Multiscale Adaptive Gabor Expansion (MAGE): Improved Detection of Transient Oscillatory Burst Amplitude and Phase

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ABBREVIATED TITLE:
MAGE Multiscale Adaptive Gabor Expansion

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Abstract

Since Denis Gabor's 1947 paper on the discrete Gabor Expansion [Gabor, 1947], time-frequency signal analysis has proven to be an important tool for many fields. In neurophysiology, time-frequency analysis has often been used to characterize and describe transient bursts in local field potential data. However, these transient bursts have a wide range of variable durations, suggesting that a time-frequency-scale dictionary composed of elementary signal "atoms" may prove useful to more accurately match recorded bursts. While overcomplete multiscale dictionaries are useful, generating a sparse code over such dictionaries is a difficult computational problem. Existing adaptive algorithms for discovering a sparse description are slow and computationally intensive. Here we describe the Multiscale Adaptive Gabor Expansion (MAGE), which uses an implicit dictionary of parametric time-frequency-scale Gabor atoms to perform fast parameter reassignment to accelerate discovery of a sparse decomposition. Using analytic expressions together with numerical computation, MAGE is a greedy pursuit algorithm similar to Matching Pursuit, restricted to a dictionary of multiscale Gaussian-envelope Gabor atoms. MAGE parameter reassignment is robust in the presence of moderate noise. By expressing a unknown signal as a weighted sum of Gabor atoms, MAGE permits a more accurate estimate of the amplitude and phase of transient bursts than existing methods.

KEYWORDS: Adaptive Gabor Expansion, Multiscale Time-Frequency Analysis, Time-Frequency Reassignment

1 Introduction

Signal processing often requires us to characterize an unknown signal $f(t)$. One way to do so is to compare the unknown signal $f(t)$ to a dictionary of known references signals G, where each element $g_k(t) \in G$ is a time-frequency signal "atom" that captures some feature of interest, such as localization in time, frequency, or scale. The inner product is a useful procedure for directly comparing an unknown target signal $f(t)$ and a known probe signal $g(t)$:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt \tag{1}$$

where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$.

The more similar f is to g, the larger the magnitude of the inner product. Conversely, if f and g are dissimilar, their inner product will tend toward zero. Therefore, the set of inner products of the signal f against the dictionary G – that is, the set $\{\langle f, g_k \rangle | g_k \in G\}$ – describes how similar the unknown signal is to every atom in the dictionary and provides a simple measurement technique for generating structured data about the unknown signal: which reference or probe atoms best describe the unknown target signal, what part of the target signal different atoms capture, and the relative importance of different atoms for accurate and robust signal reconstruction.

A surprising number of signal processing transforms can be characterized in this way, as an ordered set of inner products with a parametric dictionary [Yin Q et al, 2002]. These include 1-dimensional transforms such as the Fourier, Laplace, and fractional Fourier Transforms, 2-dimensional transforms such as the Short-Time Fourier and Wavelet Transforms, and 3-dimensional transforms such as the multi-scale Gabor transform [Qian S, 1993].

1.1 From orthogonal bases to sparse overcomplete dictionaries

If the dictionary G spans the same space as the unknown target signal f while each dictionary atom is orthogonal to every other atom (that is, $m \neq n$ implies $\langle g_m, g_n \rangle = 0$ for all $g_m, g_n \in G$), then G forms a basis for f. In this case, the analysis and synthesis of f assumes the form:

$$c_k = \langle f, g_k \rangle \text{ (analysis via orthogonal dictionary)} \tag{2}$$

$$f = \sum_k c_k g_k = \sum_k \langle f, g_k \rangle g_k \text{ (synthesis via orthogonal dictionary)} \tag{3}$$

The expansion of a signal over orthogonal dictionaries has proven to be very useful in many domains, most notably in signal compression and communication. However, in other domains such as biomedical imaging, the primary goal is to *characterize* a novel signal, rather than to *transmit* it elsewhere; in this case orthogonal dictionaries may miss signal features that we would like to describe. That is, while an orthogonal basis still provides for perfect reconstruction, describing a single feature of the target signal may require a weighted sum of several different dictionary atoms.

Therefore, for cases where signal characterization is the primary goal, it may be more appropriate to use an *overcomplete* dictionary, also known as a redundant frame. As with a basis, a frame spans the space of the signal – but unlike a basis, frame (or dictionary) atoms need not be orthogonal. Relaxing the orthogonality constraint allows us to include a larger variety of signal atoms in the dictionary, increasing the likelihood that one of the atoms will prove to be a good match to a part of the target signal. As before with the orthogonal dictionary, analysis is simple:

$$c_k = \langle f, g_k \rangle \text{ (analysis via overcomplete dictionary } G) \tag{4}$$

In many cases, such as biomedical research, the goal of signal processing is to generate the set of coefficients $\{c_k\}$ which are then examined directly for some feature of interest – that is, in many cases there is no need to explicitly reconstruct the original signal f from the set of dictionary inner products. If desired, synthesis can still be performed using a synthesis dictionary $G^\dagger$ (distinct from the analysis dictionary G), where $GG^\dagger = G^\dagger G = I$:

$$f = \sum_k c_k g_k^\dagger = \sum_k \langle f, g_k \rangle g_k^\dagger \text{ (synthesis via overcomplete dictionary } G^\dagger) \tag{5}$$

In general, however, synthesis becomes more complicated in the overcomplete setting, in that for a given analysis dictionary G, the synthesis dictionary $G^\dagger$ is not unique.

Alteratively, given a large overcomplete dictionary, one can use an iterative pursuit algorithm to generate a sparse signal description.

1.2 Sparse Signal Representation via the Matching Pursuit algorithm

Recall that given an overcomplete dictionary G, any signal can be described as a sum of weighted atoms:

$$f(t) = \sum_{k=1}^{\infty} c_k g_k(t), \tag{6}$$

where $m < n$ implies $|c_m| \geq |c_n|$. For greedy algorithms such as Matching Pursuit [Mallet and Zhang, 1993], each coefficient c_k is extracted sequentially. That is, we can express any signal $f(t)$ as a filtered signal f_N and a residual signal r_N:

$$f_N = \sum_{k=1}^{N} c_k g_k \tag{7}$$

and

$$r_N = \sum_{k=N+1}^{\infty} c_k g_k \tag{8}$$

where

$$f(t) = f_N(t) + r_N(t), \|r_N\|_2 < \epsilon \tag{9}$$

More formally, we can describe the Matching Pursuit algorithm as a function that takes in the unknown signal s to be decomposed, the dictionary D of known signal atoms, and the number N of atoms to be extracted:

$[G, c, r] = \mathrm{MP}[D, s, N]$ (Matching Pursuit Algorithm)
Inputs:
D: dictionary of normalized signal atoms; $\|g\|_2 = 1$ for all $g \in D$
s: unknown signal to be decomposed
N: number of signal atoms to extract from s
Outputs:
G: ordered list (sub-dictionary) of extracted atoms; $G = \{g_k\}_{k=1}^N \subset D$
c: ordered list of coefficients; $c \in \mathbb{C}^N$
r: residual signal; $r = s - \sum_{k=1}^N c_k g_k$
Initialization:
$G^{(0)} = \emptyset$
$c^{(0)} = \emptyset$
$r^{(0)} = s$
Main loop:
for $k = 1$ to N do
$g_k = \mathrm{argmax}_{g \in D} |\langle r^{(k-1)}, g\rangle|$ (select best atom from dictionary)
$G^{(k)} = \{G^{(k-1)}, g_k\}$ (update list of extracted atoms)
$c^{(k)} = \{c^{(k-1)}, \langle s, g_k\rangle\}$ (update list of coefficients)
$r^{(k)} = r^{(k-1)} - \langle s, g_k\rangle\ g_k$ (update residual)
end for
return $G^{(N)}, c^{(N)}, r^{(N)}$

That is, Matching Pursuit finds the next best atom to extract by computing the inner product of the unknown signal s with all atoms in the dictionary D:
1. $\rho_\gamma = \langle r^{(k-1)}, g_\gamma\rangle$ for all $g_\gamma \in D$ (compute inner product of signal with dictionary atoms)
2. $\gamma^* = \mathrm{argmax}_\gamma |\rho_\gamma|$ (find index of best atom)
3. $g_k = g_{\gamma^*}$
4. $g_k = \mathrm{argmax}_{g \in D} |\langle r^{(k-1)}, g\rangle|$ (select best atom from dictionary):

Unfortunately, steps 1 and 2 are often computatonally expensive. One solution to this computational burden is to restrict the dictionary D to only use atoms of a known parametric form, and use the analytic expression for the atom waveform and rather than storing an explicit numerical waveform. Similarly, for some parametric dictionaries the inner product also has an analytic form and can be used to speed up the discovery of the best-fit atom in matching pursuit.

1.3 The dictionary of multiscale Gaussian-envelope Gabor atoms

One signal atom that is very useful in time-frequency analysis is the Gaussian function $\varphi(t)$:

$$\varphi(t) = 2^{1/4} e^{-\pi t^2} \tag{10}$$

Recall that a signal $s : \mathbb{R} \to \mathbb{C}$ is finite-energy if and only if

$$\|s\|_2^2 = \langle s, s \rangle = \int_{-\infty}^{\infty} s(t)\overline{s(t)}\, dt = \int_{-\infty}^{\infty} |s(t)|^2\, dt = E_s < \infty \tag{11}$$

Direct integration shows that φ has unit energy,

$$E_{\varphi,t} = \langle \varphi, \varphi \rangle = \int_{-\infty}^{\infty} |\varphi(t)|^2\, dt = 1 \tag{12}$$

We can use the Gaussian energy density function to compute the average time of the signal:

$$\mu_{\varphi,t} = \int_{-\infty}^{\infty} t|\varphi(t)|^2\, dt = 0, \tag{13}$$

Similarly, we can compute the time-domain variance:

$$\sigma_{\varphi,t}^2 = \int_{-\infty}^{\infty} (t - \mu_{\varphi,t})^2 |\varphi(t)|^2\, dt = \frac{1}{4\pi}. \tag{14}$$

Recall the definition of the Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and its inverse $\mathcal{F}^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$:

$$\mathcal{F}s = \mathcal{F}[s(t)] = (\mathcal{F}s)(\nu) = \hat{s}(\nu) = \int_{-\infty}^{\infty} s(t) e^{-2\pi i \nu t}\, dt \tag{15}$$

$$\mathcal{F}^{-1}\hat{s} = \mathcal{F}[\hat{s}(\nu)] = (\mathcal{F}\hat{s})(t) = s(t) = \int_{-\infty}^{\infty} \hat{s}(\nu) e^{2\pi i \nu t}\, d\nu \tag{16}$$

which we take here as already extended to be bounded, linear, unitary operators. Applying the Fourier Transform to the time-domain Gaussian function, we find the frequency-domain expression

$$\mathcal{F}\varphi = \mathcal{F}[\varphi(t)](\nu) = \hat{\varphi}(\nu) = 2^{1/4} e^{-\pi \nu^2} \tag{17}$$

From this we determine the frequency-domain energy, mean frequency, and frequency-domain variance:

$$E_{\hat{\varphi},\nu} = \langle \hat{\varphi}, \hat{\varphi} \rangle = \int_{-\infty}^{\infty} |\hat{\varphi}(\nu)|^2\, d\nu = 1 \tag{18}$$

$$\mu_{\hat{\varphi},\nu} = \int_{-\infty}^{\infty} \nu |\hat{\varphi}(\nu)|^2\, d\nu = 0 \tag{19}$$

$$\sigma_{\hat{\varphi},\nu}^2 = \int_{-\infty}^{\infty} (\nu - \mu_{\varphi,\nu})^2 |\hat{\varphi}(\nu)|^2\, d\nu = \frac{1}{4\pi} \tag{20}$$

Here we see why Gaussian functions are so useful in time-frequency analysis. Recall that for any signal s,

$$\sigma_{s,t}\sigma_{s,\nu} \geq \frac{1}{4\pi} \tag{21}$$

meaning that there is a lower limit on how concentrated a signal can simultaneously be in both the time- and frequency-domains. In general, if the energy of s is concentrated in time, then the energy of s is spread out in frequency, and vice versa. However, Gaussian-type functions make this inequality an equality; that is,

$$\sigma_{\varphi,t}\sigma_{\varphi,\nu} = \frac{1}{4\pi} \tag{22}$$

meaning that Gaussian functions are maximally localized in time and frequency. There exist no signals which have a tighter concentration of energy in the time-frequency plane. Therefore, if you want to decompose an unknown signal by describing in terms of simple, known time-frequency "atoms," then Gaussian functions are an excellent choice, given their strong energy concentration.

1.4 The translation, modulation, and dialation operators

To be able to move φ to any location in the time-frequency plane, we introduce two operators $T_x, M_x : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, where $t, x \in \mathbb{R}$, $f(t) \in L^2(\mathbb{R})$.

The translation operator

$$T_x f = T_x[f(t)] = (T_x f)(t) = f(t - x) \tag{23}$$

shifts or translates the signal f in time, while the modulation operator

$$M_x f = M_x[f(t)] = (M_x f)(t) = f(t)\exp[2\pi i x t] \tag{24}$$

shifts or modulates the signal f in frequency. Applying these operators to φ permit us to shift the mean time and mean frequency of the localized signal atom. That is, given the simple reference function φ as a fundamental signal "atom" which is well-localized in time and frequency, together with the operators T_x and M_x that move signal atoms in time and frequency, we can fully tile an unknown signal within the time-frequency plane, comparing small segments of this unknown signal to time- and frequency-shifted version of the known reference signal φ. To describe an unknown signal f, we can compute the inner product of it with the dictionary $\{T_\tau M_\omega \varphi\}$. This is the origin of the continuous STFT (where $\tau, \omega \in \mathbb{R}$) as well as its sampled counterpart, the Gabor expansion (where $\tau = am$ and $\omega = bn$ for small, fixed $a, b \in \mathbb{R}$ and $m, n \in \mathbb{Z}$).

However, time- and frequency-shifts alone do not capture all signal features of interest. Complex signals such as the local field potential recorded in neuroelectrophysiology often involve components that reflect different durations or scales – some signal sub-components may rise and decay quickly while others evolve more slowly.

We can examine variations in scale by introducing another operator $D_x : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, where the dialation operator

$$D_x f = D_x[f(t)] = (D_x f)(t) = e^{-\frac{x}{4}} s(te^{-\frac{x}{2}}) \tag{25}$$

dialates or re-scales the input domain of f while preserving the signal energy.

Given the Gaussian function φ together with the translation, modulation, and dialation operators defined above, we can generate an overcomplete dictionary $G = \{T_a M_b D_c \varphi | a, b, c \in \mathbb{R}^3\}$. This dictionary tiles the time-frequency plane, in that for each time-frequency coordinate (a, b) we can find an atom centered at that location – in fact, centered at each time-frequency point (a, b) is a continuous family of atoms that span all possible scales or durations specified by the duration parameter c. Furthermore, since the Gaussian φ satisfies the uncertainty principle relation as an equality (Eq. ?), each such atom provides the most compact time-frequency energy concentration possible.

Specifically, let

$$\phi_k = (t_k, \nu_k, s_k) \in \mathbb{R}^3 \tag{26}$$

represent the (time, frequency, scale) parameters for a given atom $g_k \in L^2(\mathbb{R})$. Then

$$g_k = T_{t_k} M_{\nu_k} D_{s_k} \varphi, \tag{27}$$

$$g_k(t) = g(t|\phi_k) = g(t|t_k, \nu_k, s_k) = T_{t_k} M_{\nu_k} D_{s_k} \varphi(t), \tag{28}$$

such that

$$g_k(t) = \sqrt[4]{2} \exp\left[-\frac{s_k}{4} - \pi e^{-s_k} (t - t_k)^2 + 2\pi i \nu_k (t - t_k)\right]. \tag{29}$$

Applying the Fourier Transform to g_k results in

$$\hat{g}_k(\nu) = \hat{g}(\nu|\phi_k) = \hat{g}(\nu|t_k, \nu_k, s_k) = \sqrt[4]{2} \exp\left[\frac{s_k}{4} - \pi e^{s_k} (\nu - \nu_k)^2 - 2\pi i \nu t_k\right]. \tag{30}$$

Given these time- and frequency-domain formulas of the atom g_k, we can calculate the total energy, the mean time, and the mean frequency of the Gabor atom:

$$E_k = \langle g_k, g_k \rangle = \langle \hat{g}_k, \hat{g}_k \rangle = \int_{-\infty}^{\infty} |g_k(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{g}_k(\nu)|^2 \, d\nu = 1, \tag{31}$$

$$\mu_{k,t} = \int_{-\infty}^{\infty} t |g_k(t)|^2 \, dt = t_k, \tag{32}$$

$$\mu_{k,\nu} = \int_{-\infty}^{\infty} \nu |\hat{g}_k(\nu)|^2 \, d\nu = \nu_k. \tag{33}$$

This shows that the generative parameters (t_k, ν_k) correspond to the measured values of mean time and mean frequency.

Finally, as a measure of signal duration, we can calculate the time-domain and frequency-domain variance:

$$\sigma_{k,t}^2 = \int_{-\infty}^{\infty} (t - \mu_{k,t})^2 |g_k(t)|^2 \, dt = \frac{\exp[s_k]}{4\pi}, \tag{34}$$

$$\sigma_{k,\nu}^2 = \int_{-\infty}^{\infty} (\nu - \mu_{k,\nu})^2 |\hat{g}_k(\nu)|^2 \, d\nu = \frac{\exp[-s_k]}{4\pi}. \tag{35}$$

Showing that the duration parameter s_k captures the variance in signal energy in both the time and frequency domains.

1.5 Inner product relation between Gabor atoms

Using the parametric formula for g_k given above, we can identify an analytic expression for the inner product between two Gabor atoms:

$$\rho = \langle g_t, g_p \rangle = \int_{-\infty}^{\infty} g_t(t)\overline{g_p(t)} \, dt = \Phi(t_t, \nu_t, s_t, t_p, \nu_p, s_p) = A_\rho \exp[i\theta_\rho], \tag{36}$$

where

$$A_\rho = \sqrt{\operatorname{sech}\left[\frac{s_\Delta}{2}\right]} \exp\left[-\frac{\pi e^{-s_p} t_\Delta^2}{1+e^{s_\Delta}} - \frac{\pi e^{s_p} \nu_\Delta^2}{1+e^{-s_\Delta}}\right], \tag{37}$$

$$\theta_\rho = -\frac{2\pi t_\Delta \nu_\Delta}{1+e^{-s_\Delta}} - 2\pi \nu_p t_\Delta, \tag{38}$$

and the difference between target and probe Gabor parameters are given by

$$t_\Delta = t_t - t_p, \tag{39}$$

$$\nu_\Delta = \nu_t - \nu_p, \tag{40}$$

and

$$s_\Delta = s_t - s_p. \tag{41}$$

Thus, the inner product between two gabors can be computed numerically or analytically. This useful once we recall that, as in Matching Pursuit, any signal can be decomposed into a weighted sum of atoms and a residual signal:

$$\langle s, g_p \rangle = \langle \sum_{k=1}^{N} c_k g_k, g_p \rangle = \sum_{k=1}^{N} c_k \langle g_k, g_p \rangle \tag{42}$$

Since the derivate operator is conjugate linear, we also have

$$x_p \in \{t_p, \nu_p, s_p\}, \tag{43}$$

$$\frac{\partial}{\partial x_p} \langle s, g_p \rangle = \frac{\partial}{\partial x_p} \sum_{k=1}^{N} c_k \langle g_k, g_p \rangle = \sum_{k=1}^{N} c_k \frac{\partial}{\partial x_p} \langle g_k, g_p \rangle \tag{44}$$

$$\frac{\partial}{\partial x_p} \langle g_t, g_p \rangle = \langle g_t, \frac{\partial}{\partial x_p} g_p \rangle = \langle g_t, z_{x_p} g_p \rangle \tag{45}$$

$$\frac{\partial}{\partial x_p}\langle s, g_p\rangle = \sum_{k=1}^{N} c_k \langle g_k, \frac{\partial}{\partial x_p} g_p\rangle = \sum_{k=1}^{N} c_k \langle g_k, z_{x_p} g_p\rangle \tag{46}$$

for some z_{x_p}. To compute z_{x_p}, recall that the derivative of the inner product of two gabors with respect to a given Gabor parameter is equal to the inner product of a gabor atom with a derivative atom. That is, we can the exchange the derivative and inner product operators.

These derivative atoms have an analytic form. Note that the 1st derivative of probe Gabor atom with respect to the center time parameter t_p is

$$\frac{\partial}{\partial t_p} g_p(t) = z_{t_p}(t) g_p(t), \tag{47}$$

where

$$z_{t_p}(t) = 2\pi \left(e^{-s_p}\left(t - t_p\right) - i\nu_p\right), \tag{48}$$

The 1st derivative of probe Gabor atom with respect to the center frequency parameter ν_p is

$$\frac{\partial}{\partial \nu_p} g_p(t) = z_{\nu_p}(t) g_p(t), \tag{49}$$

where

$$z_{\nu_p}(t) = 2\pi i(t - t_p), \tag{50}$$

The 1st derivative of probe Gabor atom with respect to the duration parameter s_p is

$$\frac{\partial}{\partial s_p} g_p(t) = z_{s_p}(t) g_p(t), \tag{51}$$

where

$$z_{s_p}(t) = \pi e^{-s_p}(t - t_p)^2 - \frac{1}{4}, \tag{52}$$

Let m stand for numerically-measured inner product values. Note that

$$m_0, m_{x_p} : \mathbb{R}^3 \to \mathbb{C}, \tag{53}$$

$$m_0(\phi_p) = \langle s(t), g_p(t|\phi_k)\rangle = \langle s, g_p\rangle, \tag{54}$$

$$m_{x_p}(\phi_p) = \frac{\partial}{\partial x_p}\langle s(t), g_p(t|\phi_k)\rangle = \langle s, \frac{\partial}{\partial x_p} g_p\rangle. \tag{55}$$

Analytically, we have

$$\frac{\partial}{\partial x_p}\langle g_t, g_p\rangle = \zeta_{x_p}\langle g_t, g_p\rangle, \tag{56}$$

for

$$x_p \in \{t_p, \nu_p, s_p\}. \tag{57}$$

$$\zeta_{t_p} = \frac{2\pi \left(ie^{s_p}\nu_p + ie^{s_t}\nu_t + t_t - t_p\right)}{e^{s_p} + e^{s_t}}, \tag{58}$$

$$\zeta_{\nu_p} = \frac{2\pi e^{s_p} \left(e^{s_t}\left(\nu_t - \nu_p\right) - i\left(t_t - t_p\right)\right)}{e^{s_p} + e^{s_t}}, \tag{59}$$

$$\zeta_{s_p} = \frac{8\pi i \left(t_t - t_p\right) e^{s_p + s_t}\left(\nu_t - \nu_p\right) + e^{2s_t}\left(1 - 4\pi e^{s_p}\left(\nu_t - \nu_p\right)^2\right) + e^{s_p}\left(4\pi\left(t_t - t_p\right)^2 - e^{s_p}\right)}{4\left(e^{s_p} + e^{s_t}\right)^2}. \tag{60}$$

This allows us to define a reassignment function for each Gabor parameter:

$$(t, \nu, s) -> (t_t^*, \nu_t^*, s_t^*) \tag{61}$$

$$s_t^* = s_p + \log\left[\frac{2\pi e^{s_p}}{e^{s_p}\left(e^{s_p}\Re[\zeta_{t_p}]^2 - 4\pi\Re[\zeta_{s_p}] + \pi\right) - \Re[\zeta_{\nu_p}]^2} - 1\right], \tag{62}$$

$$\nu_t^* = \nu_p + \frac{\left(e^{-s_p} + e^{-s_t^*}\right)\Re[\zeta_{\nu_p}]}{2\pi}, \tag{63}$$

$$t_t^* = t_p + \frac{\left(e^{s_p} + e^{s_t^*}\right)\Re[\zeta_{t_p}]}{2\pi}. \tag{64}$$

where $\Re[z], \Im[z]$ stand for the real and imaginary parts of a complex number z.

Interestingly, not all derivatives are needed, since the derivative of the inner product of two Gabor atoms with respect to center time and center frequency are related, with the imaginary part of one correponding to the real part of the other. That is, the relation of ζ_{t_p} to ζ_{ν_p} is:

$$\Re\left[\zeta_{t_p}\right] = -\Im\left[\zeta_{\nu_p}\right] e^{-s_p}, \tag{65}$$

and

$$\Re\left[\zeta_{\nu_p}\right] = \Im\left[\zeta_{t_p}\right] e^{s_p} - 2\pi e^{s_p}\nu_p. \tag{66}$$

Thus, given an initial probe Gabor g_k that has a moderate inner product magnitude with an unknown signal s, MAGE provides parameter reassignment formulas that may represent a Gabor with a better fit to the unknown signal. Given a moderately good initial probe Gabor, the reassignment function $(t, \nu, s) -> (t_t^*, \nu_t^*, s_t^*)$ suggests the next test point in time-frequency-scale space.

1.6 From Matching Pursuit to the Multiscale Adaptive Gabor Expansion (MAGE)

This gives us an update for the Matching Pursuit algorithm to produce MAGE:

Multiscale Adaptive Gabor Expansion (MAGE) Algorithm:

$[\Phi, C, r] = \text{AGE}[s, N]$

Inputs:

s: unknown signal to be decomposed, $s \in L^2(\mathbb{R})$

N: number of signal atoms to extract from s

Outputs:

Φ: ordered list of Gabor atom parameters; $\Phi = \{\phi_k\}_{k=1}^N, \phi_k = (t_k, \nu_k, s_k) \in \mathbb{R}^3$

C: ordered list of coefficients; $C = \{c_k\}_{k=1}^N, c_k \in \mathbb{C}$

r: residual signal; $r = s - \sum_{k=1}^N c_k g_k = s(t) - \sum_{k=1}^N c_k g(t|\phi_k)$

Initialize:

$\Phi^{(0)} = \{\}$ (empty list)

$C^{(0)} = \{\}$ (empty list)

$r^{(0)} = s$

Main loop:

for $k = 1$ to N do

$\phi_k = \text{argmax}_\phi |\langle r^{(k-1)}, g(t|\phi)\rangle|$ (find parameters of best dictionary atom)

$(t_t^*, \nu_t^*, s_t^*) < -(t, \nu, s)$ (apply analytic parameter reassignment formula)

$g_k = g(t|\phi_k)$ (generate atom)

$c_k = \langle r^{(k-1)}, g_k\rangle$ (compute coefficient)

$r^{(k)} = r^{(k-1)} - c_k g_k$ (update residual)

$C^{(k)} = \{C^{(k-1)}, c_k\}$ (update coefficient list)

$\Phi^{(k)} = \{\Phi^{(k-1)}, \phi_k\}$ (update atom parameter list)

end for

return Φ, C, r

1.7 Conclusion

The goal of this paper is to introduce an improved version of fast refinement, similar to that suggested by [Yin et al, 2002]. As with their algorithm, MAGE is performing a curve-fitting operation in order to suggest an atom with a better fit to a signal to be filtered. The fast refinement algorithm of Yin et al uses finite differences to compute points for curve fitting. The spacing between points is an arbitrary hand-tuned parameter which MAGE does away with by using derivatives. Importantly, these derivatives are used to compute values for curve fitting, and are not used for an iterative hill-climbing optimization. Also, while the Yin algorithm uses 6 points for curve fitting, MAGE uses 4 points, which reduces the number of multily-add operations required.

1.8 Bibliography

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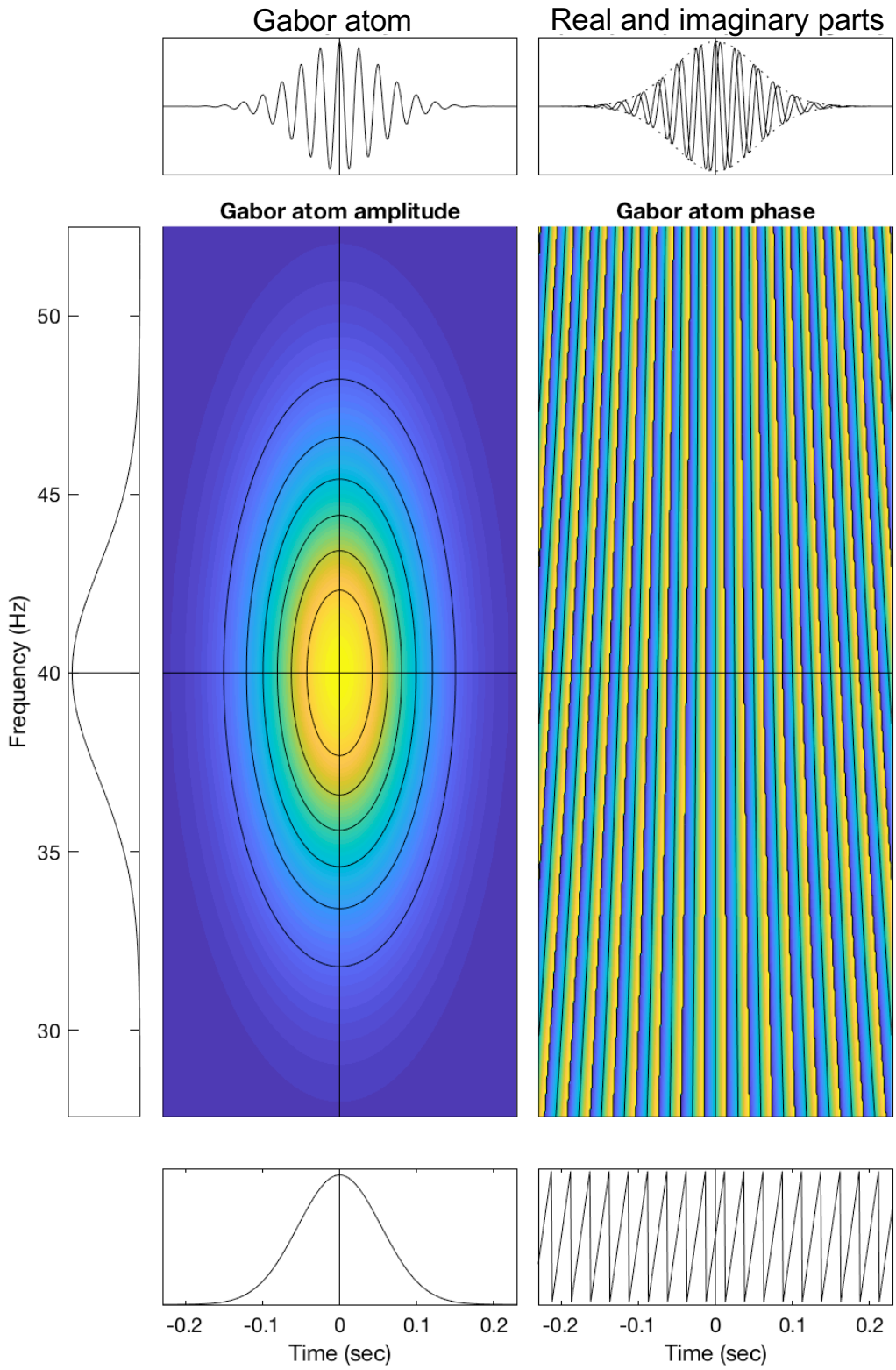

Figure 1. Gabor atom with center frequency of 40 Hz shown as time domain-traces (top) or amplitude envelope (bottom and side). Time-frequency amplitude and phase also shown.

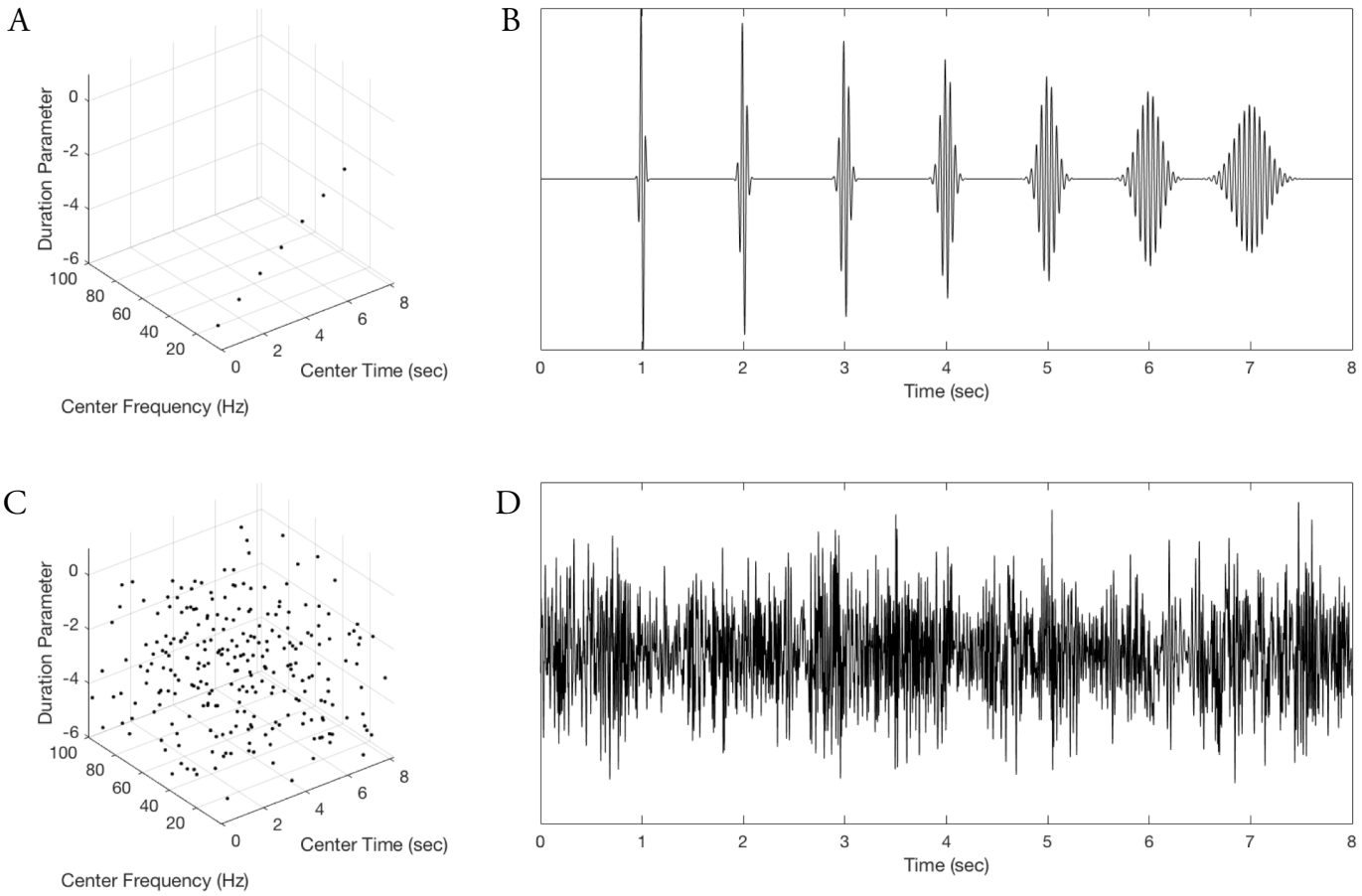

Figure 2. Example of Gabor atom parameter space and signal space. A) Parameter space (time, frequency, scale) for Gabor atoms. 7 atoms of different scales are shown. B) Signal representing sum of the 7 Gabor atoms. C) Parameter space values for 256 weighted Gabor atoms. D) Signal generated as a weighted sum of Gabor atoms given in 2c.

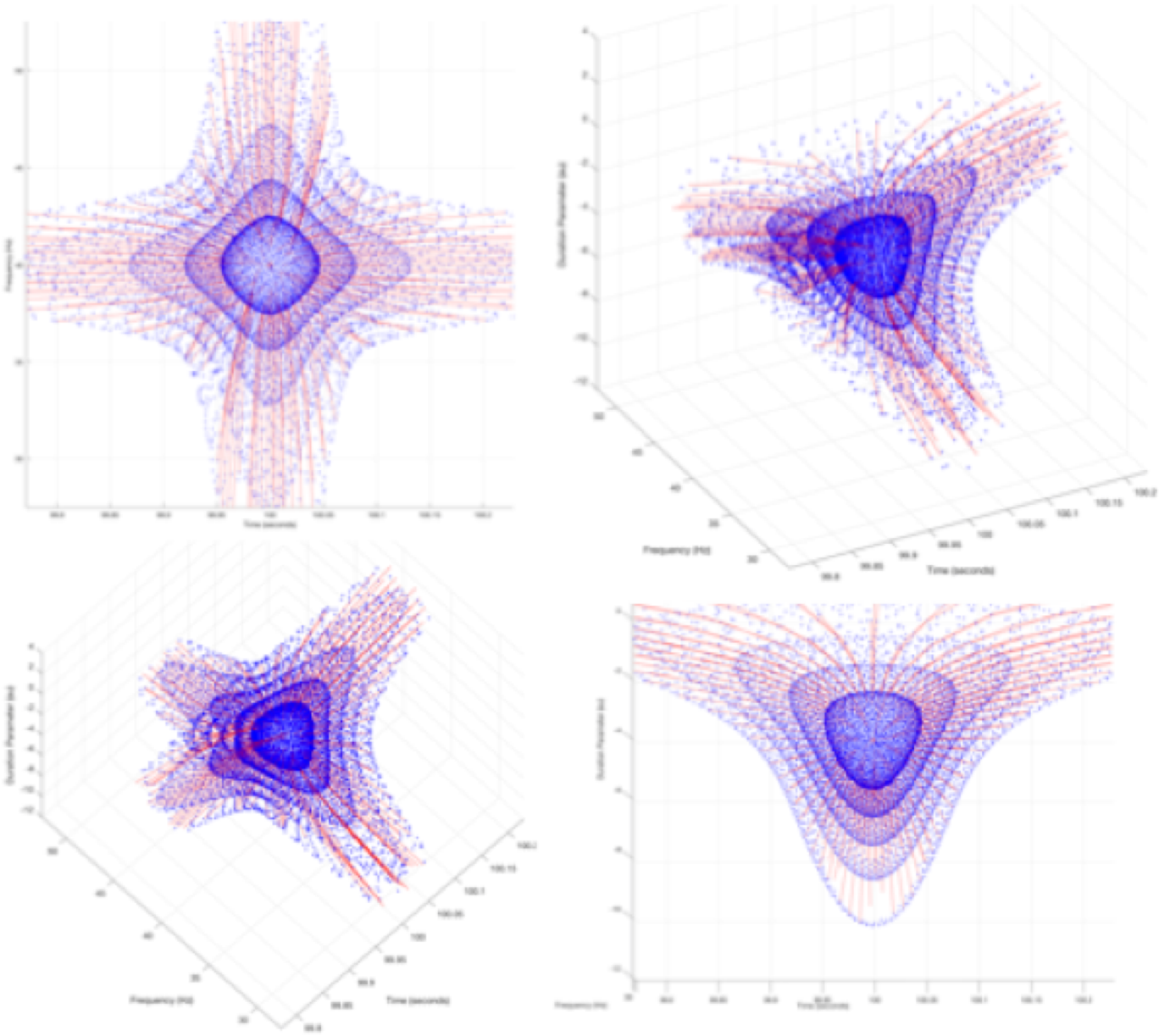

Figure 3. Concentric shells of parameter values for Gabor atoms that have a constant inner product amplitude (blue) and streamlines approaching the center (red) . 5 shells with inner product amplitude value are shown.

Figure 4.