Measurement invariance explains the universal law of generalization for psychological perception

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The universal law of generalization describes how animals discriminate between alternative sensory stimuli. On an appropriate perceptual scale, the probability of discrimination typically declines exponentially with the difference on the perceptual scale. Exceptions often follow a Gaussian probability pattern rather than an exponential pattern. Previous explanations have been based on underlying theoretical frameworks such as information theory, Kolmogorov complexity, or empirical multidimensional scaling. This articles shows that the few inevitable invariances that must apply to any reasonable perceptual scale provide a sufficient explanation for the universal exponential law of generalization. In particular, reasonable measurement scales of perception must be invariant to shift by a constant value, which by itself leads to the exponential form. Similarly, reasonable measurement scales of perception must be invariant to multiplication, or stretch, by a constant value, which leads to the conservation of the slope of discrimination with perceptual difference. In some cases, an additional assumption about exchangeability or rotation of underlying perceptual dimensions leads to a Gaussian pattern of discrimination, which can be understood as a special case of the more general exponential form. The three measurement invariances of shift, stretch, and rotation provide a sufficient explanation for the universally observed patterns of perceptual generalization. All of the additional assumptions and language associated with information, complexity, and empirical scaling are superfluous with regard to the broad patterns of perception.

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Introduction

The probability that an organism perceives two stimuli as similar typically decays exponentially with separation between the stimuli. The exponential decay in perceptual similarity is often referred to as the universal law of generalization (Shepard, 1987; Chater & Vitányi, 2003).

Generalization arises because perceived similarity may describe recognition of a general category. For example, two circles may have different sizes, colors, and shadings. Perceived similarity arises from the generalized perception of *circle* as a category.

Universal law arises because many empirical observations fit the pattern for diverse sensory modalities across different species. Typical exceptions take on a Gaussian probability pattern for perceived separation (Ghirlanda & Enquist, 2003).

Both theory and empirical analysis depend on the definition of the perceptual scale. How does one translate the perceived differences between two circles with different properties into a quantitative measurement scale?

There are many different suggestions in the literature for how to define a perceptual scale. Each of those suggestions develop very specific notions of measurement based, for example, on information theory, Kolmogorov complexity theory, or multidimensional scaling descriptions derived from observations (Shepard, 1987; Chater & Vitányi, 2003; Sims, 2018).

I focus on the minimal properties that any reasonable perceptual measurement scale must have rather than on detailed assumptions motivated by external theories of information, complexity, or empirical scaling. I express the minimal properties as simple invari-

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ances.

I show that a few inevitable invariances of any reasonable perceptual scale determine the exponential form for the universal law of generalization in perception. All of the other details of information, complexity, and empirical scaling are superfluous with respect to understanding why the universal law of generalization has the exponential form.

I also show that, when the separation between stimuli depends on various underlying perceptional dimensions, it sometimes makes sense to assume that the perceptual scale will also obey exchangeability or rotational invariance. When that additional invariance holds, the universal law takes on the Gaussian form, which I will show to be a special case of the general exponential form.

Basic problem and notation

The notation $P(R_b|S_a)$ describes the probability of a positive response, R_b, to the event b, given an initial stimulus, S_a, by the event a. A positive response expresses the perceived similarity of b to a, which may also be thought of as expressing the generalization that b and a belong to the same category.

The goal here is to understand how the perceived similarity of b to a, observed as $R_b|S_a$, translates into a continuous psychological measurement scale, $T_{b|a}$, so that

$$P(R_b|S_a) \equiv f(T_{b|a})$$

for a suitably defined mapping $R_b|S_a \mapsto T_{b|a}$ and probability distribution function, f. We seek the characteristics of the mapping and the associated function, f.

Invariant properties of measurement

There are many different suggestions in the literature for how to define a perceptual scale, $T_{b|a}$ (Shepard, 1987; Chater & Vitányi, 2003; Sims, 2018). I focus on the minimal properties that any reasonable measurement scale must have, rather than on detailed assumptions motivated by external theories (Luce & Narens, 2008; Narens & Luce, 2008). I express the minimal properties as simple invariances. Before listing the invariances, consider the simple example of temperature scales.

Suppose we wish to analyze the perception of temperature for event b, given that event a is at the freezing point for water. If we choose to measure the temperature on the Celsius scale, then $T_{a|a} = 0$ and $T_{b|a} = C$. It would make sense to assume that perceptual generalization would be identical if we assigned numerical values on a Fahrenheit scale, $\tilde{T}$, which we obtain by $\tilde{T}_{b|a} = 32 + 1.8T_{b|a}$. In other words, perception should be independent of a shift and stretch of the scale by constant values.

Formally, the scale should be shift invariant with respect to any constant, α, such that

$$f(T_{b|a}) = k_\alpha f(T_{b|a} + \alpha) \tag{1}$$

for some constant of proportionality, k_α. The scale should also be stretch invariant to any constant, β, such that

$$f(\lambda T_{b|a}) = f(\lambda_\beta \beta T_{b|a}), \tag{2}$$

for which I show below that $\lambda = \lambda_\beta \beta$ is an invariant constant that is conserved in any particular application, set by the fact that $1/\lambda$ is the average value on the perceptual scale for positive responses to varying events b for a given stimulus a.

Thus, the scale $T_{b|a}$ has the property that the associated probability pattern is invariant to the affine transformation of shift and stretch, $T_{b|a} \mapsto \alpha + \beta T_{b|a}$. I will show that affine invariance by itself determines the exponential form for the universal law of generalization in perception.

In some cases, it makes sense to assume that the perceptual scale should also obey rotational invariance, such that the Pythagorean partition

$$T_{b|a} = y_1^2(\theta) + y_2^2(\theta) \tag{3}$$

splits the measurement into components that add invariantly to $T_{b|a}$ for any value of the parameter θ. The invariant quantity $T_{b|a}$ defines a circle in the (y_1, y_2) plane with a conserved radius $R_{b|a} = \sqrt{T_{b|a}}$ that is invariant to θ, the angle of rotation around the circle, circumscribing a conserved area $\pi R_{b|a}^2 = \pi T_{b|a}$.

Rotational invariance partitions a conserved quantity into additive components, for which the order may be exchanged without altering the invariant quantity. When rotational invariance holds, the universal law takes on a Gaussian form, which we will

see to be a special case of the general exponential form.

The following sections develop the three invariances of shift, stretch, and rotation. I show that essentially all of the common properties of perceptual generalization follow from these invariances. The analysis here briefly summarizes the detailed development in Frank (2016). The novelty in this article concerns the simple understanding of widely observed psychological patterns.

Shift invariance and the exponential form

To simplify notation, denote the perceptual scale by $x \equiv T_{b|a}$ and the associated probability distribution by $f(x) \equiv f(T_{b|a})$. If we assume that the functional form for the probability distribution, f, is invariant to a constant shift of the perceptual scale, $x+\alpha$, then by the conservation of total probability

$$\int k_0 f(x)\,\mathrm{d}x = \int k_\alpha f(x+\alpha)\,\mathrm{d}x = 1 \qquad (4)$$

holds for any magnitude of the shift, α, in which the proportionality constant, k_α, changes with the magnitude of the shift, α, independently of the value of x, in order to satisfy the conservation of total probability.

From this equality for total probability, which holds for any shift α by adjustment of the constant, k_α, the condition for $x \equiv T_{b|a}$ to be a shift-invariant scale is equivalent to

$$f(x+\alpha) = \kappa_\alpha f(x), \qquad (5)$$

in which κ_α depends only on α and is independent of x. Because the invariance holds for any shift, α, it must hold for an infinitesimal shift, $\alpha = \epsilon$. We can write the Taylor series expansion for an infinitesimal shift as

$$f(x+\epsilon) = f(x) + \epsilon f'(x) = \kappa_\epsilon f(x),$$

with $\kappa_\epsilon = 1 - \lambda\epsilon$, because ϵ is small and independent of x, and $\kappa_0 = 1$. Thus,

$$f'(x) = -\lambda f(x)$$

is a differential equation with solution

$$f(x) = ke^{-\lambda x},$$

in which k is determined by the conservation of total probability. When the perceptual scale ranges over positive values, $x > 0$, then $k = \lambda$.

The assumption that a perceptual scale must be shift invariant is, by itself, sufficient to explain the exponential form of the universal law of generalization.

Stretch invariance and rate of perceptual change

If we assume that the perceptual scale is defined for positive values, $x > 0$, then the average value of λx is always one, because

$$\int_0^\infty \lambda x f(x)\mathrm{d}x = \lambda \int_0^\infty \lambda x e^{-\lambda x}\mathrm{d}x = 1.$$

Thus, for average value, $\bar{x}$, the value of λ is $1/\bar{x}$. We can think of $\bar{x}$ as the average discrimination of various events, b, relative to an initial stimulus, a, in which the set of events b corresponds to a uniform continuum along the perceptual scale, x.

It makes sense to assume that the average discrimination would not change if we arbitrarily multiplied our numerical scale for perception, x, by a constant, β. The conservation of average value and stretch invariance are equivalent, because

$$\lambda \int_0^\infty \lambda_\beta \beta x e^{-\lambda_\beta \beta x}\mathrm{d}x = 1$$

when we allow λ_β to adjust to satisfy the conservation of average value so that $\lambda = \lambda_\beta \beta$ or, equivalently, we assume stretch invariance of the scale, $x \equiv T_{b|a}$.

The constant $\lambda = 1/\bar{x}$ can be thought of as the slope or rate of change in the logarithm of discrimination, because

$$\log f(x) = -\lambda x.$$

Stretch invariance, or the conservation of average value, are sufficient to set the rate of change in the logarithm of discrimination. The average value of $-\log f(x)$ is a common definition of information or entropy, and is related to many interpretations in terms of information theory (Cover & Thomas, 1991; Sims, 2018).

Rotational invariance and Gaussian patterns

The scale, x, measures the perceptual difference between two entities or events. In some cases, the total difference, x, depends on the perceived differences along several distinct underlying dimensions. With two underlying dimensions, we may write

$$x = z_1(\theta) + z_2(\theta).$$

For a particular value of x, the parameter θ describes all of the combinations of the two underlying dimensions that add invariantly to x. If we let $x = r^2$ and let the dependence of z on θ be implicit, we can write the prior expression equivalently as

$$x = r^2 = \sqrt{z_1}^2 + \sqrt{z_2}^2,$$

which defines a circle with coordinates along the positive and negative values of $\left(\sqrt{z_1}, \sqrt{z_2}\right)$, with a constant radius r that is rotationally invariant with respect to the parametric angle, θ. Traditionally, one uses $y_i = \sqrt{z_i}$, so that the radius, r, of a sphere has the familiar definition of a Euclidean distance

$$r^2 = \sum y_i^2.$$

For each radial value, $r = \sqrt{x}$, we can write $\{y_i(\theta)\}$ as the sets indexed by the parameter θ for which the individual dimensional measures combine to the same invariant radius. If the angles of rotation with equivalent radius occur with equal probability or without prior bias, then radial values are rotationally invariant with respect to probability or prior likelihood.

I now show that rotational invariance leads to the Gaussian pattern as a special case of the general exponential form. In the exponential form derived in earlier sections, λx described the stretch-invariant perceptual scale. To express that scale in terms of a rotationally invariant radial measure, r, we note that $x = r^2$ and we let $\lambda = \pi v^2$. Thus, we can write the stretch-invariant incremental perceptual measure as

$$\lambda \mathrm{d}x = \pi v^2 \mathrm{d}r^2 = 2\pi v^2 r \mathrm{d}r.$$

The general exponential form is

$$f(x)\,\mathrm{d}x = \lambda e^{-\lambda x}\mathrm{d}x = 2\pi v^2 r e^{-\pi v^2 r^2}\mathrm{d}r.$$

At a given radius, vr, if, by rotational invariance, all combinations of values for the underlying measurement dimensions occur without bias or prior information, then the total probability in a radial increment, $v\mathrm{d}r$, is spread uniformly over the circumferential path with length $2\pi vr$.

A radial vector intersects a fraction of the total probability density in the circumferential path in proportion to $1/2\pi vr$. Thus, the probability along an increment $v\mathrm{d}r$ of the radial vector is

$$(1/2\pi vr) f(x)\mathrm{d}x = ve^{-\pi v^2 r^2}\mathrm{d}r = ve^{-\lambda r^2}\mathrm{d}r,$$

invariantly with respect to the angle of orientation of the radial vector. This expression is the Gaussian distribution, with r^2 as the squared deviation from the mean or central location, and with parameters commonly written as $\lambda = 1/2\sigma^2$ and $v = 1/\sqrt{2\pi\sigma^2}$ for variance σ^2. The variance is simply the average value of the squared radial deviations, $r^2 = x$.

We can also write the Gaussian in terms of the standard perceptual scale, x, as

$$g(x)\mathrm{d}\sqrt{x} = ve^{-\lambda x}\mathrm{d}\sqrt{x}.$$

When we consider the standard perceptual scale, x, with respect to the incremental square-root scale, $\mathrm{d}\sqrt{x}$, we obtain a Gaussian. The incremental square-root scale makes sense when we consider x as an aggregate measure of the sum of underlying perceptual dimensions. Each dimension naturally takes on a square-root scaling relative to the invariant total distance, because of the Euclidean measure of squared distance as the sum of squares along each dimension.

Discussion

Any reasonable perceptual scale must satisfy the simple affine invariances of shift and stretch. I have shown that those invariances are sufficient to explain the exponential form of the universal law of generalization. I have also shown that an additional common invariance of rotation explains why some observed patterns of generalization follow a Gaussian rather than exponential pattern. The Gaussian pattern is, in fact, a special case of exponential scaling, when the scale is a squared Euclidean distance metric over several underlying dimensions.

Previous explanations also generate the exponential pattern of the universal law (Shepard, 1987; Chater & Vitányi, 2003; Sims, 2018). The reason those explanations succeed is that they include assumptions about shift invariance, which by itself generates an exponential pattern. All of the other assumptions and language associated with those prior explanations are superfluous with respect to the exponential form. Conclusions about rate of change in discrimination typically associate with an assumption about stretch invariance or, equivalently, conservation of average value.

It is certainly true that additional assumptions will lead to more precise predictions, which may then be tested to rule out particular mechanisms. But those additional assumptions and tests do not directly bear on the general exponential form itself.

I do not know of explicit prior explanations that unify the Gaussian pattern with the universal exponential law. Such explanations, if they exist, will generally reduce to the assumption of rotational invariance. Again, additional assumptions or arguments about particular underlying mechanisms are superfluous with regard to the general pattern.

It is, of course, interesting to consider what underlying perceptual mechanisms lead to the universal law. However, almost certainly, there is no single mechanism that could explain such a widely observed pattern. General patterns require general explanations that apply broadly. The simple invariances of meaningful measurement scales provide that general explanation for the observed patterns of perceptual scaling.

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