# Damped oscillations of the probability of random events followed by absolute refractory period

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**Abstract**

Many events are followed by absolute refractoriness, when for some time after the event a repetition of a similar event is impossible. If uniform events, each of which is followed by the same period of absolute refractoriness, occur randomly, as in the Bernoulli scheme, then the event probability as a function of time exhibits damped oscillations. Here we give an exact analytical description of the oscillations, with the focus on applying in neuroscience.

**Keywords:** renewal point process, absolute refractory period, damped oscillations, neuron, stochastic spiking

## 1. Introduction

Many natural and technical events are followed by refractoriness, when for some time after the event a repetition of a similar event is unlikely (relative refractoriness) or even impossible (absolute refractoriness). A characteristic natural example is the neuron and the refractoriness of its ability to generate electrical impulses - spikes. In turn, a typical example from the technique is the existence of so-called dead time for some types of detectors (so-called the Type 1 counters [1]), especially for photodetectors. The dead time is a fixed period of time after the detector triggering during which it becomes inoperative. Both the neuronal refractoriness and the detector dead time influence essentially counting distributions. If instant events of the same type, each of which is followed by the same absolute refractory period $\tau_{ref}$, occur randomly, the average interval $T$ between the events can be represented as the sum $T = \tau_{ref} + T_0$, where $T_0$ is the value of the average interval if refractoriness is absent. A more nontrivial consequence of refractoriness are damped oscillations of the event occurrence probability as a function of time [2, 3]. Such oscillations arise due to the fact that random instant events ordered in time in the presence of refractoriness become dependent on each other. In particular, the probability of a subsequent event depends on the time elapsed from the preceding event. Random point processes of this kind are called renewal processes and are the subject of study of the renewal theory [1]. This theory has a recipe of getting an explicit analytical

description for the damped oscillations of the event probability by finding of so-called *renewal density* [2]. Specifically, this can be done for the classic example of the Poisson process modulated by the dead time or absolute refractory period [4–15]. However, the resulting analytical formula for the time-dependent probability of the event is quite cumbersome that complicates further analysis.

In this paper, for the Bernoulli process modulated by absolute refractoriness we give a compact analytical description of the damped oscillations without invoking the renewal theory. The description is presented in four equivalent forms (three kinds of a recurrence formula and one explicit formula) and is quantitatively consistent with both the results of numerical simulations and those of the renewal theory. One kind of the recurrence formula is especially simple, enabling accurate analytical calculation of the damping coefficients. Surprisingly, these are quite robust against changing the values of the model parameters.

Finally, ready-to-use MATLAB/Octave codes for performing simulations and plotting the graphs of the obtained formulas are included as supplementary material.

## 2. Formulation of the model problem

For certainty, consider a model neuron that stochastically emits spikes, each of which is followed by a period of absolute refractoriness. In particular, the neuron can spontaneously emit a spike with probability $p_s$ per unit time (i.e., in a given elementary time interval $\triangle t$) so that after the spike emission the neuron becomes temporarily inactive, i.e. it cannot emit spikes during the refractory period $\tau_{ref} = n_{ref}\triangle t$, where $n_{ref}$ is a positive integer. Then the mean rate of occurrence of events in the absence of refractoriness $\nu_0 = 1/T_0 = p_s/\triangle t$, and at $\tau_{ref} \neq 0$ from the equality $T = \tau_{ref} + T_0$ for the mean rate $\nu = 1/T$ one gets $\nu = \nu_0/(1 + \tau_{ref}\nu_0)$. In fact, this mean rate is equal to the ratio of the total number of spikes to observation time $T_{obs}$, given that $T_{obs} >> \tau_{ref}$, $T_0$. It is also useful to introduce the asymptotic value of the average probability of spike generation at each step, $\bar{p}_s = \nu\triangle t = p_s/(1 + n_{ref}p_s)$, such that, by the analogy with the formula for $\nu_0$,

$$\nu = \bar{p}_s/\triangle t = p_s/(\triangle t + p_s\tau_{ref}). \tag{1}$$

An algorithm for simulating the neuron's dynamics is extremely simple and as follows. Dividing the observation interval $T_{obs}$ by $N$ equal steps $\triangle t$, $T_{obs} = N\triangle t$, these time steps are numbered by a sequence of natural numbers starting with 1. At each step, a random number $\xi$, uniformly distributed from zero to one, is generated and compared with the given probability $p_s$ of generating a spike. If $\xi \leq p_s$, it is assumed that spike has been generated at this time step. After the spike generation, the neuron cannot emit a next spike during refractory period $\tau_{ref} = n_{ref}\triangle t$. The event

of spike generation at an arbitrary $k$-th step is further denoted by $A_k$. For definiteness, the initial state of the neuron is chosen as a moment when the neuron has just left the refractory state after emitting the spike that 'remained behind the scenes'.

Performing either a large number of repeated passes of the observation interval $T_{obs}$ for a single neuron or a single pass for the large ensemble of independent neurons, one gets a statistical distribution of the occurrence of events in the entire sequence of $N$ time intervals. Normalizing this distribution by the number of either the passes or the neurons in the ensemble, one obtains the probability distribution $P_k$ of spike generation at $k$-th elementary time step of the observation interval. Due to the refractory period, the event probability, as a function of time, exhibits damped oscillations with the average period equal to $\tau_{ref}$ (Fig. 1).

In turn, the analytical problem consists in finding the probability $P_k \equiv P(A_k)$ of spike generation at each $k$-th elementary time step so that in the asymptotic limit $k \to \infty$ one would obtain $P_k \to \bar{p}_s$.

## 3. Derivation of the exact analytical formula for $P_k$

The probability of spike generation at $k$-th time step ($k = \overline{1, N}$), $P_k = P(A_k)$, is equal to the product of $p_s$ and the probability that in the interval from $k - n_{ref}$ to $k - 1$ inclusively no spike was emitted,

$$P_k = p_s \cdot \left[1 - P(A_{k-n_{ref}} + A_{k-n_{ref}+1} + ... + A_{k-1})\right]. \tag{2}$$

Spike generation events in $n_{ref}$ consecutive time intervals are pairwise incompatible events. Therefore, according to the summation theorem for the probabilities of pairwise incompatible events, the probability of the sum in Eq. (2) equals the sum of probabilities

$$P(A_{k-n_{ref}} + A_{k-n_{ref}+1} + ... + A_{k-1}) = \sum_{j=k-n_{ref}}^{k-1} P(A_j), \tag{3}$$

and the sought-for probability at the $k$-th step is determined in a recurrent manner, with the recursion period equal to the refractory period,

$$P_k = p_s(1 - \sum_{j=k-n_{ref}}^{k-1} P_j) = p_s \left[1 - (S_{k-1} - S_{k-n_{ref}-1})\right], \tag{4}$$

where, by definition,

$$S_m = \begin{cases} 0, \ m \leq 0, \\ \sum_{j=1}^{m} P_j, \ m > 0. \end{cases} \tag{5}$$

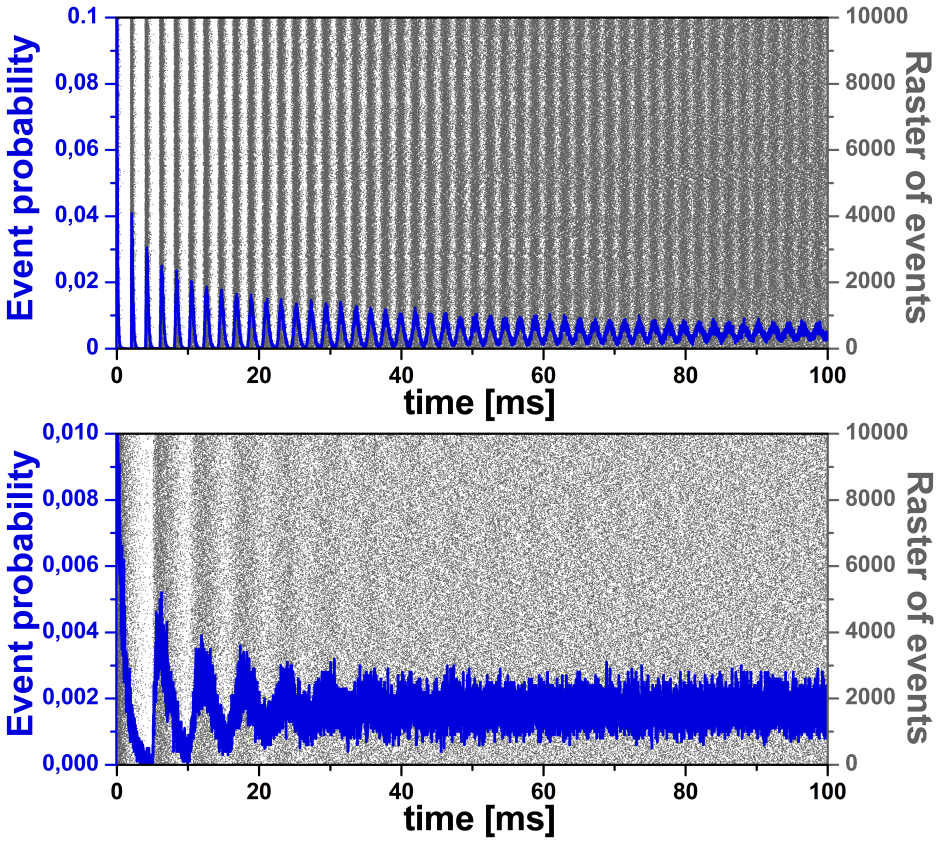


Figure 1: Numerical simulation of time dependence of the event probability, where an event is spike generation, for $10^4$ disconnected stochastically-spiking neurons at time step $\triangle t = 0.01$ ms. Top: Raster of events (gray dots, scale on the right) and the corresponding time dependence of the statistical probability of an event (blue line, scale on the left) at $p_s = 0.1$ and $\tau_{ref} = 2$ ms. It is seen that the period of damped oscillations of probability is $\tau_{ref}$. The asymptotic probability value to which the damped oscillations converge corresponds to the calculated value $\bar{p}_s = 4.76 \cdot 10^{-3}$ (or the average event frequency $\nu = 476$ Hz, see (1)). Bottom: Similar graphs for $p_s = 0.01$ and $\tau_{ref} = 5$ ms. For this case, $\bar{p}_s = 1.67 \cdot 10^{-3}$ and $\nu = 167$ Hz. Note that the parameters of the neuron model (in particular, nonphysiologically large value of $p_s$) are chosen solely for the illustrative purpose.

Such a definition of $S_m$ allows us to directly generalize the formula for $P_k$ to the range $1 \leq k \leq n_{ref}$. The closed formula for $S_k$ has the form of a linear recurrent sequence

$$S_k = P_k + S_{k-1} = p_s(1 + S_{k-n_{ref}-1}) + (1 - p_s)S_{k-1}. \tag{6}$$

The resulting formula (4) accurately describes the numerical statistics (Fig. 2).

One should note three important consequences.

First, for $1 \leq k \leq n_{ref} + 1$, where

$$S_k = p_s + (1 - p_s)S_{k-1},$$

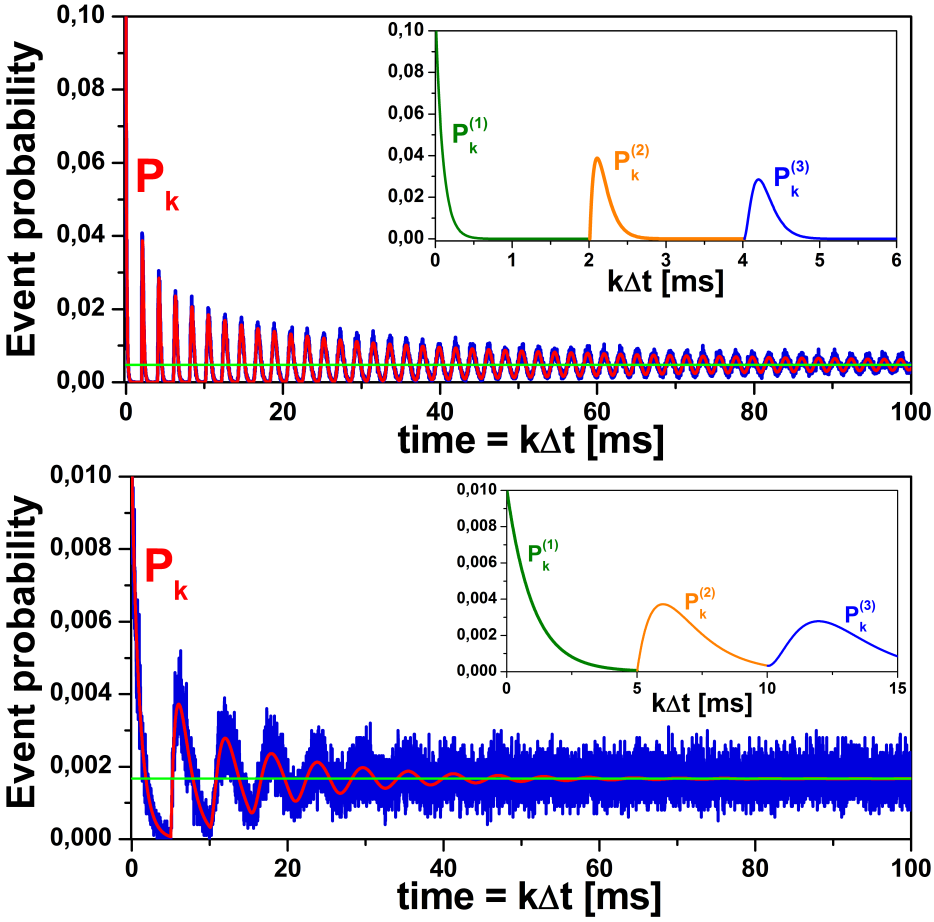


Figure 2: Comparison of the results of numerical simulations (see Fig. 1) with the calculation by mutually equivalent analytical formulas (4), (9), (15), (20) and (21). Top: Time dependence of the statistical probability of an event (blue line, data taken from the top panel of Fig. 1) and the corresponding analytic curve $P_k$ (red line) at $p_s = 0.1$ and $\tau_{ref} = 2$ ms. The asymptotic value $P_\infty = \bar{p}_s = 4.76 \cdot 10^{-3}$ is shown by the green horizontal line. Inset: Partial analytical curves for the first three refractory intervals calculated by formulas (7), (10) and (12). Bottom: Similar graphs (main graph and inset) for the parameter values $p_s = 0.01$ and $\tau_{ref} = 5$ ms, giving the asymptotic probability $P_\infty = 1.67 \cdot 10^{-3}$.

the sum $S_k$ can be easily found explicitly, as it is an arithmetic-geometric progression of the form

$$S_k = rS_{k-1} + d,$$

where $S_1 = p_s$, $r = 1 - p_s$ и $d = p_s$. According to the formula for the explicit form of the $k$-th term of this progression,

$$S_k = r^{k-1}\left[S_1 + \frac{d}{r-1}\right] - \frac{d}{r-1} = 1 - (1-p_s)^k.$$

Hence, the probability of spike generation in the interval $1 \leq k \leq n_{ref} + 1$ is equal to

$$P_k \equiv P_k^{(1)} = p_s \left[1 - S_{k-1}\right] = p_s(1 - p_s)^{k-1}. \tag{7}$$

Here and below, the upper index in parentheses indicates the number of the refractoriness interval, counted from the initial moment $t = 0$. Given the initial condition, this probability naturally coincides with the probability of the first spike generation at an arbitrary $k$-th step, such that $\sum_{k=1}^{\infty} P_k^{(1)} = 1$, and refers to the geometric distribution.

Second, the formula (4) makes it easy to obtain the asymptotic probability value $P_k$ at $k \to \infty$. The difference $S_{k-1} - S_{k-n_{ref}-1}$ contains $n_{ref}$ terms. At $k \to \infty$ the probability at the $k$-th step remains practically unchanged. Denoting it as $P_\infty$, from the general formula (4) one gets

$$P_\infty = p_s \left[1 - n_{ref} P_\infty\right],$$

whence

$$P_\infty = p_s/(1 + n_{ref} p_s). \tag{8}$$

Third, calculating the adjacent terms $P_{k+1}$ or $P_{k-1}$ similarly to the formula (6), one can exclude sums (5) from the formula (4) and obtain a linear recurrent sequence for $P_k$:

$$P_k = \begin{cases} p_s(1 - p_s)^{k-1}, \ k \leq n_{ref} + 1, \\ p_s P_{k-n_{ref}-1} + (1 - p_s) P_{k-1}, \ k > n_{ref} + 1. \end{cases} \tag{9}$$

This formula is completely equivalent with Eq. (4) and using it one can easily find the expression for $P_k$ in an explicit form within the intervals of $k$ multiples of $n_{ref} + 1$.

For example, at $n_{ref} + 1 \leq k \leq 2(n_{ref} + 1)$ one gets

$$P_k^{(2)} = (k - n_{ref} - 1) p_s^2 (1 - p_s)^{k - n_{ref} - 2} + P_k^{(1)} = P_k^{(1)} \cdot (1 + (k - n_{ref} - 1) q), \tag{10}$$

where $q = p_s(1 - p_s)^{-(n_{ref}+1)}$.

Next, at $2(n_{ref} + 1) \leq k \leq 3(n_{ref} + 1)$ one gets

$$P_k^{(3)} = \frac{1}{2}(k - 2n_{ref} - 2)(k - 2n_{ref} - 1) p_s^3 (1 - p_s)^{k - 2n_{ref} - 3} + P_k^{(2)} = \tag{11}$$

$$= P_k^{(1)} \cdot (1 + (k - n_{ref} - 1) q + \frac{1}{2}(k - 2n_{ref} - 2)(k - 2n_{ref} - 1) q^2). \tag{12}$$

It is worth noting that the numerical coefficient in the highest-order term with respect to $q$ is the so-called triangular number $j(j+1)/2$ at $j = k - 2n_{ref} - 2$. The inset in Fig. 2 shows the plots for $P_k^{(1)}$, $P_k^{(2)}$ and $P_k^{(3)}$.

Using the induction method, one can obtain a formula for $P_k^{(m+1)}$, valid within the range $m \cdot (n_{ref}+1) \leq k \leq (m+1) \cdot (n_{ref}+1)$:

$$P_k^{(m+1)} = (\frac{1}{m!}\prod_{j=1}^{m}(k - m \cdot n_{ref} - j))q^m P_k^{(1)} + P_k^{(m)}. \tag{13}$$

The general explicit expression for $P_k$ in the polynomial form is as follows:

$$P_k = P_k^{(1)} \cdot (1 + a_k^{(1)} p_s (1-p_s)^{-(n_{ref}+1)} + a_k^{(2)} p_s^2 (1-p_s)^{-2(n_{ref}+1)} + \ldots) = \tag{14}$$

$$= P_k^{(1)} \cdot (1 + \sum_{i=1}^{m} a_k^{(i)} q^i), \tag{15}$$

where $m$ is the integer part of the rational number $k/n_{ref}$ rounded off to a smaller value, $q = p_s(1-p_s)^{-(n_{ref}+1)}$, and the coefficients

$$a_k^{(m)} \equiv \frac{\theta(k - m \cdot (n_{ref}+1))}{m!}\prod_{j=1}^{m}(k - m \cdot n_{ref} - j), \tag{16}$$

where the unit step function $\theta(x) = 1$ at $x \geq 0$ and $\theta(x) = 0$ otherwise.

Taking into account the equality

$$\frac{1}{m!}\prod_{j=1}^{m}(n-j) = \frac{(m+1)}{n} C_n^{m+1} = C_{n-1}^m, \; n > m \geq 1, \tag{17}$$

where $C_n^k$ is the standard binomial coefficient,

$$C_n^k = \frac{n!}{k!(n-k)!} = \binom{n}{k}, \tag{18}$$

the coefficients $a_k^{(m)}$ are directly expressed through the binomial coefficients:

$$a_k^{(m)} = C_{k-m \cdot n_{ref}-1}^m \cdot \theta(k - m \cdot (n_{ref}+1)). \tag{19}$$

Another equivalent formula for $P_k$ can be found in a different way, as follows. Denote $P(A_k|A_j)$ the conditional probability of spike generation at the $k$-th step, provided that the previous spike was generated at the $j$-th step. At times greater than the refractory period, i.e. at $k > n_{ref}$, the probability of generating a subsequent spike depends only on the moment of a previous spike. Therefore, taking into account the initial condition, $P(A_k|A_j) = P_{k-j-n_{ref}}^{(1)}$, if $k - j > n_{ref}$, and $P(A_k|A_j) = 0$, if $k - j \leq n_{ref}$.

At $k > n_{ref}+1$, the probability $P_k$ can be written as the sum of two terms: the probability $P_k^{(1)}$ that a spike will be emitted for the first time at the $k$-th step and the probability that at least one

spike has been emitted previously. The latter has the form of a convolution and follows from the total probability formula.

$$P_k = P_k^{(1)} + \sum_{j=1}^{k-1} P_j \cdot P(A_k|A_j) = P_k^{(1)} + \sum_{j=1}^{k-n_{ref}-1} P_j \cdot P_{k-j-n_{ref}}^{(1)}. \tag{20}$$

Notably, using substitution $i = k - j - n_{ref}$, one can virtually swap the indices of the multipliers under the sign of the sum in (20), while the formula does not change its numerical value:

$$P_k = P_k^{(1)} + \sum_{i=1}^{k-n_{ref}-1} P_{k-i-n_{ref}} \cdot P_i^{(1)}. \tag{21}$$

Despite the different appearance in relation to the formulas (4) and (9), the formulas (20), (21) lead to the same numerical results and can be derived from the recurrent formula (9). In particular, the formula (21) can be straightforwardly obtained from (9) by successively substituting in the latter the values $P_{k-1}$, $P_{k-2}$, ... , $P_{k-n_{ref}-1}$ and taking into account the definition (7) for $P_k^{(1)}$.

## 4. Damping of the oscillations

The formulas (10) and (12) allow one to analytically calculate the relative damping of the second and third peaks of $P_k$. Finding the location of these peaks corresponds to solving a linear and quadratic algebraic equation, respectively. In particular, for $(n_{ref} + 1) \leq k \leq 2(n_{ref} + 1)$, calculating $dP_k^{(2)}/dk = 0$ by the explicit formula (10) we get the location of the second peak,

$$k_{\max 2} = n_{ref} + 1 + R, \tag{22}$$

where

$$R = 1/u - 1/q, \tag{23}$$

$$q = p_s(1 - p_s)^{-(n_{ref}+1)}, \tag{24}$$

$$u = \ln \frac{1}{(1 - p_s)}. \tag{25}$$

Substituting $k_{\max 2}$ into Eq. (10), we get the amplitude of the second peak

$$P_{\max}^{(2)} \equiv P_{k_{\max 2}}^{(2)} = P_{k_{\max 2}}^{(1)} \frac{q}{u} = \frac{p_s^2}{u}(1 - p_s)^{R-1}. \tag{26}$$

The damping can be traced simply by the ratio of amplitudes for the adjacent peaks,

$$D_{i+1} = P_{\max}^{(i+1)}/P_{\max}^{(i)}. \tag{27}$$

Given that $k_{\max 1} = 1$ and $P^{(1)}_{k_{\max 1}} = p_s$, for the second peak we get

$$D_2 = \frac{p_s}{u}(1-p_s)^{R-1}. \tag{28}$$

For the third refractoriness interval, $2(n_{ref}+1) \le k \le 3(n_{ref}+1)$, calculation of $dP^{(3)}_k/dk = 0$ by the formula (12) results in a quadratic equation,

$$xk^2 - yk - z = 0, \tag{29}$$

where

$$x = \frac{1}{2}q^2 u, \tag{30}$$

$$y = q^2 + (2n_{ref} + \frac{3}{2})q^2 u - qu, \tag{31}$$

$$z = -(2n_{ref} + \frac{3}{2})q^2 - (n_{ref}+1)(2n_{ref}+1)q^2 u + q + (n_{ref}+1)qu - u. \tag{32}$$

A suitable solution of this equation is the root

$$k_{\max 3} = \frac{y + \sqrt{y^2 + 4xz}}{2x} = 2(n_{ref}+1) + R + X, \tag{33}$$

where

$$X = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{u^2} - \frac{(2n_{ref}+1)}{q} - \frac{1}{q^2}}. \tag{34}$$

Substituting $k_{\max 3}$ into the formula (12) for $P^{(3)}_k$ yields

$$P^{(3)}_{k_{\max 3}} = P^{(1)}_{k_{\max 3}} \cdot \left(1 + (n_{ref} + 1 + R + X)q + \frac{1}{2}(R+X)(1+R+X)q^2\right). \tag{35}$$

After elementary but cumbersome calculations one can get a compact analytical expression for the damping coefficient of the third peak,

$$D_3 = P^{(3)}_{k_{\max 3}}/P^{(2)}_{k_{\max 2}} = p_s(1-p_s)^X \cdot (\frac{1}{u} + X + \frac{1}{2}). \tag{36}$$

Numerical calculations have confirmed the validity of the obtained formulas. Below we have also listed the numerical values of the relevant quantities, calculated by the above formulas, for the two examples shown in Figs. 1 and 2.

For the first example, at $\tau_{ref} = 2$ ms ($n_{ref} = \tau_{ref}/\triangle t = 200$ at $\triangle t = 0.01$ ms) and $p_s = 0.1$, we get $q \approx 1.6 \times 10^8$, $u \approx 0.1$, $R \approx 9.5$, $k_{\max 2} = 210$, $P^{(2)}_{k_{\max 2}} \approx 0.04$, $k_{\max 3} = 420$, $P^{(3)}_{k_{\max 3}} \approx 0.03$, $D_2 \approx 0.39$, and $D_3 \approx 0.74$.

For the second example, at $\tau_{ref} = 5$ ms ($n_{ref} = 500$) and $p_s = 0.01$, we get $q \approx 1.5$, $u \approx 0.01$, $R \approx 99$, $k_{\max 2} = 599$, $P^{(2)}_{k_{\max 2}} \approx 0.004$, $k_{\max 3} = 1196$, $P^{(3)}_{k_{\max 3}} \approx 0.003$, $D_2 \approx 0.37$, and $D_3 \approx 0.75$.

It is seen that the numerical values $D_2 \approx 0.4$ and $D_3 \approx 0.75$ are fairly robust against changing the parameters. Thus, the magnitude of the second peak is approximately equal to 40% of the magnitude of the first. In turn, the value of the third peak is approximately 75% of the value of the second or 30% of the value of the first peak.

## 5. Correspondence with the renewal theory

In the framework of the renewal theory [1, 2], when the intervals between events are independent random variables, knowing the distribution density of such intervals, one can find the time dependence of the event probability, provided that an event occurred at the initial moment of time. In particular, the sought-for probability is expressed through the so-called renewal density $h(t)$,

$$P_k^{(rd)} = h(k \cdot \triangle t) \cdot \triangle t. \tag{37}$$

In turn, the renewal density $h(t)$ is determined by the density $f(\tau)$ of the distribution of intervals $\tau$ between successive events [1, 2],

$$h(t) = \sum_{n=1}^{\infty} f_n(t), \tag{38}$$

where $f_1(t) \equiv f(t)$ and for $n \geq 2$ functions $f_n(t)$ are given by the recursive convolution

$$f_n(t) = \int_0^{+\infty} f_{n-1}(x) f(t-x) dx. \tag{39}$$

Qualitatively, functions $f_n(t)$ are the distribution densities of so-called $n$-th order intervals between events [2]: denoting as a first-order interval the elapsed time from some given event to the next following event, the second-order interval is defined as the elapsed time between the given event and the second following event, etc. An $n$-th order interval is therefore the sum of $n$ consecutive first-order intervals and is spanned by $(n+1)$ consecutive events.

At asymptotically large time $t \to +\infty$, $h(t)$ is saturated [1],

$$\lim_{t \to +\infty} h(t) = \rho. \tag{40}$$

Here $\rho$ is the mean rate of events, defined as the inverse mean interval between the successive events,

$$\rho^{-1} = \int_0^{+\infty} \tau f(\tau) d\tau. \tag{41}$$

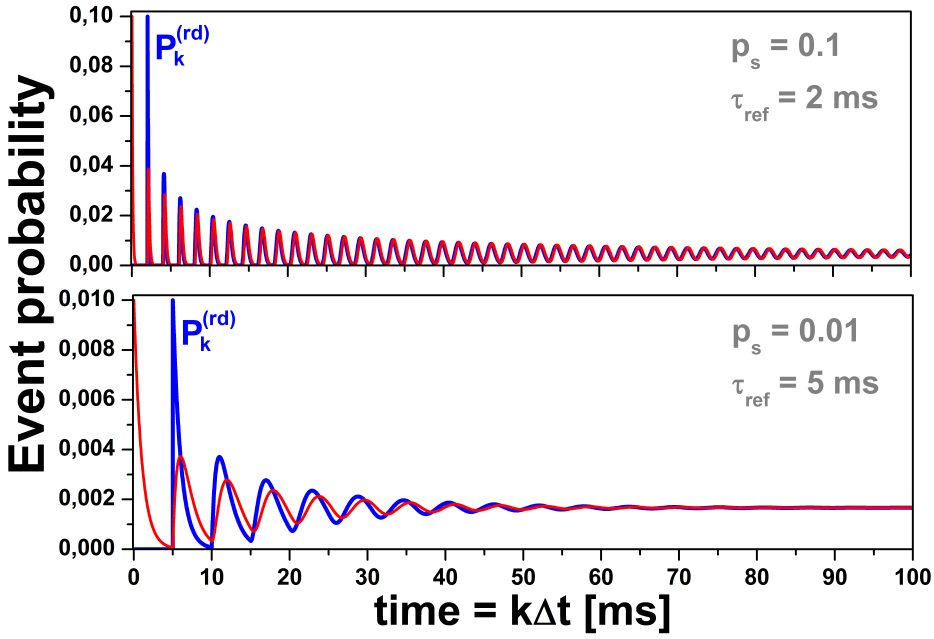


Figure 3: Comparison of the time dependencies for the event probability $P_k$ calculated by mutually equivalent formulas (4), (9), (15), (20), (21) in Section 3 (red curves) and the event probability $P_k^{(rd)}$ calculated within the renewal theory approach by formula (37) in Section 5 (blue curves). The red and blue curves, accurate to an offset equal to the refractoriness period, completely coincide. The offset arises due to the different initial condition (exit from the refractoriness period for the red curves and spike generation for the blue ones) and is left intentionally in order to make the curves distinguishable. Top plot is for $p_s = 0.1$ and $\tau_{ref} = 2$ ms. Bottom plot is for $p_s = 0.01$ and $\tau_{ref} = 5$ ms.

There are many, likely independent, examples of applying the renewal theory results to the case, where the events occurrence is the Poisson process modulated by the constant time of inoperativeness (dead time or, in our notations, absolute refractory period $\tau_{ref}$) or, equivalently, the distribution density of intervals between events has the form of a displaced exponential distribution,

$$f(t) = \nu_0 \cdot \exp(-\nu_0(t - \tau_{ref})) \cdot \theta(t - \tau_{ref}), \tag{42}$$

where $\nu_0 = p_s/\triangle t$ and $\theta(\ldots)$ is the Heaviside step function.

In particular, to the best of our knowledge, the first relevant paper dates back to 1947 [4] (see also Exercise No. 9 in [1], page 129) and has been followed by in-depth studies, classic [5–8] and more recent [11–13], and the applications to neuroscience [2, 9, 10, 14–18] (see also [19] and [20]). Below, we briefly outline and compare the previous results with our findings.

For $f(t)$ given by (42), using the Laplace transform, one can reduce the formula (39) to the

following expression

$$f_n(t) = \nu_0 \cdot \frac{(\nu_0(t - n\tau_{ref}))^{n-1}}{(n-1)!} \cdot \exp(-\nu_0(t - n\tau_{ref})) \cdot \theta(t - n\tau_{ref}), \tag{43}$$

which is the probability density function for the Erlang/gamma distribution, enabling to compute the renewal density $h(t)$ and the sought-for probability (37). The plot of the function $P_k^{(rd)}$ with

$$h(t) = \sum_{n=1}^{n_{\max}} f_n(t), \tag{44}$$

where $n_{\max} = 10^3$ and $f_n(t)$ is determined by (43), is shown in Fig. 3. The time dependence, accurate to an offset equal to the refractoriness period, completely coincides with that of $P_k$ calculated by mutually equivalent formulas (4), (9), (15), (20), (21) in Section 3. The offset arises due to the different initial condition: an exit from the refractoriness period at $t = 0$ in our model and an event occurrence at $t = 0$ in the standard renewal-density approach.

Finally, using (40), (41) and (42), one gets the asymptotic value of $h(t)$,

$$\lim_{t \to +\infty} h(t) = \nu_0/(1 + \nu_0\tau_{ref}), \tag{45}$$

which is consistent with formulas (1) and (8) for $\nu$ and $P_\infty$, respectively.

## 6. Conclusion

The model considered in this paper is a Bernoulli scheme supplemented by the condition of absolute refractoriness. This formally refers to the renewal theory. However, being quite simple, the model allows obtaining useful results without invoking this formalism. In particular, four equivalent analytical descriptions of the damped oscillations of the event probability have been given: 1) a recurrent formula through the difference of two sums, 2) a closed recurrent formula, 3) an explicit formula in the form of a polynomial, and 4) a recurrent convolution-type formula. It has also been shown that these results accurately coincide with the one of the renewal theory. Finally, using the closed recurrent formula, the relative damping coefficients for the second and third peaks of the event probability have been found in exact analytical form.

It should be noted that an analogous model was previously briefly discussed in [21], where a recurrent formula for the event probability, similar to the formula (4), was presented without derivation.

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