), and so on, up to the probability that there are Q kin and between them all one offspring is produced. The last entry is $\psi_s(Q)k_s(1) + \psi_s^{*2}[Q]k_s(2) + \cdots + \psi_s^{*Q}[Q]k_s(Q)$ and reads: the probability that there is one kin who has Q offspring, plus

the probability that there are two kin and between them produce Q offspring (irrespective of ordering), and so on, up to the probability that there are Q kin and between them all Q offspring are produced. We adopt the convention that $F(s) = \mathbf{I}, \ \forall s < 0$.

2.6 Total kin versus age-distributed kin

As mentioned in Section 2.2, we write $\mathbf{k}_s(y)$ to represent an age-specific kin-number pmf for Focal's kin of age s when Focal is age y. Suppose however, we wish to know when Focal is y years old, the probability that she has a total number $\bar{Q} \leq Q$ of these kin (over all possible ages $s = 0, 1, \ldots, n$), i.e., the $\bar{Q} + 1$ entry of the pmf $\mathbf{k}_\Sigma(y)$.

Here we face two cases which define total kin. If Focal can have any integer number of kin at any point in her life (i.e., non-ancestor) then the following theorem (a proof of which is given in Appendix A) describes the pmf of the total kin-number:

Theorem 1. *For kin of Focal which are not direct ancestors, the probability that Focal of age y has exactly $\bar{j}$ kin whose ages can range between z_1 and z_n is given by the $(\bar{j}+1)$-th entry of $\boldsymbol{k}_{z_1}(y) * \cdots * \boldsymbol{k}_{z_n}(y)$. Moreover, $\boldsymbol{k}_\Sigma(y) = \boldsymbol{k}_{z_1}(y) * \cdots * \boldsymbol{k}_{z_n}(y)$.*

As a special case, if Focal can have only one kin at any point in her life (i.e., an ancestor) then we have the following lemma which describes the probability mass function of the total kin-number:

Lemma 1. *For kin of Focal which are direct ancestors, the probability that when Focal is aged y this ancestor is alive is given by $p = \sum_i [\boldsymbol{k}_i(y)]_2$ (where $[.]_j$ represents the j-th entry), resulting in $\boldsymbol{k}_\Sigma(y) = (1-p, p, 0, \ldots, 0)^\dagger$.*

Lemma (1) is intuitive since Focal has, with probability $k_s^{(y)}(2)$ (the second entry of $\mathbf{k}_s(y)$), one living ancestor of age s. Over all possible ages i of the

ancestor, we see the probability that the kin is alive is given by p above (and the probability the kin is not alive by $1-p$).

3. Towards the kin-formulae

Using all of the above ingredients and observations, here we provide the explicit formulae for kin. We break down the formula into four theoretically distinct regimes, corresponding to the branches of the tree in Figure 1 that extend left (younger lineages), right (older lineages), up (ancestors) and down (descendants). In the next sections we respectively deal with these cases in turn. A more detailed break down of the formulae and how they relate to the established stochastic kinship model of Caswell (2024) is given in Appendix B. We provide an illustrative application of our framework in Section 4.

3.1 Kin which descend through older sisters of Focal's $(q-1)$-th ancestor

Focal's g, q kin which descend from an older sister of Focal's $(q-1)$-th ancestor, and which are of age s_g when Focal is age y are denoted by ${}^{\mathcal{O}}\mathbf{k}^{g,q}_{s_g}(y)$. A closed form for the the age-number probability distribution of these kin is given by

$$
\begin{aligned}
{}^{\mathcal{O}}\mathbf{k}^{g,q}_{s_g}(y) =& U(0,s_g)\prod_{i=1}^{g}\left\{ \left|_{s_{i-1}=n^{\min}_{i-1}}^{n^{\max}_{i-1}} F(s_{i-1}-s_i)U(0,s_{i-1}-s_i)\right.\right\}\times \\
& \sum_{b_1}\cdots\sum_{b_q}\rho_{b_1}\cdots\rho_{b_q}\boldsymbol{\psi}_{\beta_q-s_1} \\
n^{\min}_{i-1} =& \begin{cases} \underline{n}+s_i, & \text{for } i\neq 2 \\ \beta_{q-1}+1, & \text{for } i=2 \end{cases} \quad \text{and} \quad n^{\max}_{i-1} = \begin{cases} 0, & \text{for } i=1 \\ \bar{n}+s_i, & \text{for } i>1 \end{cases}
\end{aligned}
\tag{13}
$$

Notice, in Eq (13), the conditioning on kin descending through older sisters of Focal's $(q-1)$-th ancestor. If we only consider $g=1$ generations of descent

then $n^{\max} = 0$ (we don't need to convolve). Otherwise, if $g > 1$ then the limits over which the convolution over s_1 is taken (i.e. when $i \geq 2$) ensures that kin descend through older sisters of Focal's $(q-1)$-th ancestor. Taking the example of cousins from Focal's aunt older than mother; $g = 2, q = 2$, then the bound $n_1^{\min} = b_1 + y + 1$ ensures that aunt's current age s_1 is greater than current age of Focal's mother, $\beta_1 = b_1 + y$, and thereby, when Focal's aunt produced Focal's cousin (at age $s_1 - s_2$), she was older than Focal's mother (at age $b_1 + y - s_1$). As such, the possible ages at which aunt produced cousin are $s_1 - s_2 = b_1 + y + 1 - s_2, b_1 + y + 2 - s_2, \ldots, \bar{n}$, where the bound $n_{\max} = \bar{n} + s_2$ on s_1 caps the possible fertile ages of $s_1 - s_2$ at $\bar{n}$.

The conditioning on older-sisters of Focal's $(q-1)$-th ancestor only applies to the first descendant of Focal's q-th ancestor of age s_1, i.e., when $i = 2$ in Eq (13). For each descendant of the older sister of Focal's $(q-1)$-th ancestor, the convolution is taken over the plausible fertility interval; age of reproduction $\Delta_i = s_{i-1} - s_i$ with $s_{i-1} \in \{\underline{n} + s_i, \bar{n} + s_i\} \implies \Delta_i \in \{\underline{n}, \bar{n}\}$.

Eq (13) might at first appear unwieldy. As such, we refer the reader to simple examples of the equation: to derive Focal's older sisters, and then nieces through these older sisters, respectively described with annotations in Appendix B.1 and Appendix B.2.

3.2 Kin which descend through younger sisters of Focal's $(q-1)$-th ancestor

We denote by ${}^{\mathcal{Y}}\mathbf{k}_{s_g}^{g,q}(y)$, Focal's g, q kin which descend from a younger sister of Focal's $(q-1)$-th ancestor, and which are of age s_g when Focal is age y. A closed

form for the the age-number probability distribution of these kin is given by

$$
\begin{aligned}
{}^{\mathcal{Y}}\mathbf{k}_{s_g}^{g,q}(y) =& U(0,s_g)\prod_{i=1}^{g}\left\{ \left|_{s_{i-1}=n_{i-1}^{\min}}^{n_{i-1}^{\max}} F(s_{i-1}-s_i)U(0,s_{i-1}-s_i)\right.\right\}\times \\
& \sum_{b_1}\cdots\sum_{b_q}\rho_{b_1}\ldots\rho_{b_q}F(\beta_q-s_1)U(b_q,\beta_q-s_1)\mathbf{e}_2 \\
n_{i-1}^{\min} =& \underline{n}+s_i, \text{ and } \quad n_{i-1}^{\max} = \begin{cases} 0, \text{ for } i=1 \\ \beta_{q-1}-1, \text{ for } i=2 \\ \bar{n}+s_i, \text{ for } i>2 \end{cases} .
\end{aligned}
\tag{14}
$$

In Eq (14) the actions from right to left are interpreted as follows. First, we know with certainty that Focal's q-th ancestor produces Focal's $(q-1)$-th ancestor at age b_q. At this reproduction event the ancestor has a pmf defined through a unit vector with mass in the 1-kin-number-class, $\mathbf{e}_2$. Pre-multiplying through $U(b_q,\beta_q-s_1)$ projects the ancestor to survive up to age β_q-s_1 at which time it produces the younger sister of Focal's $(q-1)$-th ancestor, the reproduction event defined through multiplication of $F(\beta_q-s_1)$. Here it is instructive to recall that β_q-s_1 gives the age of Focal's $q-$th ancestor when producing the younger sister of Focal's $(q-1)$-th ancestor (b_q is the age when Focal's q-th ancestor produces Focal's $(q-1)$-th ancestor, β_q is age of Focal's q-th ancestor when Focal is y, and s_1 is the age of the younger sister of Focal's $(q-1)$-th ancestor when Focal is y). While $g>1$, the convolution over $s_1<\beta_{q-1}-1$ ensures that when the sister of Focal's $(q-1)$-th ancestor produces offspring (occurring at age (s_1-s_2) she is strictly younger than Focal's $(q-1)$-th ancestor at this time (whose age would be β_q-s_2). Subsequent convolutions over $s_i, i>1$ simply ensure the age $s_{i-1}-s_i$ at which Focal's q-th ancestor's $(i-1)$-th descendant produces its i-th descendent are contained within the fertility interval $[\underline{n},\bar{n}]$.

The most basic applications of Eq (14) derive Focal's younger sisters, and

then nieces through these younger sisters, respectively described with annotations in Appendix B.5 and Appendix B.6.

3.3 Descendants of Focal

Descendants of Focal can be calculated through Eq (13) with $q := 0$. Recall s_i is the age of Focal's $i-$th direct descendant. We obtain the closed form equation for the pmf of Focal's $g-$th descendant of age s_g when Focal is y, through

$$
\begin{aligned}
&{}^{\mathcal{D}}\mathbf{k}^{g,0}_{s_g}(y) = U(0, s_g) \prod_{i=1}^{g} \left\{ |_{s_{i-1}=n^{\min}_{i-1}}^{n^{\max}_{i-1}} F(s_{i-1} - s_i) U(0, s_{i-1} - s_i) \right\} \boldsymbol{\psi}_{(y-s_1)} \\
&n^{\min}_{i-1} = \underline{n} + s_i, \text{ and } n^{\max}_{i-1} = \begin{cases} 0, \text{ for } i = 1 \\ \min\{y, \bar{n} + s_i\}, \text{ for } i > 1 \end{cases}
\end{aligned}
\tag{15}
$$

Eq (15) tells us that (Focal who is immortal) will have g-generation descendants of age s_g if she gives birth to offspring at age $y - s_1$ who in turn survive to $s_1 - s_2$ and reproduce at this age, continuing in this manner up to Focal's $(g-1)$-th generation descendant, who survives from birth to age $s_{g-1} - s_g$ at which point producing Focal's g-th descendant (who survives to age s_g). Because at each generation i of descent, Focal can have multiple descendants, the $(i+1)$-th generation descendants are given by the convolution of pmfs of the i-th generation over the fertile bound (recall Theorem (1)).

3.4 Ancestors of Focal

Descendants of Focal correspond to the case whereby $g := 0$. Recall q_i is the age of Focal's i-th direct ancestor when producing the $(i-1)$-th. We obtain the closed form equation for the pmf of Focal's $q-$th ancestor of age s_0 when Focal

is y, through

$$^{\mathcal{A}}\mathbf{k}_{s_0}^{0,q}(y) = (1-\mu)\mathbf{e}_1 + \sum_{\beta_q = s_0} \mu U(b_q, s_0)\mathbf{e}_2, \;\; \mu = \prod_{i=1}^{q} \rho_{b_i}. \tag{16}$$

The sum is taken over all possible arrangements of the ages b_i at which Focal's i-th ancestor produced Focal's $(i-1)$-th, such that the age of Focal's q-th ancestor when Focal is y, $\beta_q = b_1 + \ldots b_q + y$ is equal to s_0. The vector $\mu\mathbf{e}_2$ in Eq (16) assign binomial probabilities that Focal's $q-th$ ancestor is indeed defined through the genealogical sequence of b_i. We project the q-th ancestor to survive from age b_q – when last we knew it was in existence (producing $(q-1)$-th ancestor) to survival to her age $s_0 = b_1 + b_2 + \cdots + b_q + y$ when Focal is y. The vector $(1-\mu)\mathbf{e}_1$ re-normalises by adding to the pmf, the probability that Focal's q-th ancestor was not defined by the sequence of b_i (i.e., is not of exact age s_0).

4. Application

Here, we present a simple exposition of the above-proposed methods for calculating probabilities of kin. To apply the model we make the following simplifying assumptions: births follow a Poisson distribution and deaths follow a Bernoulli distribution. Note that these distributional assumptions in no way restrict the model; they are simply chosen for progress in producing results. We could equally use empirical distributions of multiple births for single year of age data.

Under the Bernoulli assumption, given age-specific survival probability u_i, we use the distribution $\{u_i, 1-u_i\}$ for the probabilities of survival or death. Regarding the Poisson assumption, given an age-specific fertility rate f_i for age i, we use the distribution $\{f_i^k e^{-f_i}/k!\}_{k=0}^{Q}$ for age-specific probabilities of having k offspring where we set $Q = 6$ as an upper bound for lifetime reproduction (and thus a limit on kin-number). Of course we could use a larger upper limit for

the Poisson number of kin, however, it would be unlikely for an individual to experience more than 6 sisters, aunts or cousins, during their life[1].

We use single year of age fertility and mortality data sourced from the Human Fertility and Mortality Databases (Human Fertility Collection, 2024; Human Mortality Database, 2024). We use period based rates from 1974 in the UK. Further details of the model as well as code and data can be found at [link redacted for anonymous review]

4.1 Unconditional distributions of kin

4.1.1 Sisters

Using Eq (13) and Eq (14) respectively, we are able to produce age-specific kin-number pmfs for Focal's younger and older sisters of age s_1, for each age y of Focal's life:

$$\begin{aligned} {}^{\mathcal{O}}\mathbf{k}^{1,1}_{s_1}(y) &= U(0,s_1)\sum_{b_1}\rho_{b_1}\boldsymbol{\psi}_{b_1+y-s_1} \\ {}^{\mathcal{Y}}\mathbf{k}^{1,1}_{s_1}(y) &= U(0,s_1)\sum_{b_1}\rho_{b_1}F(b_1+y-s_1)U(b_1,b_1+y-s_1)\mathbf{e}_2 \end{aligned} \tag{17}$$

Given these distributions, it is a simple case of applying Theorem (1) to obtain the overall distributions of kin (of all possible kin ages when Focal is age y):

$$\begin{aligned} {}^{\mathcal{O}}\mathbf{k}^{1,1}_{\Sigma}(y) &= |_{s_1=y+1}^{n}\Big\{U(0,s_1)\sum_{b_1}\rho_{b_1}\boldsymbol{\psi}_{b_1+y-s_1}\Big\} \\ {}^{\mathcal{Y}}\mathbf{k}^{1,1}_{\Sigma}(y) &= |_{s_1=0}^{y-1}\Big\{U(0,s_1)\sum_{b_1}\rho_{b_1}F(b_1+y-s_1)U(b_1,b_1+y-s_1)\mathbf{e}_2\Big\}. \end{aligned} \tag{18}$$

Figure 2 provides visual examples of the age-specific pmfs of younger and older sisters of Focal – per Eq (17) – for each possible age of kin when Focal is aged 40. Plotted are the expected number ± standard deviation of kin (readily obtained

[1] Based on the data we use here. In other demographies individuals may well experience such numbers of kin.

from the pmfs). Notice that discontinuities about the vertical dotted line at age 40; kin are strictly younger/older than Focal.

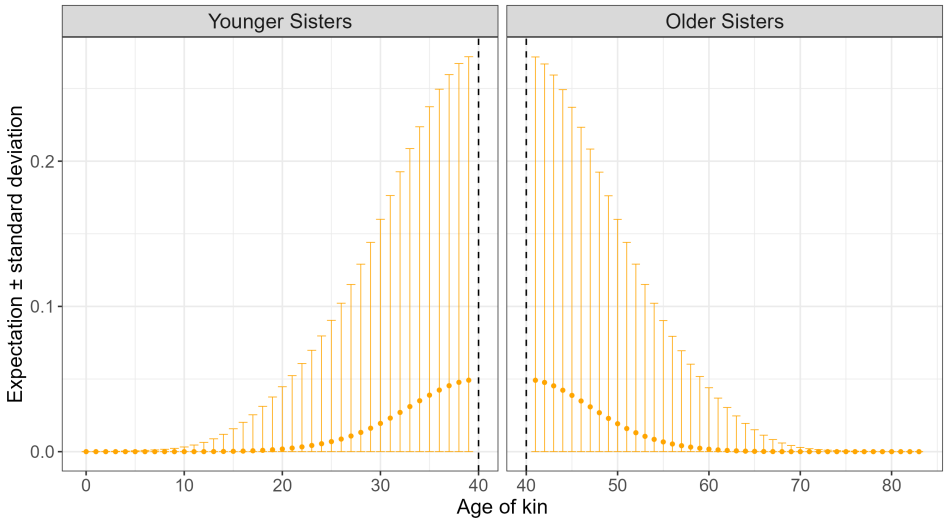

Figure 2: Theoretically predicted age-specific pmfs for Focal's sisters when Focal is aged 40. Left; younger sisters, right; older sisters. Points show expectations and error-bars expectations ± standard deviation. The dashed (blue) verticals show Focal's age.

Figure 3 illustrates the theoretically predicted pmfs for the total number of younger and older sisters of Focal at age 40. Regarding younger sisters, we theoretically predict that Focal experiences none of this kin type with a probability 0.648. Regarding older sisters we theoretically predict that Focal experiences no such kin with a probability 0.651. The expected numbers of sisters are illustrated through vertical lines: younger; 0.436, and older; 0.433. Horizontal error-bars display the range spanning from the expected number of sisters ± standard deviation in the number of sisters: (std. dev.) younger; 0.664, and older; 0.664.

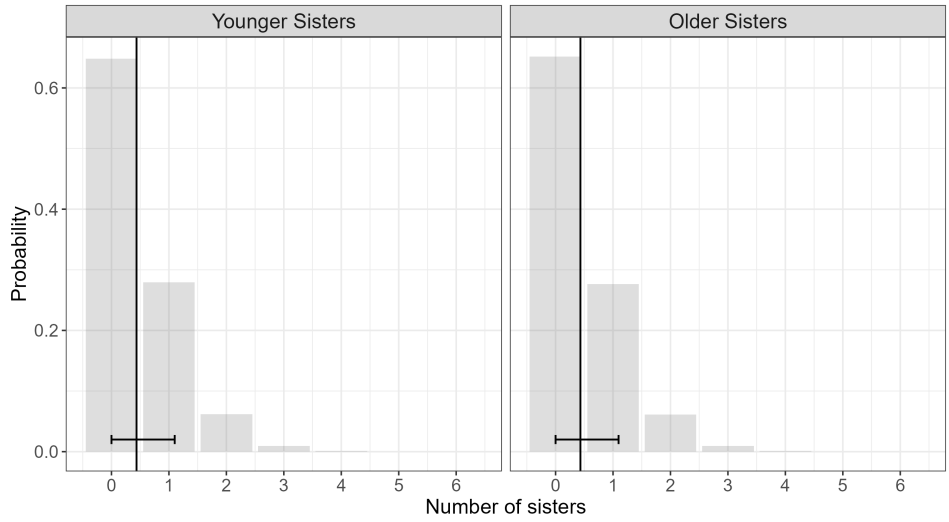

Figure 3: Accumulated-kin pmfs for Focal's sisters when Focal is aged 40. Left: younger sisters; right: older sisters. Bars show the probability that Focal has a given number of sisters. Vertical lines show the expected numbers of sisters and horizontal error-bars show expected numbers ± standard deviation in kin-number.

In Figure 4 we use Eq (18) to compare the overall kin-number pmfs of total younger and older sisters of Focal. In the plot, square points show the expected kin-number and crosses the standard deviation in kin-number. Also plotted are the fifth and ninety-fifth percent quantiles in kin-number, respectively shown by solid and dashed lines.

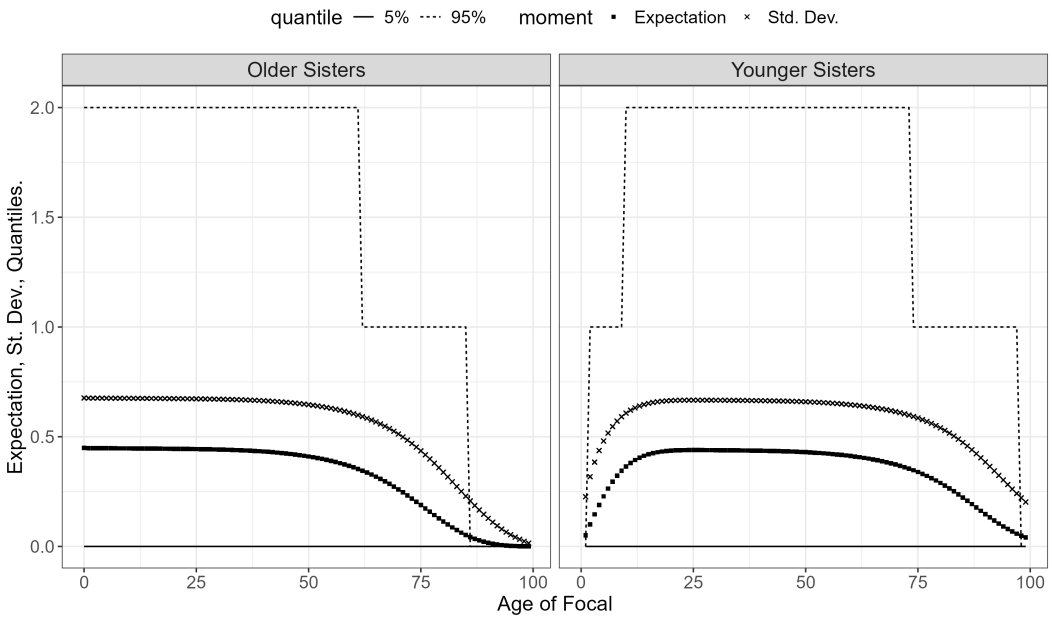

Figure 4: Accumulated sisters over Focal's lifetime. Left: younger sisters of Focal; right: older sisters. Squares show expected numbers of kin while crosses standard deviation in kin-number. Solid and dashed line show 5% and 95% quantiles respectively.

4.1.2 Ancestors and descendants

In order to obtain age-specific pmfs for Focal's mother of age s_0 when Focal is y, and daughters of age s_1 when Focal is age y, we respectively apply Eq (16) and Eq (15):

$$\begin{aligned} {}^{\mathcal{A}}\mathbf{k}^{0,1}_{s_0}(y) =& (1-\rho_{s_0-y})\mathbf{e}_1 + \rho_{s_0-y}U(s_0-y,s_0)\mathbf{e}_2 \\ {}^{\mathcal{D}}\mathbf{k}^{1,0}_{s_1}(y) =& U(0,s_1)\boldsymbol{\psi}_{y-s_1}. \end{aligned} \tag{19}$$

In Figure 5 we illustrate the expected kin-numbers $\pm$ standard deviation in kin-numbers obtained from these pmfs at two different ages in Focal's life course; $y = 20$ and $y = 50$.

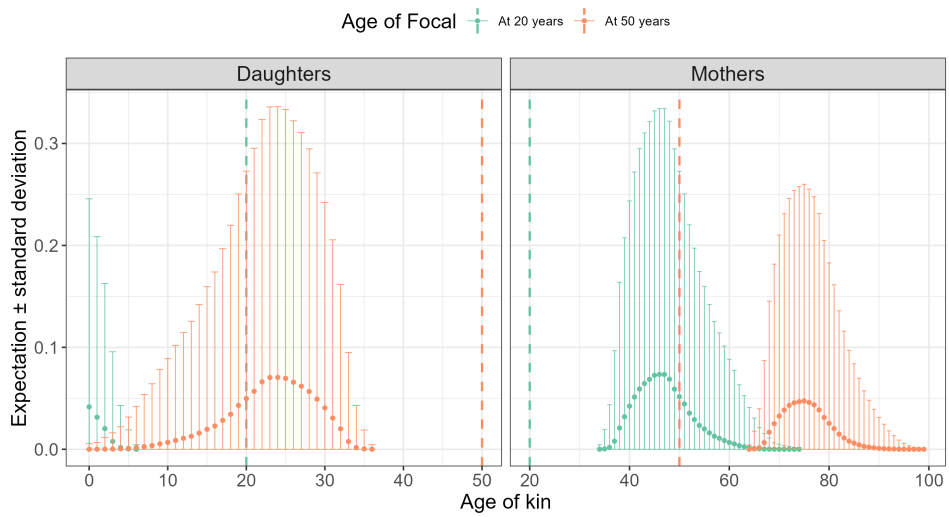

Figure 5: Theoretically predicted age-specific pmfs for Focal's daughters and mother when Focal is aged 20 (blue) and 50 (red). Left; daughters, right; mother. Points show expectations and vertical error-bars expectations ± standard deviation. The dashed verticals show Focal's ages.

In the same vein as in Section 4.1.1, to obtain the pmf of total kin by ages $y = 20, 50$ we apply Theorem (1):

$$\begin{aligned} {}^{\mathcal{A}}\mathbf{k}_{\Sigma}^{0,1}(y) &= |_{s_0=y+\underline{n}}^{n}\Big\{(1-\rho_{s_0-y})\mathbf{e}_1 + \rho_{s_0-y}U(s_0-y,s_0)\mathbf{e}_2\Big\} \\ {}^{\mathcal{D}}\mathbf{k}_{\Sigma}^{1,0}(y) &= |_{s_1=0}^{y-\underline{n}}\Big\{U(0,s_1)\boldsymbol{\psi}_{y-s_1}\Big\}. \end{aligned} \tag{20}$$

In Eq (20), recall that $\underline{n}$ is the minimum fertile age, and note the convolution limits. Regarding Focal's mother, we ensure that her possible age range is bounded to be at least $\underline{n}$ years older than Focal. Regarding Focal's daughters, the bounds ensure that daughters cannot yet be of age $y - \underline{n}$.

Figure 6 illustrates the pmfs of kin of Focal at the two ages in her life (as illustrated in Figure 5). In each panel, the bars show the probabilities that Focal experiences $j = 0, 1, \ldots, 6$ of kin. The left column shows Focal's daughters, while

the right Focal's mother. Overlaid we plot the expected kin-number (vertical dashed) $\pm$ standard deviation in kin-number (horizontal error-bars) obtained from the pmfs.

Notice that by comparing Focal at ages 20 and 50 (the rows of Figure 6), how the probable numbers of kin dramatically change by age of Focal. Regarding daughters, when Focal is aged 20 she is just beginning her reproductive cycle, whereas at age 50 she will have completed fertility. At age 20, our theoretical model predicts that Focal has one or more daughters with probability 0.098. Contrastingly, at age 50 the theoretical model predicts Focal to experience one or more daughters with probability 0.606. Corresponding to these age-dependent increases in the probability Focal has one or more daughters, Focal's expected number of daughters increases from 0.103 to 0.934, and standard deviation in number of daughters increases from 0.321 to 0.966.

The probability that Focal's mother is alive follows a Bernoulli distribution. At age 20, the theoretical model expects Focal's mother to be alive with probability 0.970. By the time Focal is aged 50, the probability will have dropped to 0.562. Notice that the standard deviation in the probability that Focal's mother is alive increases from 0.170 to 0.496 as Focal moves from age 20 to age 50.

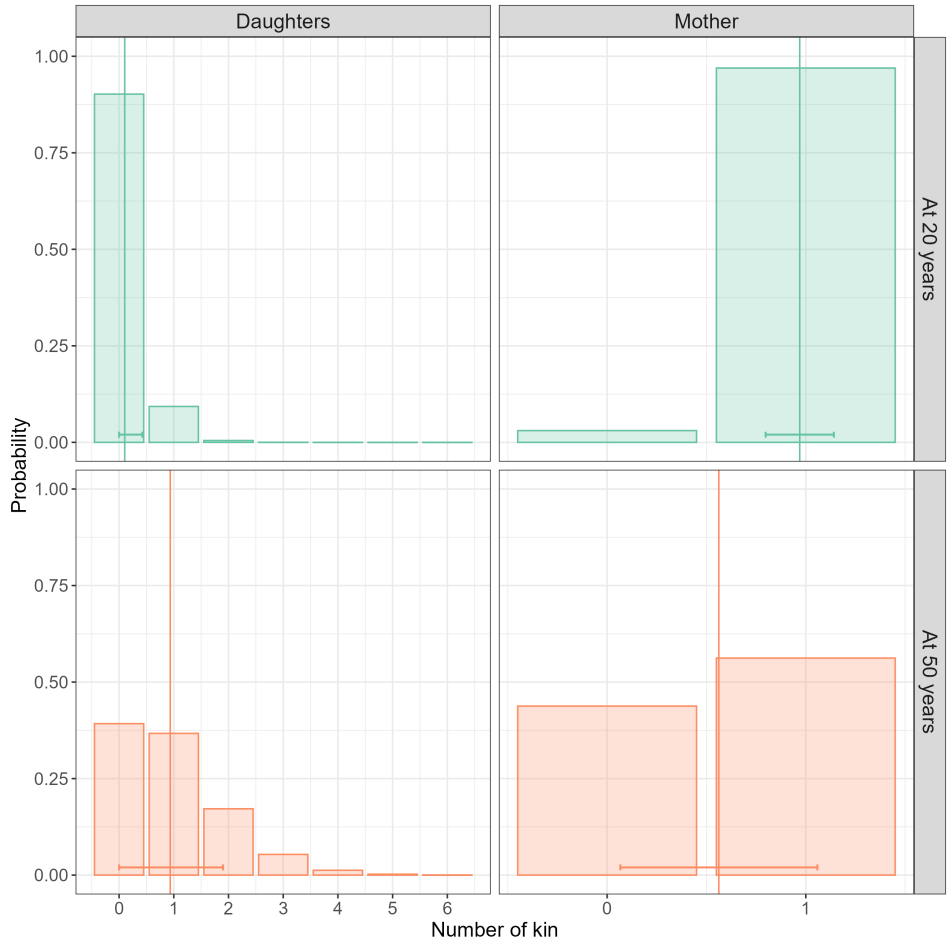

Figure 6: Accumulated-kin pmfs for Focal's daughters and mother when Focal is aged 20 and 50. Rows show age of Focal. Left; daughters, right; mother. Bars give the probability that Focal has a given number of kin; vertical lines show the expected numbers of kin; horizontal error-bars show expected kin-number $\pm$ standard deviation in kin-number.

By utilising the expressions in Eq (20), here we explore how the number of kin Focal experiences changes over her life course. In Figure 7 we plot, by age of Focal, the expected kin-number (squares) and standard deviation in kin-number (crosses) for Focal's daughters (left panel) and mother (right panel). The fifth and ninety-fifth percent quantiles in kin number are illustrated respectively

through solid and dashed lines.

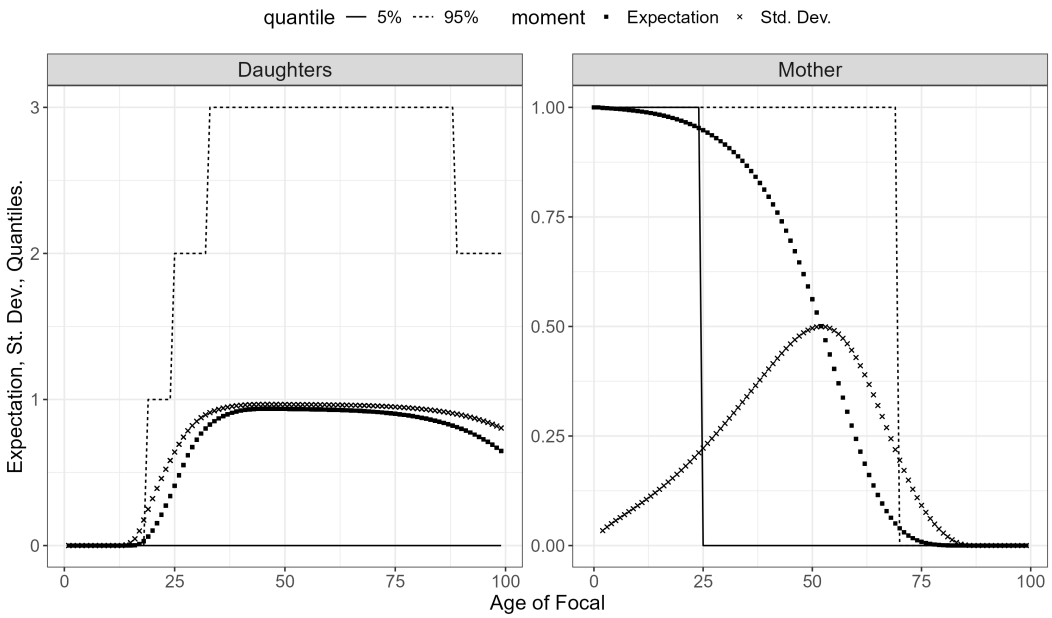

Figure 7: Accumulated daughters and the probability that mother is alive, by age of Focal. Left: Focal's daughters; right: Focal's mother. Squares show expected numbers of kin while crosses standard deviation in kin-number. Solid and dashed line show 5% and 95% quantiles respectively.

4.2 Conditional distributions of kin

Our method allows one to condition the probability that Focal experiences some number of one kin-type on life-history events pertaining to other kin-types. For example, consider Focal's sisters. The probability distributions for Focal's sisters are found through the reproduction of Focal's mother, averaged over all possible ages at which she had Focal, $b_1 \in [\underline{n}, \bar{n}]$, by respective probability weights ρ_{b_1}. Focal's mother, however, is one age when having Focal. It is therefore interesting to remove the weighting; to view the conditional pmfs of Focal's sisters under the certain (error-free) knowledge of Focal's mother's age when she had Focal.

Illustration: Focal's sib-ship conditioned on age of mother

Conditional on Focal's mother being of age b_1 when she had Focal, the distributions representing the probability that Focal has $j \in \{0, 1, \ldots, Q\}$ older/younger sisters, are also functions of mother's age:

$$\begin{aligned} {}^{\mathcal{O}}\mathbf{k}_{s_1}^{1,1}(y; b_1) =& U(0, s_1)\boldsymbol{\psi}_{b_1+y-s_1} \\ {}^{\mathcal{Y}}\mathbf{k}_{s_1}^{1,1}(y; b_1) =& U(0, s_1)F(b_1 + y - s_1)U(b_1, b_1 + y - s_1)\mathbf{e}_2 \end{aligned} \tag{21}$$

For each fixed age whereby we condition Focal's mother being at birth of Focal, $b_1 = \underline{n}, \ldots, \bar{n}$, Focal has a different probable number of sisters, as illustrated by Figure 8. Plotted are the conditional probabilities of Focal having $0, 1, \ldots, 6$ younger (left panel) or older (middle panel) sisters, at different ages in the life course (rows). The probability that Focal experiences a combined number (right panel) of sisters is given by ${}^{\mathcal{Y}}\mathbf{k}_{s_1}^{1,1}(y; b_1) *^{\mathcal{O}} \mathbf{k}_{s_1}^{1,1}(y; b_1)$. When looking at the left hand panel, we see that the younger Focal is, the less time her mother has to produce her younger sisters, and the less likely Focal experiences such kin. At older ages of Focal, we see presence of younger sisters are more likely, but depend on whether Focal was born earlier or later in the reproductive ages of the mother. Contrastingly, the middle panel reflecting number-probabilities of older sisters, remains invariant with respect to Focal's age, and only changes with the age at which Focal's mother had Focal (b_1). This is because Focal has accumulated all of her older sisters before birth. The right hand panel gives the probability that Focal experiences some given number of sisters irrespective of whether the kin are older or younger then Focal.

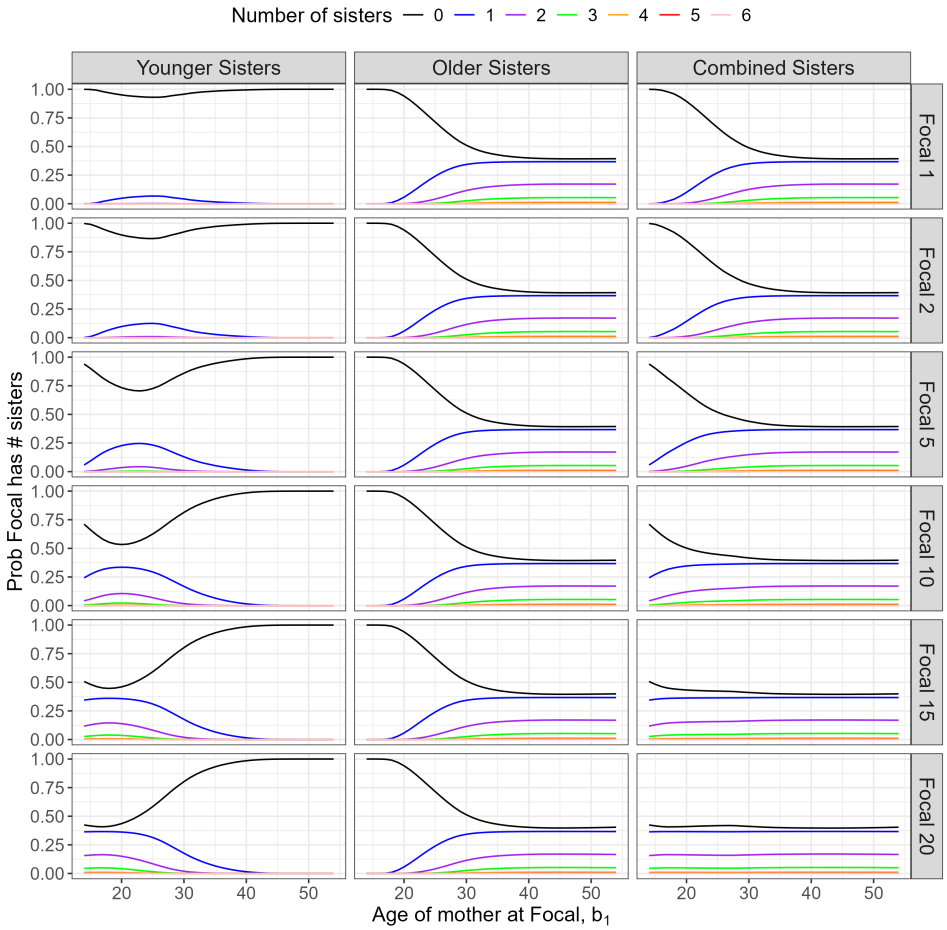

Figure 8: The probable number of sisters of Focal as a function of the age of Focal's mother, at different times in Focal's life. Rows give the age of Focal. Left: younger sisters; middle: older sisters; right: combined sisters.

We might also consider the probability that Focal has no older or younger sisters, or no sisters at all. Again, these probabilities depend on the age of Focal's mother when Focal was born; see Figure 9. Here, the left panel, illustrating the probability that Focal has no younger sisters, demonstrates that the older Focal's mother is at birth of Focal, the less likely she experiences such kin. In this case, Focal simply doesn't accumulate younger sisters since mother is nearing the end

of her reproductive interval. The more probable instances of Focal experiencing a younger sister occur when her mother is relatively young when having Focal, and when Focal is currently at an age whereby her mother will have completed reproduction. Interestingly, for Focal up to age 10, if her mother gave birth to her at a very young age (e.g., $b_1 = 15$) we find that it more probable for Focal to have no younger sisters compared to if Focal's mother gave birth to her at a mid-fertile age (e.g., $b_1 = 25$). Because age-specific fertility is unimodal in our example (1974; UK), if Focal's mother is young when mothering Focal she retains a low probability of reproduction in the immediate years after; thus Focal has to wait before accumulating younger sisters (e.g., 10 years in this example so that mother is aged $b_1 = 25$).

Note that in the present example we do not account for birth interval in the fertility modelling, or else we might expect to see more complex patterns in these heat maps, for example indicating that transition to gaining a first younger sister is less likely as Focal ages, separately from the fertility decline as mother ages.

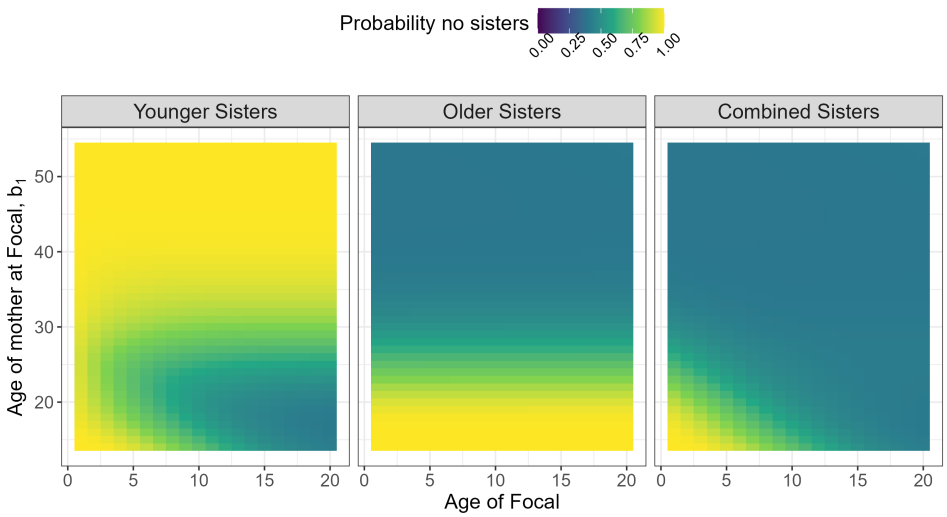

Figure 9: The probability that Focal has no sisters as a function of age, and conditional age of mother at Focal. Left: no younger sisters; middle: no older sisters; right: no younger or older sisters.

Let Focal's sib-ship size S (excluding Focal) be a random variable, conditioned on the age of mother at birth of Focal (b_1). As the conditional pmfs of accumulated sisters are

$$\begin{aligned} {}^{\mathcal{O}}\mathbf{k}_{\Sigma}^{1,1}(y;b_1) &= |_{s_1=y+1}^{n} U(0,s_1)\boldsymbol{\psi}_{b_1+y-s_1} \\ {}^{\mathcal{Y}}\mathbf{k}_{\Sigma}^{1,1}(y;b_1) &= |_{s_1=0}^{y-1} U(0,s_1)F(b_1+y-s_1)U(b_1,b_1+y-s_1)\mathbf{e}_2. \end{aligned} \tag{22}$$

Focal's overall sib-ship size at age y, conditional on mother giving birth to Focal at age b_1, is given by ${}^{\mathcal{Y}}\mathbf{k}_{\Sigma}^{1,1}(y;b_1) *^{\mathcal{O}} \mathbf{k}_{\Sigma}^{1,1}(y;b_1)$. In Figure 10 we plot for Focal of age $y = 20$, variation in the pmfs for her sib-ship size for different values of b_1.

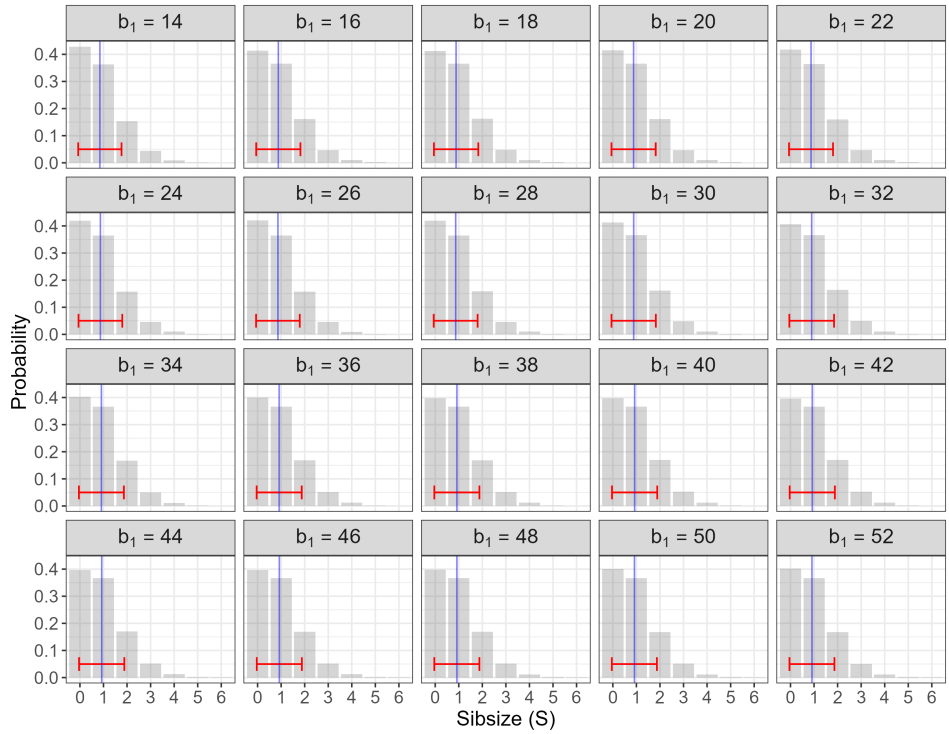

Figure 10: Probability distributions for Focal's sib-size S (her number of sister excluding herself) as a function of her mother's age when having her (b_1). Blue vertical lines show the conditionally expected kin number and red error-bars show the expected number $\pm$ standard deviation in kin-number.

Given any sib-size S, the birth rank of Focal can also be conditioned on the age at which her mother produced her. Let the variable $X_{i,j}$ indicate the probability that Focal has i older and j younger sisters (i.e., $i + j = S$). Thus, for each possible sib-ship size, $j = S - i$, and as such

$$X_{i,S-i} = \left[{}^{\mathcal{O}}\mathbf{k}^{1,1}_{\Sigma}(y;b_1)\right]_i \left[{}^{\mathcal{Y}}\mathbf{k}^{1,1}_{\Sigma}(y;b_1)\right]_{S-i} \text{ where } \sum_{S\geq 0}\sum_{i=1}^{S+1} X_{i,S-i} = 1 \qquad (23)$$

with the right hand side guaranteed by the conditioning on mother's age at birth of Focal. The variable $X_{i,S-1}$ allows us to calculate the probability of Focal's

birth rank, as illustrated in Figure 11. Here we plot birth rank probabilities for Focal of age $y = 20$: (i) conditional on mother's age b_1 (by row) and (ii) as a weighted mix over probable ages of mother (bottom row). Note that each of the rows sum to one because conditional on mother's age, they provide probabilities of Focal having $S = 0, 1, \ldots$ sisters, and given that, Focal's birth order. We might also condition on sib-ship size, in which case we would simply re-normalise each panel.

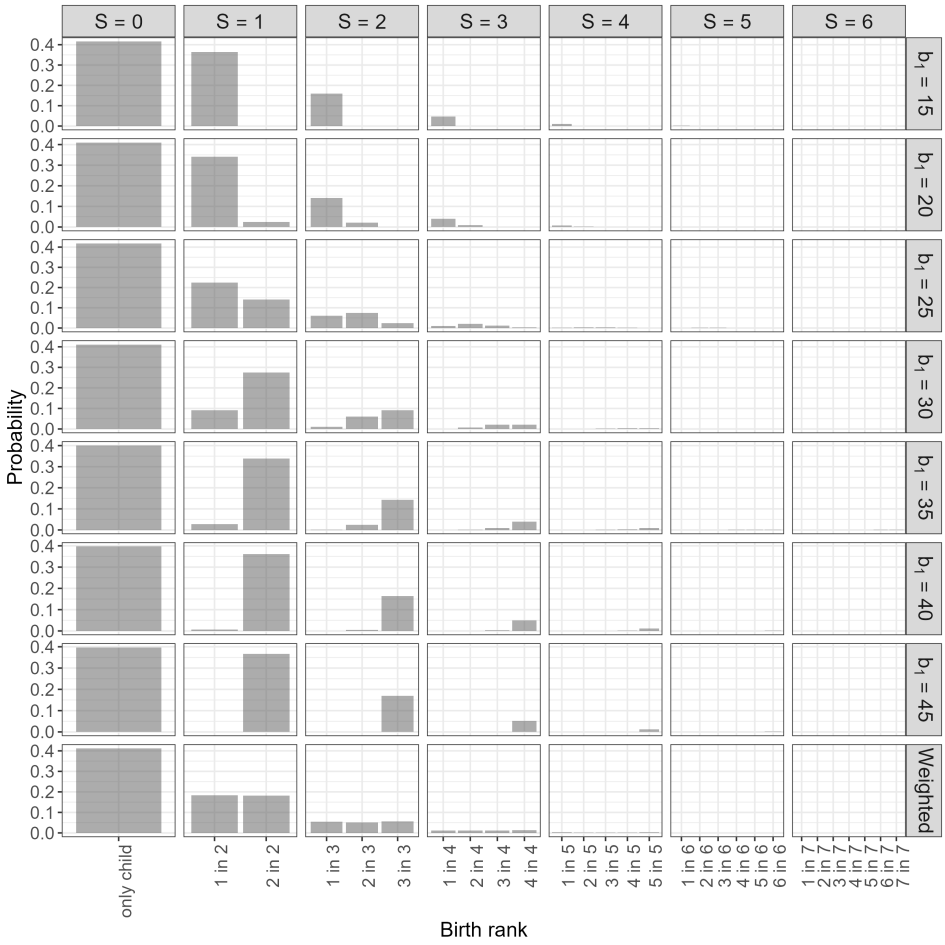

Figure 11: Probability that Focal experiences S sisters, and from them i are older; $S - i$ are younger. Conditioning on the age of Focal's mother at birth of Focal (b_1) is illustrated by rows (the bottom row shows the weighted average over all b_1). Columns give sizes of sib-ship (S).

5. Discussion

The research presented in this paper extends and enriches recent advances in modelling demographic stochasticity within kinship networks (Caswell, 2024). By projecting a complete probability distribution of the number of kin, our framework

provides a comprehensive analysis of variation within kinship structures. In particular, we are not restricted to examine the mean and variance in kin numbers; from complete probability distributions, higher-order moments or other quantities of interest (such as quantiles) can be readily obtained. The probabilistic nature of our framework allows conditioning kinship networks on particular life-history events, e.g., as demonstrated in Section 4.2. The approach presented here thus allows analytical results regarding demographic uncertainty that approach the richness of those obtained via micro-simulation, providing population distributions of numbers of kin by age of both Focal and kin, for arbitrary kin types.

Our proposed framework treats probability distributions for age-specific mortality and fertility as model inputs. From these, the probability distributions of kin naturally emerge. This innovation allows one flexibility to choose from empirical distributions (e.g., single-year-of-age offspring probabilities) as well as parametric distributions. Contrastingly, fitting empirical distributions using the mean and variance in kin numbers – outputs of the state-of-the-art stochastic kinship model – would be challenging. Another innovation made here is that we reduce the dimensionality of the matrix projections. The size of the state-space here is defined by life-time maximum kin-number, Q. Although, as argued below, this does not necessarily offer a computational advance, this feature of our model alleviates the need for big state-space matrices.

In principle, our model utilises the theory of branching processes with incorporated age-structure. We appeal to the pleasing recursive nature of kin-structures, quite similar to the seminal work of Goodman, Keyfitz, and Pullum (1974) – but with convolutions replacing integrals. Waugh (1981) provides a way to construct a joint probability generating function (p.g.f) for generation size over multiple generations in a Galton-Watson process, and then samples a typical

"Ego" (rather than Focal) from a fixed generation (see e.g., Section 7 in that paper). In this way, the author elegantly calculates what they refer to as Ego's "sorority" (sib-ship) size. The addition of age-structure in our framework builds on such work and allows one to calculate probabilities of birth order as well as overall sib-ship size. Moreover, in order to apply the methods proposed by Waugh (1981), one requires an analytic p.g.f (they use a fractional linear functional form). For progress here, we avoid such restrictions and maintain a discrete probability distribution which is easier to extract from empirical data.

From a computational perspective we cannot compete with the leading matrix-projection model Caswell (2024). The manner in which distributions of collateral relatives are calculated requires summations over ancestral reproductions and convolutions over their descendants. However, our equations are substantially faster to implement than an individual-based micro-simulation model is, and is able to yield results of comparable complexity. Indeed, when comparing to a direct stochastic simulation, not presented here for space constraints (but available to explore at [link redacted for anonymous review]) , we find consonance in results. Kin which form ancestors and descendants in Focal's family tree have almost the exact same distributions in the simulation. We find that the expected numbers of sisters agree, however, the variance in the number of such kin is slightly greater in the simulation. The most probable explanation for this discrepancy is Monte-Carlo error, although another source of uncertainty could be demographic stochasticity. In finite populations, fluctuations in births and deaths will result in the realised proportions of individuals in each age class deviating from the stable distribution. Consider a population of size $N(t)$ at time t structured by age classes i so that $N(t) = \sum_i n_i(t)$. Recall from Section 2.1 that F_i is a distribution of offspring for an individual in age-class i. Using $\bar{F}_i = (\sum_{j=1}^{n_i} F_i)/n_i$ to represent the average age-class reproductive value,

the deviation in individual reproduction from the Leslie model is $f_i - \bar{F}_i = \epsilon_i$ and moreover, $E[\epsilon_i^2] = \mathrm{var}[\bar{F}_i|f_i] = \mathrm{var}[F_i]/n_i$. Such demographic deviations mean that the proportions of mothers in each age-class will fluctuate around some stationary distribution. That is, denoting $R_i(t) = \sum_{j=1}^{n_i} F_i$ as the total reproduction from age-class i and $R(t)$ as the total reproduction over all age-classes, the observed proportion of mothers of age i in the simulation, $R_i(t)/R(t)$, will not exactly correspond to ρ_i in the theoretical model. A comprehensive investigation of the effects of population size in simulated processes will provide a further avenue for further research, but is beyond the scope of the present article.

The present work is restricted by the assumptions of time-invariance, one-sex and age-structured populations. Extending to time-variant situations should be trivial; the only theoretical demand being to probabilistically sample a mother in a non-stable population through $\rho_{x,t}$. Methods in Butterick et al. (2025) pertaining to time-inhomogeneous genealogical Markov chains can readily achieve as much. The remainder of moving to time-varying demographics should merely a case of careful indexing. Incorporating two sexes should also be relatively straightforward. Model ingredients requiring change would include projecting (i) survival independently by sex and (ii) the numbers of newborns distributed by sex. Constructing block-structured matrices similar to those used in Caswell (2022), with $U^{\mathrm{f,m}}, F^{\mathrm{f,m}}$ *in lieu* of the projection matrices therein, would allow such progress. In this case, projecting older and younger lineages (Eq (13) and Eq (14)) would require little change; except for imposing a female dominance in direct ancestor reproduction, and the output being a block-structured vector of female and male kin-pmfs. Similarly, Focal's descendants would simply be represented by a block-structured vector of pmfs. One notable change would be that the probable ages at which ancestors reproduce would be sex-specific (i.e.,

$\rho^{\mathrm{f}}_{b_1}, \rho^{\mathrm{m}}_{b_1}$ respectively represent the probable ages of Focal's mother and father at her birth). As such, in order to project direct ancestors, Eq (16) would involve a double summation over both variables (both sexes). We have already made good progress in extending the model to accommodate both sexes, and hope that this will be the focus of our next paper.

An interesting but challenging prospect is to extend the present research to a multi-state model. Such progress would require more in-depth consideration. For instance, construction of the matrices in Section 2.4 and Section 2.5 that form the core of this method would become complicated. As well as a function of age, the probability that $j = 1, 2, \ldots$ kin die at some given time will also depend on their stages. Elements of the projection matrix $U(s', s, g, f)$ under a multi-state model would have to provide the probabilities that so many kin survive from age s' to s, starting from stage g and ending in stage f. Progress in this direction would allow for a very rich investigation of kinship structures, perhaps paralleling the complexity of simulation. Including stage would allow for example, one to condition fertility profiles ($\boldsymbol{\psi}$) as dependent on elapsed time since last birth (as well as age). Not only would such an innovation provide a more accurate description for human kinship where it is known that time-proximity between births can create a "sibling constellation" (Morosow and Kolk, 2017), but would also be highly relevant in other species which experience postpartum infertility. Implementing stochastic age×stage structured kinship remains an open research problem.

In summary, the methodology proposed in this paper has, for the first time, enabled projecting a probability distribution of the (i) total number of kin and (ii) age-specific distribution of kin, of a typical population member. Having a complete distribution of kin allows for detailed analysis of probabilities of given numbers of relatives present at different parts of the life course, which is

important from the point of view of studying the presence of support networks, availability of care, and so on. In terms of the methodological contribution, as well as providing the mean and variance in kin numbers, the approach above naturally yields higher-order moments in kinship networks.

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References

Alburez-Gutierrez, D., Barban, N., Caswell, H., Kolk, M., Margolis, R., Smith-Greenaway, E., Song, X., Verdery, A.M., and Zagheni, E. (2022). Kinship, Demography, and Inequality: Review and Key Areas for Future Development. doi:10.31235/osf.io/fk7x9. URL https://osf.io/fk7x9_v1.

Alburez-Gutierrez, D., Williams, I., and Caswell, H. (2023). Projections of human kinship for all countries. *Proceedings of the National Academy of Sciences* 120(52): e2315722120. doi:10.1073/pnas.2315722120. URL https://www.pnas.org/doi/10.1073/pnas.2315722120.

Allen, L.J.S. (2010). *An Introduction to Stochastic Processes with Applications to Biology*. New York: Chapman and Hall/CRC.

Butterick, J., Hilton, J., Bijak, J., and Smith, P.W. (2025). A mathematical framework for time-variant multi-state kinship modelling [in press, pre-proof]. *Theoretical Population Biology* doi:doi.org/10.1016/j.tpb.2025.02.002.

Caswell, H. (2019). The formal demography of kinship: A matrix formulation. *Demographic Research.* 41: 679–712. doi:10.4054/DemRes.2019.41.24.

Caswell, H. (2020). The formal demography of kinship II: Multistate models, parity, and sibship. *Demographic Research.* 42(38): 1097–1146. doi:10.4054/DemRes.2020.42.38.

Caswell, H. (2022). The formal demography of kinship IV: Two-sex models and their approximations. *Demographic Research.* 47(13): 359–396. doi:10.4054/DemRes.2022.47.13.

Caswell, H., Margolis, R., and Verdery, A.M. (2023). The Formal Demography of Kinship V: Kin Loss, Bereavement, and Causes of Death. *Demographic Research.* 49(41): 1163–1200. doi:10.4054/DemRes.2023.49.41.

Caswell, H. and Song, X. (2021). The formal demography of kinship III: Kinship dynamics with time-varying demographic rates. *Demographic Research.* 45(16): 517–546. doi:10.4054/DemRes.2021.45.16.

Caswell, H. (2024). The formal demography of kinship VI: Demographic stochasticity and variance in the kinship network. *Demographic Research.* 51(39): 1201–1256. doi:10.4054/DemRes.2024.51.39.

Coste, C.F.D., Bienvenu, F., Ronget, V., Ramirez-Loza, J.P., Cubaynes, S., and Pavard, S. (2021). The kinship matrix: inferring the kinship structure of a population from its demography. *Ecology Letters.* 24(12): 2750–2762. doi:10.1111/ele.13854.

Crump, K.S. and Mode, C.J. (1968). A general age-dependent branching process. I. *Journal of Mathematical Analysis and Applications* 24(3): 494–508. doi: 10.1016/0022-247X(68)90005-X.

Demetrius, L. (1975). Natural selection and age-structured populations. *Genetics* 79(3): 535–544. doi:10.1093/genetics/79.3.535.

Goldman, N. (1978). Estimating the intrinsic rate of increase of population from the average numbers of younger and older sisters. *Demography* 15: 499–507. doi:10.2307/2061202.

Goodman, L.A., Keyfitz, N., and Pullum, T.W. (1974). Family formation and the frequency of various kinship relationships. *Theoretical Population Biology.* 5(1): 1–27. doi:10.1016/0040-5809(74)90049-5.

Hammel, E.A. (2005). Demographic dynamics and kinship in anthropological populations. *Proceedings of the National Academy of Sciences* 102(6): 2248–2253. doi:10.1073/pnas.0409762102. URL https://www.pnas.org/doi/10.1073/pnas.0409762102.

Human Fertility Collection, . (2024). Max Planck Institute for Demographic Research (Germany) and Vienna Institue of Demography (Austria). URL https://www.fertilitydata.org.

Human Mortality Database, . (2024). University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). URL https://www.mortality.org.

Jagers, P. (1982). How probable is it to be first born? and other branching-process applications to kinship problems. *Mathematical Biosciences* 59(1): 1–15. doi:10.1016/0025-5564(82)90105-5.

Jagers, P. and Nerman, O. (1984). The growth and composition of branching populations. *Advances in Applied Probability* 16(2): 221–259. doi:10.2307/1427068.

Joffe, A. and Waugh, W.A.O. (1982). Exact distributions of kin numbers in a Galton-Watson process. *Journal of Applied Probability* 19(4): 767–775. doi:10.2307/3213829.

Keyfitz, N. and Caswell, H. (1997). *Applied Mathematical Demography*. New York: Springer Cham.

Kolk, M. (2017). A Life-Course Analysis of Geographical Distance to Siblings, Parents, and Grandparents in Sweden. *Population, Space and Place* 23(3): e2020. doi:10.1002/psp.2020. URL https://onlinelibrary.wiley.com/doi/abs/10.1002/psp.2020.

Margolis, R., Sheftel, M.G., Wang, H., Chandler, R.F., Newmyer, L., and Verdery, A.M. (2024). Older Adults' Descendants and Family Networks in the Context of Global Educational Expansion. *Population and Development Review* n/a(n/a). doi:10.1111/padr.12681.

URL https://onlinelibrary.wiley.com/doi/abs/10.1111/padr.12681. _eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1111/padr.12681.

Margolis, R. and Verdery, A.M. (2019). A Cohort Perspective on the Demography of Grandparenthood: Past, Present, and Future Changes in Race and Sex Disparities in the United States. *Demography* 56(4): 1495–1518. doi:10.1007/s13524-019-00795-1.

Mode, C.J., Jacobson, M.E., and Pickens, G.T. (1987). A Stochastic Population Projection System Based on General Age-Dependent Branching Processes. *Journal of Applied Probability* 24(1): 1–13. doi:10.2307/3214054.

Morosow, K. and Kolk, M. (2017). How Does Birth Order and Number of Siblings Affect Fertility? A Within-Family Comparison Using Swedish Register Data. *European Journal of Population* 36(2): 197–233. doi:10.1007/s10680-019-09525-0.

Pittavino, M., Arpino, B., and Pirani, E. (2025). Kinlessness at Older Ages: Prevalence and Heterogeneity in 27 Countries. *The Journals of Gerontology: Series B* 80(1): gbae180. doi:10.1093/geronb/gbae180. URL https://doi.org/10.1093/geronb/gbae180.

Pollard, J.H. (1966). On the use of the direct matrix product in analysing certain stochastic population models. *Biometrika* 53(3-4): 397–415. doi:10.2307/2333647.

Pullum, T.W. (1982). The eventual frequencies of kin in a stable population. *Demography.* 19: 549–565. doi:10.2307/2061018.

Thiele, T., Alburez-Gutierrez, D., Calderón-Bernal, L. P., Snyder, M., and Zagheni, E. (2023). rsocsim. URL https://github.com/MPIDR/rsocsim.

Tuljapurkar, S. (1993). Entropy and convergence in dynamics and demography. *J. Math. Biol.* 31: 253–271. doi:10.1007/BF00166145.

Tuljapurkar, S., Zuo, W., Coulson, T., Horvitz, C., and Gaillard, J.M. (2020). Skewed distributions of lifetime reproductive success: beyond mean and variance. *Ecology Letters* 23(4): 748–756. doi:10.1111/ele.13467. URL https://onlinelibrary.wiley.com/doi/abs/10.1111/ele.13467.

Verdery, A.M. and Margolis, R. (2017). Projections of white and black older adults without living kin in the United States, 2015 to 2060. *Proceedings of the National Academy of Sciences of the United States of America* 114(42): 11109–11114. doi:10.1073/pnas.1710341114.

Vlachantoni, A., Evandrou, M., Falkingham, J., and Qin, M. (2024). Dynamics of unmet need for social care in England. *Ageing & Society* 44(6): 1247–1265. doi:10.1017/S0144686X22000745. URL https://eprints.soton.ac.uk/467908/.

Wachter, K.W., Hammel, E.A., and Laslett, P. (eds.) (1978). *Statistical Studies of Historical Social Structure*. Academic Press. doi:10.1016/B978-0-12-729150-5.50001-8. URL https://www.sciencedirect.com/science/article/pii/B9780127291505500018.

Waugh, W.A.O. (1981). Application of the Galton-Watson process to the kin number problem. *Advances in Applied Probability* 13(4): 631–649. doi:10.2307/1426964.

Appendices

1. Proof of Theorem 1

Proof. Recall that the j-th entry of $\mathbf{k}_s(y)$ gives the probability that Focal has a number $j-1$ of these kin of age s when she is aged y, i.e., $k_s^{(y)}(j-1)$. Let $P_{\bar{Q}}$ be the probability that Focal has exactly $\bar{Q}$ kin at age y. Suppose that the ages of these kin can only be aged z_1 and z_2 for example. Then

$$P_{\bar{Q}} = \sum_{q_1=0}^{\bar{Q}} k_{z_1}^{(y)}(q_1) k_{z_2}^{(y)}(\bar{Q}-q_1) = \big(k_{z_1}^{(y)} * k_{z_2}^{(y)}\big)[\bar{Q}] = (\mathbf{k}_{z_1}(y) * \mathbf{k}_{z_2}(y))_{\bar{Q}+1}. \quad \text{(A-1)}$$

Thus, $(\mathbf{k}_{z_1} * \mathbf{k}_{z_2})$ is a discrete probability distribution with j-th entry the probability that Focal has exactly $j-1$ kin when aged y. If kin can be of ages z_1, z_2, z_3 then

$$\begin{aligned} P_{\bar{Q}} = \sum_{q_1=0}^{\bar{Q}} k_{z_1}^{(y)}(q_1) \sum_{q_2=0}^{\bar{Q}-q_1} k_{z_2}^{(y)}(q_2) k_{z_3}^{(y)}(\bar{Q}-q_1-q_2) &= \sum_{q_1=0}^{\bar{Q}} k_{z_1}^{(y)}(q_1)\big(k_{z_2}^{(y)} * k_{z_3}^{(y)}\big)[\bar{Q}-q_1] \\ &= \big(k_{z_1}^{(y)} * k_{z_2}^{(y)} * k_{z_3}^{(y)}\big)[\bar{Q}] \\ &= (\mathbf{k}_{z_1} * \mathbf{k}_{z_2} * \mathbf{k}_{z_3})_{\bar{Q}+1}. \end{aligned} \quad \text{(A-2)}$$

For any age y of Focal, the above reasoning leads us to believe that we can extend the convolutions of age-specific pmfs $\mathbf{k}_z(y)$ over all kin potential ages $z_i, i = 1, \ldots, n$ to obtain the number distribution. We use induction to show that if above holds for kin up to age z_{m-1} then it holds for z_m (for notation the

superscript y is omitted):

Suppose that z_i holds for $i = m-1$:

$$\sum_{j_1=0}^{\bar{j}} k_{z_1}(j_1) \sum_{j_2=0}^{\bar{j}-j_1} k_{z_2}(j_2) \cdots \sum_{j_{m-2}=0}^{\bar{j}-\sum_{i=1}^{m-2} j_i} k_{z_{m-2}}(j_{m-2}) k_{z_{m-1}}(\bar{j} - \sum_{i=1}^{m-2} j_i)$$

$= \big(k_{z_1} * k_{z_2} * \cdots * k_{z_{m-1}}\big)[\bar{j}]$. Now shift indices $i \to i+1$:

$$\sum_{j_2=1}^{\bar{j}} k_{z_2}(j_2) \sum_{j_3=1}^{\bar{j}-j_2} k_{z_3}(j_3) \cdots \sum_{j_{m-1}=0}^{\bar{j}-\sum_{i=2}^{m-2} j_i} k_{z_{m-1}}(j_{m-1}) k_{z_m}(\bar{j} - \sum_{i=2}^{m-2} j_i)$$

$= \big(k_{z_2} * k_{z_3} * \cdots * k_{z_m}\big)[\bar{j}]$ (by hypothesis). Subtract j_1 from each k_i and sum over j_1:

$$\sum_{j_1=0}^{\bar{j}} k_{z_1}(j_1) \sum_{j_2=0}^{\bar{j}} k_{z_2}(j_2 - j_1) \sum_{j_3=0}^{\bar{j}-j_2} k_{z_3}(j_3 - j_1) \cdots \times$$

$$\sum_{j_{m-1}=0}^{\bar{j}-\sum_{i=1}^{m-1} j_i} k_{z_{m-1}}(j_{m-1} - j_1) k_{z_m}(\bar{j} - \sum_{i=1}^{m-1} j_i)$$

$$= \sum_{j_1=0}^{\bar{j}} k_{z_1}(j_1) \big(k_{z_2} * k_{z_3} * \cdots * k_{z_m}\big)[\bar{j} - j_1] = \big(k_{z_1} * \cdots * k_{z_m}\big)[\bar{j}] = \big(\mathbf{k}_{z_1} * \cdots * \mathbf{k}_{z_m}\big)_{\bar{j}+1}$$

□

2. Illustration of the formulae in relation to Caswell's approach.

In this section we illustrate our theoretical approach with annotation for ease of interpretation. We also relate our model to the established one of Caswell (2024) by referring to the notation used therein.

2.1 Older sisters (m) or $(g = 1, q = 1)$-kin

Using notation of the established kinship model (see Figure 1), through Eq (13) our model produces the following pmf for Focal's older sisters of age s_1, when

Focal is aged y:

$$\mathbf{k}_{s_1}^{1,1}(y) \equiv \mathbf{m}_{s_1}(y) = U(0, s_1) \sum_{b_1} \rho_{b_1} \boldsymbol{\psi}_{b_1+y-s_1}. \tag{B-1}$$

Eq (B-1) is a function of Focal's age y and Focal's older sister's age s_1. We interpret the above by conditioning on the age of Focal's mother at Focal: at each possible age at which Focal's mother had Focal, b_1, for Focal to experience an older sister of age s_1 when she is age y, mother must produce this kin at age $b_1 + y - s_1$. Given that mother has Focal at age b_1 with probability ρ_{b_1}, the pmf of older sisters is then the weighted average of the summation.

2.2 Nieces from older sisters (p) or $(g = 2, q = 1)$-kin

Appealing to Eq (13), when Focal is of age y, the age-specific pmfs of her nieces aged s_2, born from her older sisters are given through

$$\begin{aligned}
\mathbf{k}_{s_2}^{2,1}(y) \equiv \mathbf{p}_{s_2}(y) &= U(0, s_2) \,|_{s_1=y+1}^{\bar{n}+s_2} F(s_1 - s_2) U(0, s_1 - s_2) \sum_{b_1} \rho_{b_1} \boldsymbol{\psi}_{b_1+y-s_1} \\
&= U(0, s_2) \,|_{\Delta_1=(y-s_2)+1}^{\bar{n}} F(\Delta_1) U(0, \Delta_1) \sum_{b_1} \rho_{b_1} \boldsymbol{\psi}_{b_1+(y-s_2)-s_1} \\
&= \underbrace{U(0, s_2)}_{\text{Niece survives to } s_2} \overbrace{|_{\Delta_1=(y-s_2)+1}^{\bar{n}} \underbrace{F(\Delta_1) \mathbf{m}_{\Delta_1}(y - s_2)}_{\substack{\text{Reproduction of sister} \\ \text{when she was age } \Delta_1 \\ \text{when Focal was } y - s_1}}}^{\text{All ages } \Delta_1 \text{ when sister fertile}}.
\end{aligned} \tag{B-2}$$

From right to left, Eq (B-2) effectively calculates the completeage-specific pmfs for older sisters of Focal for ages whereby they could potentially produce a niece of Focal, i.e., when Focal was aged $y - s_2$. The convolution over s_1 calculates the total reproduction of Focal's older sister – yielding a pmf of offspring produced by sister of age $s_1 - s_2$. The matrix multiplication through $U(0, s_2)$ projects the offspring (i.e., newborns nieces of Focal) up to age s_2 when Focal is y. The change

of variables to Δ_1 in the last line demonstrates how, as should be expected, nieces of age s_2 of Focal age y are produced by Focal's older sisters at ages Δ_1 (at least $y+1-s_2$) when Focal was age $y-s_2$. Nieces then survive from age 0 up to age s_2.

2.3 Aunts older than mother (r) or $(g=1, q=2)$-kin

Appealing to Eq (13) we obtain the age-specific pmf for aunts who are older than Focal's mother:

$$\mathbf{k}^{1,2}_{s_1}(y) \equiv \mathbf{r}_{s_1}(y) = U(0,s_1)\sum_{b_1}\sum_{b_2}\rho_{b_1}\rho_{b_2}\boldsymbol{\psi}_{b_1+b_2+y-s_1}. \tag{B-3}$$

Eq (B-3), through the double summation calculates the weighted average of the pmfs of grandmother's reproduction at age $b_1+b_2+y-s_2$, "after" producing Focal's mother. The pmf of the newborns, constituting Focal's aunts older than mother, are then projected through the U matrix to survive to age s_1.

2.4 Cousins from aunts older than mother (t) or $(g=2, q=2)$-kin

Appealing to Eq (13) we obtain the age-specific pmf for cousins who descend through aunts who are older than Focal's mother:

$$\begin{aligned}\mathbf{k}^{2,2}_{s_2}(y) \equiv \mathbf{t}_{s_2}(y) &= U(0,s_2)\,\Big|_{s_1=y+b_1+1}^{\bar{n}+s_2} F(s_1-s_2)U(0,s_1-s_2)\times \\ &\quad \sum_{b_1}\sum_{b_2}\rho_{b_1}\rho_{b_2}\boldsymbol{\psi}_{b_1+b_2+y-s_1} \\ &= U(0,s_2)\,\Big|_{\Delta_1=(y-s_2)+b_1+1}^{\bar{n}} \underbrace{F(\Delta_1)\mathbf{r}_{\Delta_1}(y-s_2)}_{\substack{\text{Reproduction of aunt}\\ \text{when she was age } \Delta_1\\ \text{when Focal was } y-s_2}}.\end{aligned} \tag{B-4}$$

Eq (B-4) reads that cousins of Focal at age s_2 when Focal is y were produced by Focal's older aunts when Focal was $y-s_2$ and when aunt was age Δ_1 (at

least $y + b_1 + 1 - s_2$ or older). In more detail, the double summation calculates the the pmf representing Focal's grandmother's reproduction "after" producing Focal's mother. The convolution over s_1 of the pmf pre-multiplied through $F(s_1 - s_2)U(0, s_1 - s_2)$ yields the total reproduction of aunts who survive up to and reproduce at age $\Delta_1 = s_1 - s_2$. Focal's aunt's possible age is constrained such she is older than Focal's mother (i.e., $s_1 - s_2 > y - s_2 + b_1$), explaining the limits $\Delta_1 = (y - s_2) + b_1 + 1, (y - s_2) + b_1 + 2, \ldots, \bar{n}$. Reproduction of aunt results in the newborn cousins of Focal. Through $U(0, s_2)$ the pmf of newborn cousins is projected to age s_2.

2.5 Younger sisters (n) or $(q = 1, g = 1)$-kin

In relation to the established model Caswell (2019), we have from appealing to Eq (14), the pmf of younger sisters of age s_1 of Focal age y:

$$\mathbf{k}^{1,1}_{s_1}(y) \equiv \mathbf{n}_{s_1}(y) = U(0, s_1) \sum_{b_1} \rho_{b_1} F(b_1 + y - s_1) U(b_1, b_1 + y - s_1) \mathbf{e}_2. \quad \text{(B-5)}$$

Eq (B-5) reads: at age b_1 Focal's mother has Focal and at that age her pmf is a unit vector with mass in the 1-kin-number-class (mother is certainly alive when having Focal). Mother survives $y - s_1$ years (through the U-matrix), up to age $b_1 + y - s_1$, when she produces a younger sister of Focal (through the F-matrix). The younger sister survives from birth up to age s_1.

2.6 Nieces from younger sisters (q) or $(q = 1, g = 2)$-kin

We have from Eq (14):

$$\begin{aligned} \mathbf{k}^{2,1}_{s_2}(y) \equiv \mathbf{q}_{s_2}(y) = & U(0, s_2) \,|_{s_1 = \underline{n} + s_2}^{y-1} F(s_1 - s_2) U(0, s_1 - s_2) \times \\ & \sum_{b_1} \rho_{b_1} F(y + b_1 - s_1) U(b_1, y + b_1 - s_1) \mathbf{e}_2. \end{aligned} \quad \text{(B-6)}$$

Eq (B-6) reads: at age b_1 Focal's mother had Focal, and is a unit vector with mass in the 1-kin-number-class (mother is certainly alive when having Focal). Mother then survives up to age $b_1 + y - s_1$ (through the U-matrix in the summand) at which time she produces a younger sister of Focal (through the F-matrix). The younger sister survives to age $s_1 - s_2$, at which point she produces offspring – nieces of Focal, who survive to be age s_2 when Focal is age y.

2.7 Aunts younger than Focal's mother (s) or $(q = 2, g = 1)$-kin

Appealing to Eq (14) we see that the age-specific pmf for an aunt younger than Focal's mother who is of age s_1 when Focal is age y, is given through

$$\mathbf{k}^{1,2}_{s_1}(y) \equiv \mathbf{s}_{s_1}(y) = U(0, s_1) \sum_{b_1} \sum_{b_2} \rho_{b_1} \rho_{b_1} F(y+b_1+b_2-s_1) U(b_2, y+b_1+b_2-s_1) \mathbf{e}_2. \tag{B-7}$$

2.8 Cousins from aunts younger than Focal's mother (v) or $(q = 2, g = 1)$-kin

We have from Eq (14):

$$\mathbf{k}^{1,2}_{s_2}(y) \equiv \mathbf{v}_{s_2}(y) = U(0, s_2) \,|_{s_1=\underline{n}+s_2}^{y+b_1-1} F(s_1 - s_2) U(0, s_1 - s_2) \times \Big\{ \sum_{b_1} \sum_{b_2} \rho_{b_1} \rho_{b_1} F(y + b_1 + b_2 - s_1) U(b_2, y + b_1 + b_2 - s_1) \mathbf{e}_2 \Big\}. \tag{B-8}$$

Eq (B-8) reads that cousins of Focal at age s_2 when Focal is y were produced by Focal's younger aunts when Focal was $y - s_2$ and when aunt was age Δ_1 (at least $y + b_1 + 1 - s_2$ or older).

2.9 Daughters (a) or $(g = 1, q = 0)$-kin

At each age of Focal's reproductive life she gives birth to a non-negative integer $j = 0, 1, \ldots, Q$ of offspring with respective probabilities $\psi_i(j)$. The pmfs for

daughters aged s_1 when Focal is aged y are calculated using Eq (15):

$$\mathbf{k}_{s_1}^{0,1}(y) \equiv \mathbf{a}_{s_1}(y) = U(0, s_1)\boldsymbol{\psi}_{y-s_1} \tag{B-9}$$

Notice that for Focal's daughter to be aged s_1 when Focal is aged y, Focal's reproductive probability mass function must have been defined when Focal was at age $y - s_1$, hence $\boldsymbol{\psi}_{y-s_1}$

2.10 Grand daughters (b) or $(g = 2, q = 0)$-kin

We apply Eq (15) with $g = 2$ to obtain the age-specific pmf for grand daughters of age s_2 when Focal is y. The equation below demonstrates that the grand daughters of Focal currently aged s_2 are produced through Focal's daughter over all possible ages of s' when Focal was age $y - s_2$ (in the past):

$$\begin{aligned}\mathbf{k}_{s_2}^{2,0}(y) \equiv \mathbf{b}_{s_2}(y) =& U(0, s_2) \mid_{s_1=\underline{n}+s_2}^{y-s_2} F(s_1 - s_2)U(0, s_1 - s_2)\boldsymbol{\psi}_{y-s_1} \\ =& U(0, s_2) \mid_{\Delta_1=\underline{n}}^{y-1} F(\Delta_1)\mathbf{a}_{\Delta_1}(y - s_2)\end{aligned} \tag{B-10}$$

with the change of variables $\Delta_1 = s_1 - s_2$ rendering Focal's daughter's age Δ_1 at reproduction of Focal's grand daughter (at which time Focal was age $y - s_2$), and described by the pmf $\mathbf{a}_{\Delta_1}(y - s_2)$.

2.11 Mother (d) or $(g = 0, q = 1)$-kin

From Eq (16) we see that when Focal is age y, the age-specific probability distribution for a mother of age s_0 reduces to

$$\mathbf{k}_{s_0}^{0,1}(y) \equiv \mathbf{d}_{s_0}(y) = \begin{pmatrix} 1 - \rho_{b_1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + U(b_1, s_0) \begin{pmatrix} 0 \\ \rho_{b_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{B-11}$$

where the summation term in Eq (16) is over one element (i.e., reduced to $\sum_{b_1 = s_0 - y}$) because the age at which mother gives birth to Focal is uniquely defined by mother's age s_0 when Focal is y.

2.12 Grandmother (g) or $(g = 0, q = 2)$-kin

From Eq (16) we see that when Focal is age y, the age-specific probability distribution for a mother of age s_0 is given by:

$$\mathbf{k}_{s_0}^{0,2}(y) \equiv \mathbf{g}_{s_0}(y) = \begin{pmatrix} 1 - \rho_{b_1}\rho_{b_2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sum_{b_1 + b_2 = s_0 - y} U(b_2, s_0) \begin{pmatrix} 0 \\ \rho_{b_1}\rho_{b_2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{B-12}$$

Note that now we are summing over possible ages at which grandmother produced mother (b_2) and mother produced Focal (b_1) which result in grandmother being age s_0 when Focal is y.