Complex dynamics in an SIS epidemic model induced by nonlinear incidence

Ruixia Yuan¹ , Zhidong Teng² , Jinhui Li¹ *

1 School of Mathematics and Statistics, Central China Normal University, Wuhan, Hubei 430079, China

2 College of Mathematics and system Sciences, Xinjiang University, Urumqi, Xinjiang Uygur Autonomous Region 830046, China

These authors contributed equally to this work.

* lijinhui2050@163.com

Abstract

We study an epidemic model with nonlinear incidence rate, describing the saturated mass action as well as the psychological effect of certain serious diseases on the community. Firstly, the existence and local stability of disease-free and endemic equilibria are investigated. Then, we prove the occurrence of backward bifurcation, saddle-node bifurcation, Hopf bifurcation and cusp type Bogdanov-Takens bifurcation of codimension 3. Finally, numerical simulations, including one limit cycle, two limit cycles, unstable homoclinic loop and many other phase portraits are presented. These results show that the psychological effect of diseases and the behavior change of the susceptible individuals may affect the final spread level of an epidemic.

Introduction

In generic, the population, for an SIS model, is always separated into two compartments, susceptible and infective individuals. And, in most SIS epidemic models (see Anderson and May [2]), the incidence takes the mass-action form with bilinear interactions. However, in practical, to describe the transmission process more realistic, it is necessary to introduce the nonlinear contact rates [1].

Actually, various forms of nonlinear incidence rates have been proposed recently. For example, in order to incorporate the effect of behavioral changes, Liu, Levin, and Iwasa [3] used a nonlinear incidence rate of the form

$$g(I)S = \frac{\kappa I^\iota S}{1 + \alpha I^h},$$

where κI^ι represents the force of infection of the disease, $1/(1 + \alpha I^h)$ is a description of the suppression effect from the behavioral change of susceptible individuals when the infective population increases. ι, h and κ are all positive constants, and α is a nonnegative constant. See also Hethcote and van den Driessche [4], Moghadas [5] and Alexander and Moghadas [6, 7], etc.

In [10], Ruan discuss a specific infection force

$$g_1(I) = \frac{\kappa I}{1 + \alpha I^2},$$

Fig 1. Graphs of different incidence rate functions. (a) Nonmonotone incidence $g_1(I) = \frac{\kappa I}{1+\alpha I^2}$ ($\kappa = \alpha = 1$). (b) Nonlinear incidence $g_2(I) = \frac{\kappa I^2}{1+\alpha I^2}$ ($\kappa = \alpha = 1$). (c) Nonlinear and nonmonotone incidence rate $G(I) = \frac{aI^2}{I^2+c} + \frac{bI}{I^2+c}$ ($a = c = 1, b = 0.5$).

see Fig. 1(a), which can describe the effects of psychology function caused by protection measures and intervention policies. Obviously, $g_1(I)$ is increasing with small I and decreasing with large I , that is, $g_1(I)$ is non-monotone. The function $g_1(I)$ can be used to interpret the psychological effect: for a very large number of infective individuals, the infection force may decrease as the number of infective individuals increases, since large number of infective may lead to the reducing of the number of contacts per unit time. For example, in 2003, epidemic outbreak of severe acute respiratory syndrome (SARS) had such psychological effects on the general public (see [11]), and aggressive measures and policies had been taken, such as border screening, mask wearing, quarantine, isolation, etc. And also [11] showed that either the number of infective individuals tends to zero as time evolves or the disease persists.

Furthermore, Li, Zhao and Zhu (see [20]) studied the following SIS model, which describes behavior change effect of susceptible individual when infectious population increases

$$\begin{cases} \frac{dS}{dt} = \Lambda - d_0 S - g_2(I)S + \delta I, \\ \frac{dI}{dt} = g_2(I)S - (d_0 + d' + \delta)I, \end{cases} \quad (1)$$

where

$$g_2(I) = \frac{\alpha I^2}{1 + \beta_0 I^2},$$

see Fig. 1(b). By the qualitative and bifurcation analyses, they showed that the maximal multiplicity of weak focus is 2, and proved that the model can undergo a Bogdanov-Takens bifurcation of codimension 2. These results illustrate that the behavior change of the susceptible individuals may affect the final spread level of an epidemic.

Actually, both the effect of psychological and behavior affect of susceptible individuals have influence on the transmission of the disease. Thus, motivated by the above researches, we consider a nonlinear incidence rate of a SIS model as follows

$$\begin{cases} \frac{dS}{dt} = \Lambda - SG(I) - dS + \sigma I, \\ \frac{dI}{dt} = SG(I) - (d + \mu + \sigma)I, \end{cases} \quad (2)$$

where

$$G(I) = \frac{aI^2}{c + I^2} + \frac{bI}{c + I^2},$$

see Fig. 1(c), which can describe the effect of psychology and behavior change of susceptible individuals. $S(t)$ and $I(t)$ represent the number of susceptible individuals and infected individuals at time t , respectively. Λ is the recruitment rate of population, d is the natural death, μ is the disease-induced death rate, and σ represents the recovered rate, a, b and c are all positive constants.

The organization of this paper is as follows. First of all, we analyze the existence of equilibria and local stability of equilibria. Then, we study the existence of Hopf bifurcation around the positive equilibrium at the critical value under the conditions of $R_0 < 1$ and $R_0 > 1$. We also show that these positive equilibria can be weak focus for some parameter values and a cusp type of Bogdanov-Takens bifurcation of codimension 3. Finally, we give some brief conclusion.

Types and Stability of equilibria

In this section, our aim is to perform an elaborative analysis of equilibria for system (2).

Firstly, for convenience, we make scalings:

$(\Lambda', a', d', \sigma', c') = (\frac{\Lambda}{b}, \frac{a}{d+\mu+\sigma}, \frac{db}{d+\mu+\sigma}, \frac{\sigma}{d+\mu+\sigma}, \frac{c(d+\mu+\sigma)^2}{b^2})$, and $(x, y, \tau) = (\frac{d+\mu+\sigma}{b}S, \frac{d+\mu+\sigma}{b}I, (d+\mu+\sigma)t)$. To avoid the abuse of mathematical notation, we still denote $(\Lambda', a', d', \sigma', c', \tau)$ by $(\Lambda, a, d, \sigma, c, t)$. Then model (2) becomes

$$\begin{cases} \frac{dx}{dt} = \Lambda - \frac{xy(ay+1)}{y^2+c} - dx + \sigma y, \\ \frac{dy}{dt} = \frac{xy(ay+1)}{y^2+c} - y, \end{cases} \quad (3)$$

with $d + \sigma < 1$.

Lemma 1 *The set $D = \{(x, y) | x \geq 0, y \geq 0, x + y \leq \frac{\Lambda}{d}\}$ is an invariant manifold of system (3), which is attracting in the first octant of $\mathbb{R}^2$.*

Proof Summing up the two equations in (3), we can get

$$\frac{d(x+y)}{dt} = \Lambda - d(x+y) - (1-d-\sigma)y \leq \Lambda - d(x+y).$$

Thus, $\limsup_{t \rightarrow \infty} (x+y) \leq \frac{\Lambda}{d}$, which implies the conclusion.

Obviously, system (3) always has a unique disease-free equilibrium $E_0 = (\frac{\Lambda}{d}, 0)$. The positive equilibria of (3) can be gained by solving the following algebraic equations

$$\Lambda - \frac{xy(ay+1)}{y^2+c} - dx + \sigma y = 0, \quad \frac{xy(ay+1)}{y^2+c} - y = 0,$$

which yields

$$(d + a(1 - \sigma))y^2 + (1 - \sigma - a\Lambda)y + dc - \Lambda = 0. \quad (4)$$

Denote the basic reproduction number as follows,

$$R_0 = \frac{\Lambda}{dc}.$$

Computing the discriminant of (4), we get

$$\Delta = [1 - \sigma - a\Lambda]^2 - 4(d + a(1 - \sigma))(dc - \Lambda).$$

Furthermore, define

$$R^* = 1 - \frac{[(1 - \sigma) - a\Lambda]^2}{4dc(d + a(1 - \sigma))},$$

then $\Delta > 0$ if and only if $R_0 > R^*$, $\Delta = 0$ if and only if $R_0 = R^*$ and $\Delta < 0$ if and only if $R_0 < R^*$. It is obvious that $R^* < 1$ and we can obtain the following theorem.

Theorem 1 Model (3) always has a disease-free equilibrium E_0 and the following conclusions hold.

- (i) When $R_0 < 1$, we have
 - (a) if $R_0 < R^*$, then system (3) has no positive equilibrium;
 - (b) if $R_0 = R^*$ and $\Lambda > (1 - \sigma)/a$, then system (3) has a unique positive equilibrium $E_1(x_1, y_1)$, where $x_1 = \frac{y_1^2 + c}{ay_1 + 1}$ and $y_1 = \frac{a\Lambda - (1 - \sigma)}{d + a(1 - \sigma)}$;
 - (c) if $R_0 > R^*$ and $\Lambda > (1 - \sigma)/a$, then system (3) has two positive equilibria $E_2(x_2, y_2)$ and $E_3(x_3, y_3)$, where $x_k = \frac{y_k^2 + c}{ay_k + 1}$ ($k = 2, 3$) and $y_2 = \frac{a\Lambda - (1 - \sigma) - \sqrt{\Delta}}{2(d + a(1 - \sigma))}$, $y_3 = \frac{a\Lambda - (1 - \sigma) + \sqrt{\Delta}}{2(d + a(1 - \sigma))}$;
- (ii) When $R_0 = 1$ and $\Lambda > (1 - \sigma)/a$, then system (3) has a unique positive equilibrium $E_4(x_4, y_4)$, where $x_4 = \frac{y_4^2 + c}{ay_4 + 1}$ and $y_4 = \frac{a\Lambda - (1 - \sigma)}{2(d + a(1 - \sigma))}$;
- (iii) When $R_0 > 1$, then system (3) has a unique positive equilibrium $E_5(x_5, y_5)$, where $x_5 = \frac{y_5^2 + c}{ay_5 + 1}$ and $y_5 = \frac{a\Lambda - (1 - \sigma) + \sqrt{\Delta}}{2(d + a(1 - \sigma))}$.

In the following, we discuss the local stability of $E_k(x_k, y_k)$ ($k = 0, 1, 2, 3, 4, 5$) and present the corresponding phase portrait. By directly calculating, the Jacobian matrix at equilibrium E_k is

$$J_k = \begin{pmatrix} -d - \frac{y_k(ay_k + 1)}{y_k^2 + c} & \sigma - \frac{-y_k^2 + 2acy_k + c}{(y_k^2 + c)^2} x_k \\ \frac{y_k(ay_k + 1)}{y_k^2 + c} & -1 + \frac{-y_k^2 + 2acy_k + c}{(y_k^2 + c)^2} x_k \end{pmatrix}.$$

We have

$$Tr J_k = \frac{\psi_k}{(y_k^2 + c)(ay_k + 1)}, \quad Det J_k = \frac{y_k \phi_k}{(y_k^2 + c)(ay_k + 1)},$$

where

$$\begin{aligned} \psi_k &= -a(d + 1 + a)y_k^3 - (d + 2 + 2a)y_k^2 - (acd + 1 - ac)y_k - cd, \\ \phi_k &= a(d + a(1 - \sigma))y_k^2 + 2(d + a(1 - \sigma))y_k + (1 - \sigma) - dac. \end{aligned}$$

In addition, after some complicated computations, we get

$$\phi_2 = \frac{\sqrt{\Delta}(a(\sqrt{\Delta} - (a\Lambda + 1 - \sigma)) - 2d)}{2(d + a(1 - \sigma))}$$

and

$$\phi_k = \frac{\sqrt{\Delta}(a(\sqrt{\Delta} + (a\Lambda + 1 - \sigma)) + 2d)}{2(d + a(1 - \sigma))} > 0, \quad (k = 3, 4, 5).$$

Theorem 2 The disease free equilibrium E_0 of system (3) is

- (i) an attracting node if $R_0 < 1$;
- (ii) a hyperbolic saddle if $R_0 > 1$;
- (iii) a saddle-node of codimension 1 if $R_0 = 1$ and $2a - \frac{1 - \sigma}{dc} \neq 0$; a repelling node if $R_0 = 1$ and $2a - \frac{1 - \sigma}{dc} = 0$.

Proof Obviously, at equilibrium E_0 , we have $\det(J_0) = d(1 - R_0)$ and $\text{Tr}(J_0) = -d - (1 - R_0)$. Therefore, E_0 is a stable node if $R_0 < 1$ and a hyperbolic saddle if $R_0 > 1$, and degenerate if $R_0 = 1$.

When $R_0 = 1$, we let $u = x - \frac{\Lambda}{d}$, $v = y$, then system (3) becomes

$$\begin{cases} \frac{du}{dt} = -du - (1 - \sigma)v - \frac{1}{c}uv - 2av^2 + \mathcal{O}(|(u, v)|^3), \\ \frac{dv}{dt} = 2av^2 + \frac{1}{c}uv + \mathcal{O}(|(u, v)|^3). \end{cases} \quad (5)$$

Indeed, if $R_0 = 1$, the Jacobian J_0 is diagonalizable with eigenvalues $\lambda_1 = -d$ and $\lambda_2 = 0$ and respective eigenvectors $\nu_1 = (1, 0)$ and $\nu_2 = (-\frac{1-\sigma}{d}, 1)$. By the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1-\sigma}{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

system (5) can be rewritten as

$$\begin{cases} \frac{dx}{dt} = -dx - (\frac{1}{c} + \frac{1-\sigma}{d})xy - (\frac{1-\sigma}{cd} - 2a + \frac{2a(1-\sigma)}{d})y^2 + \mathcal{O}(|(x, y)|^3), \\ \frac{dy}{dt} = (2a - \frac{1-\sigma}{dc})y^2 + \frac{1}{c}xy + \mathcal{O}(|(x, y)|^3). \end{cases}$$

If $2a - \frac{1-\sigma}{dc} \neq 0$, according to the calculation of center manifold [15], we know that the center manifold $x = h(y)$ of (5) begins with quadratic term of y . In addition, from the second equation of (5), we can easily obtain that the equation restricted to the center manifold as follows

$$\frac{dy}{dt} = (2a - \frac{1-\sigma}{dc})y^2 + \mathcal{O}(y^3).$$

By applying the Theorem 7.1 in Zhang et al. [15], E_0 is a saddle-node.

If $2a - \frac{1-\sigma}{dc} = 0$, then the center manifold turns into

$$\frac{dy}{dt} = \frac{1}{c}y^3 + \mathcal{O}(y^4).$$

Since $\frac{1}{c} > 0$ and the first nonzero item is uneven. Thus, the equilibrium E_0 is a repelling node, according to Theorem 7.1 in Zhang et al. [15].

From the expression of ψ_1 and ϕ_1 , we can see that one of the eigenvalues of characteristic matrix of E_1 is zero and the other is nonzero if $\psi_1 \neq 0$. The type of E_1 can be directed proved by checking the conditions in Zhang et al. ([15], Theorem 7.1-7.3). So, we have the following results.

Theorem 3 *If $R_0 = R^*$, then system (3) has a unique positive equilibrium E_1 . More Precisely,*

- (a) *if $\psi_1 \neq 0$, then E_1 is a saddle-node;*
- (b) *if $\psi_1 = 0$, then E_1 is a cusp.*

Theorem 4 *Suppose that $R^* < R_0 < 1$ and $a\Lambda > 1 - \sigma$, then system (3) has two positive equilibria E_2 and E_3 , and equilibrium E_2 is a hyperbolic saddle for all permissible choices of parameters, equilibrium E_3 is not degenerate. Moreover,*

- (i) *E_3 is a stable focus or node if $\psi_3 < 0$;*
- (ii) *E_3 is a weak focus or center if $\psi_3 = 0$;*
- (iii) *E_3 is an unstable focus or node if $\psi_3 > 0$.*

Proof Note that ϕ_2 is less than zero since $\Delta < (a\Lambda + 1 - \sigma)^2$, then E_2 is a hyperbolic saddle for any choices of parameters. And at E_3 , we have $\phi_3 > 0$. Thus, the stability of equilibrium E_3 depends on the sign of ψ_3 .

Theorem 5 *When $R_0 = 1$ and $c > \frac{1-\sigma}{a}$, then system (3) has a unique positive equilibrium $E_4(x_4, y_4)$, and the equilibrium E_4 is stable if $\psi_4 < 0$.*

Proof In fact, when $R_0 = 1$ and $c > \frac{1-\sigma}{a}$, then $\phi_4 > 0$. Thus, the stability of E_4 is determined by the sign of ψ_4 .

Theorem 6 Assume $R_0 > 1$, then system (3) has a unique positive equilibrium E_5 . Moreover,

- (i) E_5 is stable if $\psi_5 < 0$;
- (ii) E_5 is a weak focus or center if $\psi_5 = 0$;
- (iii) E_5 is unstable if $\psi_5 > 0$.

Proof Obviously, when $R_0 > 1$, then $\phi_5 > 0$, and then E_5 is stable if $\psi_5 < 0$.

Lemma 2 From the expression of ψ_k ($k = 1, 3, 5$), we can obtain that E_k ($k = 1, 3, 5$) is always stable if $d \geq 1 - \frac{1}{ac}$.

When $\psi_i \neq 0$ ($i = 3, 5$), the dynamics of system (3) can be easily seen in Fig. 2, Fig. 3 and Fig. 4, respectively. The dynamical behaviors of system (3) when $\psi_i = 0$ ($i = 3, 5$) will be discussed in detail in the next section.

Fig 2. When $R^* < R_0 < 1$, $\Lambda > (1 - \sigma)/a$. (a): $\psi_3 < 0$, equilibria E_0 and E_3 are locally stable and E_2 is unstable. (b): $\psi_3 > 0$, equilibrium E_0 is locally stable and E_2 and E_3 are unstable.

Remark 1 In fact, Fig. 2 (a) shows the occurrence of bi-stability, in which solution may converge to one of the two equilibria, depending on the initial conditions. And in practical, this interesting phenomenon implies that initial states determine whether the disease can dies out or not.

Remark 2 From Fig. 2 (a), we can see there exists two seperatrices. All solutions tend to the disease-free equilibrium E_0 except the two green lines tend to equilibrium E_2 .

Theorem 7 Suppose that $R_0 = 1$ and $\Lambda > \frac{1-\sigma}{a}$, then system (3) has a unique positive equilibrium E_4 . If $\psi_4 > 0$, then there exists at least one stable limit cycle in the interior of the first quadrant.

Proof Indeed, the Jacobian J_0 is diagonalizable with eigenvalues $\lambda_1 = -d$ and $\lambda_2 = 0$ and respective eigenvectors $\nu_1 = (1, 0)$ and $\nu_2 = (-\frac{1-\sigma}{d}, 1)$. By the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1-\sigma}{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

Fig 3. When $R_0 > 1$. (a): $\psi_5 < 0$, the disease-free equilibrium E_0 is unstable and the unique positive equilibrium E_5 is globally stable. (b): $\psi_5 > 0$, E_0 and E_5 are unstable and there exists a stable limit cycle.

Fig 4. When $R_0 = 1$, and $\Lambda > (1 - \sigma)/a$. (a): $\psi_4 < 0$, the disease-free equilibrium E_0 is unstable and the unique positive equilibrium E_4 is globally stable. (b): $\psi_4 > 0$, E_0 and E_4 are unstable and there exists a stable limit cycle.

system (5) becomes

$$\begin{cases} \frac{dx}{dt} = -dx - \left(\frac{1}{c} + \frac{1-\sigma}{d}\right)xy - \left(\frac{1-\sigma}{cd} - 2a + \frac{2a(1-\sigma)}{d}\right)y^2 + \mathcal{O}(|(x,y)|^3), \\ \frac{dy}{dt} = \left(2a - \frac{1-\sigma}{dc}\right)y^2 + \frac{1}{c}xy + \mathcal{O}(|(x,y)|^3). \end{cases} \quad (6)$$

The theorem of Chocitaichvili [21] yields directly that system (6) is topologically equivalent to the system:

$$\begin{cases} \frac{dx}{dt} = -dx, \\ \frac{dy}{dt} = \left(2a - \frac{1-\sigma}{dc}\right)y^2 + \mathcal{O}(|y|^3). \end{cases}$$

It is easy to see that $2a - \frac{1-\sigma}{dc} > 0$ according to the condition of $R_0 = 1$ and $\Lambda > \frac{1-\sigma}{a}$, and then we can gain that there exist a unique repelling equilibrium E_4 in the region

D_1 shown in (a) of Fig.5. Consequently, by the Poincaré-Bendixson theorem, at least one stable limit cycle appears in the interior of the first quadrant. Similarly, when $\psi_5 > 0$, we have the following result.

Theorem 8 Suppose that $R_0 > 1$. If $\psi_5 > 0$, then there exists at least one stable limit cycle in the interior of the first quadrant.

Proof Indeed, the Jacobian J_5 has eigenvalues $\lambda_1 = -d$ and $\lambda_2 = R_0 - 1 > 0$, when $R_0 > 1$. Thus, we can gain that there exist a E_5 , which is the unique repelling equilibrium in the region D_2 shown in (b) of Fig.5. Consequently, by the Poincaré-Bendixson theorem, at least one stable limit cycle appears in the interior of the first quadrant.

Fig 5. When $R_0 = 1$, and $\Lambda > (1 - \sigma)/a$. (a): When $R_0 = 1$, $\Lambda > \frac{1-\sigma}{a}$ and $\psi_4 > 0$, there exists a stable limit cycle. (b): When $R_0 > 1$ and $\psi_5 > 0$, there exists a stable limit cycle.

Remark 3 From (b) of Fig. 3, we can see clearly that there exist a stable limit cycle enclosing the equilibrium E_5 even though E_0 is a saddle-node. Similarly, from (b) of Fig. 4, we can also gain that there exists a stable limit cycle if E_4 is unstable.

Bifurcations

Backward bifurcation

Theorem 9 When $R_0 = 1$ and $\Lambda > \frac{1-\sigma}{a}$, model (3) exhibits a backward bifurcation at equilibrium E_0 .

Remark 4 When $R_0 = 1$ and $\Lambda > \frac{1-\sigma}{a}$, system (3) exhibits a unique positive equilibrium E_4 , which means that once R_0 crosses 1, the disease can invade to a relatively high level. And this is one of the main characters of backward bifurcation [19].

Remark 5 Actually, backward bifurcation did not emerge with $a = 0$, which is considered in [20]. This indicates that introducing the non-monotonic incidence into model (1) makes the epidemic model more complex and exhibit richer dynamical behaviors.

Hopf bifurcation

In this subsection, we will study the Hopf bifurcation of system (3) for (i) $R^* < R_0 < 1$ and $\Lambda > (1 - \sigma)/a$; (ii) $R_0 > 1$. From the discussion in Section 2, it can be seen that Hopf bifurcation may occur at E_3, E_5 . As the expressions of equilibria E_3 and E_5 are same, no considering the values of every parameters. Based on Theorem 4, Theorem 5 and Theorem 6, we know that the stability of E_3 and E_5 are similar and when $\psi_k = 0$, $E_k (k = 3, 5)$ is a weak focus or center. Thus, we show the existence of Hopf bifurcation around $E_k (k = 3, 5)$.

Theorem 10 Suppose $E_k (k = 3, 5)$ exist, then model (2) undergoes a Hopf bifurcation at equilibrium E_k if $\psi_k = 0$. Moreover,

- (a) if $\eta < 0$, there is a family of stable periodic orbits of model (3) as ψ_k decreases from 0;
- (b) if $\eta = 0$, there are at least two limit cycles in (3), where η will be defined below;
- (c) if $\eta > 0$, there is a family of unstable periodic orbits of (3) as ψ_k increases from 0.

Proof From the above discussions, we can see that $\text{tr} J_k = 0$ if and only if $\psi_k = 0$, and $\det J_k > 0$ when equilibrium E_k exists. Therefore, the eigenvalues of J_k are a pair of pure imaginary roots if $\psi_k = 0$. From direct calculations we have that

$$\left. \frac{d(\text{tr} J_k)}{d\psi_k} \right|_{\psi_k=0} = \frac{1}{(y_k^2 + c)(y_k + 1)} \neq 0.$$

By Theorem 3.4.2 in [16], $\Psi_k = 0$ is the Hopf bifurcation point for (3).

Next, introduce a new time variable τ by $dt = (y^2 + c)d\tau$. By rewriting τ as t , we obtain the following equivalent system of (3)

$$\begin{cases} \frac{dx}{dt} = \Lambda(y^2 + c) - xy(ay + 1) - dx(y^2 + c) + \sigma y(y^2 + c), \\ \frac{dy}{dt} = xy(ay + 1) - y(y^2 + c). \end{cases} \quad (7)$$

Let $X = x - x_k$ and $Y = y - y_k$, still use (x, y) to express (X, Y) , then system (7) becomes

$$\begin{cases} \frac{dx}{dt} = b_{11}x + b_{12}y + c_1xy + c_2y^2 + c_3xy^2 + c_4y^3, \\ \frac{dy}{dt} = b_{21}x + b_{22}y + c_5xy + c_6y^2 + c_7xy^2 + c_8y^3, \end{cases}$$

where

$$\begin{aligned} b_{11} &= -d((y_k)^2 + c) - y_k(ay_k + 1), \quad b_{21} = y_k(ay_k + 1), \quad b_{22} = (ax_k - 2y_k)y_k, \\ b_{12} &= 2\Lambda y_k - 2(a + d)x_k y_k - x_k + \sigma(3(y_k)^2 + c), \quad c_1 = -1 - 2(a + d)y_k, \\ c_2 &= \Lambda + 3\sigma y_k - (a + d)x_k, \quad c_3 = -(a + d), \quad c_4 = \sigma, \quad c_5 = 2ay_k + 1, \\ c_6 &= ax_k - 3y_k, \quad c_7 = a, \quad c_8 = -1. \end{aligned}$$

Let $\hat{E}$ denote the origin of $x - y$ plane. Since E_k satisfies Eq. (3), we obtain

$$\det J(\hat{E}) = b_{11}b_{22} - b_{12}b_{21} = \frac{y_k\psi_k}{y_k + b} > 0,$$

and it is easy to verify that $b_{11} + b_{22} = 0$ if and only if $\psi_k = 0$. Let $\omega = (\det J(\hat{E}))^{\frac{1}{2}}$, $u = -x$ and $v = \frac{b_{11}}{\omega}x + \frac{b_{12}}{\omega}y$, then the normal form of system (7) reads

$$\begin{cases} \frac{du}{dt} = -\omega v + f(u, v), \\ \frac{dv}{dt} = \omega u + g(u, v), \end{cases} \quad (8)$$

where

$$\begin{aligned}
 f(u, v) &= \left(\frac{b_{11}c_1}{b_{12}} - \frac{b_{11}^2c_2}{b_{12}^2}\right)u^2 + \omega\left(\frac{c_1}{b_{12}} - \frac{2b_{11}c_2}{b_{12}^2}\right)uv - \frac{c_2\omega^2}{b_{12}^2}v^2 + b_{11}^2\left(\frac{c_3}{b_{12}^2} - \frac{b_{11}c_4}{b_{12}^3}\right)u^3 \\
 &\quad + b_{11}\omega\left(\frac{2c_3}{b_{12}^2} - \frac{3b_{11}c_4}{b_{12}^3}\right)u^2v + \omega^2\left(\frac{c_3}{b_{12}^2} - \frac{3b_{11}c_4}{b_{12}^3}\right)uv^2 - \frac{c_4\omega^3}{b_{12}^3}v^3, \\
 g(u, v) &= \frac{1}{\omega}\left(\frac{b_{11}^3c_2}{b_{12}^2} - \frac{b_{11}^2c_1}{b_{12}} - c_5b_{11} + \frac{b_{11}^2c_6}{b_{12}}\right)u^2 + \left(\frac{2b_{11}^2c_2}{b_{12}^2} - \frac{b_{11}c_1}{b_{12}} + \frac{2b_{11}c_6}{b_{12}} - c_5\right)uv \\
 &\quad + \omega\left(\frac{b_{11}c_2}{b_{12}^2} + \frac{c_6}{b_{12}}\right)v^2 + \frac{b_{11}^2}{b_{12}\omega}\left(\frac{b_{11}^2c_4}{b_{12}^2} - \frac{b_{11}c_3}{b_{12}} - c_7 + \frac{b_{11}c_8}{b_{12}}\right)u^3 \\
 &\quad + \left(\frac{3b_{11}^3c_4}{b_{12}^3} - \frac{2b_{11}^2c_3}{b_{12}^2} - \frac{b_{11}c_7}{b_{12}} + \frac{3b_{11}^2c_8}{b_{12}^2}\right)u^2v \\
 &\quad + \omega\left(\frac{3b_{11}^2c_4}{b_{12}^3} - \frac{b_{11}c_3}{b_{12}^2} - \frac{c_7}{b_{12}} + \frac{3b_{11}c_8}{b_{12}^2}\right)uv^2 + \omega^2\left(\frac{b_{11}c_4}{b_{12}^3} + \frac{c_8}{b_{12}^2}\right)v^3.
 \end{aligned}$$

Set

$$\begin{aligned}
 \Gamma &= \frac{1}{16}[f_{uuu} + f_{uvv} + g_{uvv} + g_{vvv}] \\
 &\quad + \frac{1}{16\omega}[f_{uv}(f_{uu} + f_{vv}) - g_{uv}(g_{uu} + g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{vv}],
 \end{aligned}$$

where f_{uv} denotes $\frac{\partial^2 f}{\partial u \partial v}(0, 0)$, etc. Then, by computing we obtain

$$\Gamma = \frac{\eta}{8b_{12}^2\omega^2},$$

where

$$\begin{aligned}
 \eta &= \left[c_3 + 3c_8 - \frac{c_2(c_1 + 2c_6)}{b_{12}}\right]\omega^4 + \left[(c_3 + 3c_8 - \frac{2c_2(c_1 + 2c_6)}{b_{12}})b_{11}^2\right. \\
 &\quad \left.+ b_{11}(c_1^2 - 2b_{12}c_7 + c_2c_5 + c_1c_6 - 2c_6^2) + b_{12}c_5c_6\right]\omega^2 \\
 &\quad + \left[c_5 - \frac{b_{11}(c_1 + 2c_6)}{b_{12}}\right][b_{11}^2(b_{12}c_6 - c_1) + b_{11}(b_{11}^2c_2 - b_{12}^2c_5)].
 \end{aligned}$$

By Theorem 3.4.2 and Theorem 3.4.11 in [16], the rest claims in Theorem 10 are true.

Remark 6 What need to be noted here is that the expression b_{12} in Theorem 10 is nonzero. Otherwise, we have $\det J_k = b_{11}b_{22} < 0$, since $b_{11} + b_{22} = 0$, which is an contradiction.

Next, we present examples to show that equilibrium E_k can be a stable weak focus of multiplicity two, and under a small perturbation, system (3) undergoes a degenerate Hopf bifurcation and produces two limit cycles.

Firstly, fix $y_k = 1/2$ and solve for $a = -2 + \frac{6}{(2-2\Lambda-\sigma)}$. Also, fix $x_k = 1$, based on $a = -2 + \frac{6}{(2-2\Lambda-\sigma)}$, we can get $d = \frac{1}{2(2\Lambda+1-\sigma)}$. Then $\Psi_k = 0$ if and only if $c = \frac{-10-2\Lambda-\sigma}{4(2\Lambda+\sigma-2)}$. Secondly, substitute these expressions into η and through complicated computation, we can obtain

$$\begin{aligned}
 \eta \triangleq L_1 &= \frac{81(1+2\Lambda+\bar{\sigma})^2}{64(1-2\Lambda+\bar{\sigma})^6}(32\Lambda^5 - 80\Lambda^4(-1 + \bar{\sigma}) + 8\Lambda^3(-39 + 2\bar{\sigma}(-13 + 5\bar{\sigma})) \\
 &\quad + 4\Lambda^2(-103 + \bar{\sigma}(99 + 2(24 - 5\bar{\sigma})\bar{\sigma})) + \bar{\sigma}(140 - \bar{\sigma}(139 + \bar{\sigma}(-21 \\
 &\quad + (-11 + \bar{\sigma})\bar{\sigma}))) + 2\Lambda(-80 + \bar{\sigma}(242 + \bar{\sigma}(-81 + \bar{\sigma}(-38 + 5\bar{\sigma}))))),
 \end{aligned}$$

with $\bar{\sigma} = 1 - \sigma > 0$, which is the first Liapunov number of the equilibrium $(0, 0)$ of (8). Then, we fix $\bar{\sigma} = 4/5$ and solve equation $\eta = 0$, then we get only one suitable value

0.5859 for Λ . That is to say, if $(\sigma, \Lambda, d, a, c) = (1/5, 5859, 0.2, 8, 4.741)$, then $L_1 = 0$. Furthermore, it can be seen that $E_k = E_3$ under this group of parameters.

In the following, we further compute the second Liapunov number of the equilibrium $(0, 0)$ of system (8) by the successor function method. It is convenient to introduce polar coordinates (r, θ) and rewrite system (8) in polar coordinates by $x = r \cos \theta$, $y = r \sin \theta$. It is clear that in a small neighborhood of the origin, the successor function $D(c_0)$ of system (8) can be expressed by

$$D(c_0) = r(2\pi, c_0) - r(0, c_0),$$

where $r(\theta, c_0)$ is the solution of the following Cauchy problem

$$\begin{aligned} \frac{dr}{d\theta} &= R_2(\theta)r^2 + R_3(\theta)r^3 + R_4(\theta)r^4 + R_5(\theta)r^5 + \dots, \\ r(0) &= c_0, \quad 0 < |c_0| \ll 1, \end{aligned}$$

where $R_i(\theta)$ ($i = 1, 2, 3, \dots$) is a polynomial of $(\sin \theta, \cos \theta)$, whose coefficient can be expressed by the coefficients of system (8). We omit them here, since the expressions are too long.

With the aid of Mathematica, we get

$$L_2 \doteq -0.334275,$$

and

$$\left| \begin{array}{cc} \frac{\partial \text{tr} J_3}{\partial \Lambda} & \frac{\partial \text{tr} J_3}{\partial c} \\ \frac{\partial L_1}{\partial \Lambda} & \frac{\partial L_1}{\partial c} \end{array} \right| = -42982.27165,$$

when $(\sigma, \Lambda, d, a, c) = (1/5, 5859, 0.2, 8, 4.741)$. Therefore, the interior equilibrium E_3 is a stable weak focus of multiplicity two if $(\sigma, \Lambda, d, a, c) = (1/5, 5859, 0.2, 8, 4.741)$. The phase portraits of system (3) under this group of parameters are shown in Fig. 6.

Fig 6. Bold the figure title. (a): One Limit cycles enclosing an unstable hyperbolic focus E_3 . (b): Two Limit cycles enclosing an unstable hyperbolic focus E_3 , and the small one is stable, the large one is unstable.

Besides, we give the numerical simulation graphs for one limit cycle and two limit cycles under small perturbations of some parameters. From Fig. 6 (a), we can see that there exist only one limit cycle around the endemic E_3 , and Fig. 6 (b) shows us that a new limit cycle emerge with small perturbations of parameters Λ and c . It is worth

Fig 7. Equilibrium E_3 is stable and there exists an unstable homoclinic loop.

Fig 8. (a): One Limit cycles enclosing an unstable hyperbolic focus E_5 . (b): Two Limit cycles enclosing an unstable hyperbolic focus E_5 , and the small one is stable, the large one is unstable.

emphasizing that if we change the values of parameters Λ and c , an unstable homoclinic loop arise, which is shown in Fig.7.

Around equilibrium E_5 , we can gain the same result from Fig. 8. Under the condition that parameter a, d, σ take value 5.6923, 0.11, 0.2 and change value of Λ and c from 0.45102, 3.6062 to 0.51, 3.5972, respectively, which is a minor change, then the number of limit cycles will add one. Thus, the interior equilibrium E_5 is a stable weak focus of multiplicity two if $(\sigma, \Lambda, d, a, c) = (0.2, 0.51, 0.11, 5.6923, 3.5962)$.

Remark 7 As a matter of fact, the reproduction number is equal to zero in [8, 20], which simplifies the condition that Hopf bifurcation occur. In our model, we also comprehensively discuss the existence of Hopf bifurcation when $R_0 < 1$, $R_0 = 1$ and $R_0 > 1$. Besides, authors in [20] did not show the appearance of homoclinic loop, which is an interesting bifurcation phenomenon given in Fig.7.

Bogdanov-Takens bifurcation

In this subsection, we investigate the Bogdanov-Takens bifurcation in system (3). The following Lemma 3 is from Perko [17], and Lemma 4 is Proposition 5.3 in Lamontagne et al. [18].

Lemma 3 *The system*

$$\begin{cases} \frac{dx}{dt} = y + Ax^2 + Bxy + Cy^2 + \mathcal{O}(|(X, Y)|^3), \\ \frac{dy}{dt} = Dx^2 + Exy + Fy^2 + \mathcal{O}(|(X, Y)|^3), \end{cases}$$

is equivalent to the system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = Dx^2 + (E + 2A)xy + \mathcal{O}(|(X, Y)|^3), \end{cases}$$

in some small neighborhood of (0, 0) after changes of coordinates.

Lemma 4 *The system*

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = x^2 + a_{30}x^3 + a_{40}x^4 + y(a_{21}x^2 + a_{31}x^3) + y^2(a_{12}x + a_{22}x^2) + \mathcal{O}(|(x, y)|^4), \end{cases}$$

is equivalent to the system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = x^2 + (a_{31} - a_{30}a_{21})xy + \mathcal{O}(|(X, Y)|^3), \end{cases}$$

in some small neighborhood of (0, 0) after changes of coordinates.

From Theorem 1, we can see that there exists a unique positive equilibrium $E_1(x_1, y_1)$ when $R_0 = R^*$, where

$$x_1 = \frac{y_1^2 + c}{ay_1 + 1}, \quad y_1 = \frac{a\Lambda - (1 - \sigma)}{2(d + a(1 - \sigma))}.$$

From the proving process of Theorem 4, it is easily to know that $\det J_1 = 0$. And Theorem 3 suggests that the characteristic matrix J_1 possesses a zero eigenvalue with multiplicity 2 when $\psi_1 = 0$, which shows that system (3) may admit a Bogdanov-Takens bifurcation. Thus, we can prove the following theorem.

Define two functions:

$$\begin{aligned} f(y) &= (1 + a + d)y^4 + (1 - ac(3 + 2a + 2d))y^3 - 3c(1 + a)y^2 \\ &\quad - c(1 + ac(-1 + 2d))y - c^2d, \\ g(y) &= c^2d + 2c(1 + 6a - 2c + cd)y + 6c(2 + a)y^2 \\ &\quad + 2(2a^2c + acd - 1)y^3 - (4 + 2a + d)y^4. \end{aligned}$$

Theorem 11 *Suppose that $R_0 = R^*$ and $\psi_1 = 0$, then the only interior equilibrium E_1 of system (3) is a cusp. Moreover,*

- (a) *if $f(y_1)g(y_1) \neq 0$, then E_1 is a cusp of codimension 2;*
- (b) *if $f(y_1)g(y_1) = 0$, then E_1 is a cusp of codimension greater than or equal to 3.*

Proof Changing the variables as $X = x - x_1$, $Y = y - y_1$, system (3) becomes

$$\begin{cases} \frac{dX}{dt} = b_{11}X + b_{12}Y + c_1XY + c_3Y^2 + \mathcal{O}(|(X, Y)|^3), \\ \frac{dY}{dt} = b_{21}X + b_{22}Y - c_1XY + c_2Y^2 + \mathcal{O}(|(X, Y)|^3), \end{cases} \quad (9)$$

where

$$\begin{aligned} b_{11} &= -d - \frac{y_1(ay_1+1)}{y_1^2+c}, \quad b_{12} = \sigma - \frac{-y_1^2+2acy_1+c}{(y_1^2+c)^2}x_1, \quad b_{21} = \frac{y_1(ay_1+1)}{y_1^2+c}, \\ b_{22} &= -1 + \frac{-y_1^2+2acy_1+c}{(y_1^2+c)^2}x_1, \quad c_1 = -\frac{-y_1^2+2acy_1+c}{(y_1^2+c)^2}, \\ c_2 &= \frac{2(y_1^3-3acy_1^2-3cy_1+ac^2)x_1}{(y_1^2+c)^3}, \quad c_3 = -\frac{x_1(ac^2-3cy_1-3acy_1^2+y_1^3)}{(c+y_1^2)^3}. \end{aligned}$$

Make the non-singular linear transformation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{b_{11}}{b_{21}} & \frac{1}{b_{21}} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then system (9) is transformed into

$$\begin{cases} \frac{dx}{dt} = y + b_1x^2 + b_2xy, \\ \frac{dy}{dt} = b_3x^2 + b_4xy + Q_2(x, y), \end{cases} \quad (10)$$

where $Q_2(x, y)$ is a smooth function in (x, y) at least of the third order and

$$b_1 = c_2 - \frac{c_1b_{11}}{b_{21}}, \quad b_2 = -\frac{c_1}{b_{21}}, \quad b_3 = b_{21}c_3 + b_{11}(c_1 - c_2) + \frac{b_{11}^2c_1}{b_{21}}, \quad b_4 = \frac{-dc_1}{b_{21}}.$$

By Lemma 3, we obtain an topologically equivalent system of system (10)

$$\begin{cases} \frac{du}{dt} = v, \\ \frac{dv}{dt} = b_3u^2 + (b_4 + 2b_1)uv + Q_3(u, v), \end{cases}$$

where

$$b_3 = \frac{df(y_1)}{(c+y_1^2)^3}, \quad b_4 + 2b_1 = -\frac{g(y_1)}{y_1(ay_1+1)(y_1^2+c)^2}.$$

Therefore, E_1 is a cusp of codimension 2 if $f(y_1)g(y_1) \neq 0$, by the results in Perko [17], or else, E_1 is a cusp of codimension at least 3.

Remark 8 In fact, Theorem 11 (b) include the following three cases:

- (1) if $f(y_1) \neq 0$ and $g(y_1) = 0$, E_1 is a cusp point;
- (2) if $f(y_1) = 0$ and $g(y_1) \neq 0$, E_1 is nilpotent focus/elliptic point;
- (3) if $f(y_1) = g(y_1) = 0$, E_1 is a nilpotent focus.

Unfortunately, because of the complexity of $f(y_1)$ and $g(y_1)$, we can not determine which of these three situations can occur theoretically. But we will show for some parameter values, $f(y_1) \neq 0$ and $g(y_1) = 0$, i.e. E_1 is a cusp point.

In the following, we will give an example to show that the case (b) of Theorem 11 occur.

In the first place, fix $y_1 = 1/5$, then from $y_1 = \frac{a\Lambda-(1-\sigma)}{2(d+a(1-\sigma))}$, we can solve for $\sigma = 1 - \frac{-2d+5a\Lambda}{5+2a}$. By assumptions $\psi_1 = 0$ and $R_0 = R^*$, then from

$$R_0 = \frac{\Lambda}{dc}, \quad R^* = 1 - \frac{[(1-\sigma) - a\Lambda]^2}{4dc(d+a(1-\sigma))},$$

we can get d and Λ , where

$$d = \frac{(5+a)^2 \Lambda}{-1+25c+10ac}, \quad \Lambda = \frac{(35+a(11+a-25c))(1-5(5+2a)c)}{(5+a)^3(1+25c)}.$$

Then

$$f(y_1) = -\frac{(1+25c)(1+a^2c)}{125(5+a)} < 0$$

for any $a, c > 0$. We solve $g(y_1) = 0$ for parameter c denoted by $c_{\pm}^g$, where

$$c_{\pm}^g = \frac{(375+a(1630+a(323+2a)) \pm \sqrt{(5+a)(28125+a(236775+a(531415+a(104449+4a(318+a))))))}{50a(15+2a)}.$$

Actually, if $c = c_{-}^g$, we have $d = d(a) < 0$ for any $a > 0$. Thus, $g(y_1) = 0$ if and only if $c = c_{+}^g \triangleq c^g(a)$. In addition, with the help of *mathematica*, for any $0 < a < 1.86433$ or $a > 123.449$, one can have all other parameters positive and satisfying $d + \sigma < 1$.

For example, take $a = 1$, then we can get

$$\begin{aligned} (a, d, c, \Lambda, \sigma) &= (1, \frac{61+\sqrt{37585}}{6(197+\sqrt{37585})}, \frac{2330+12\sqrt{37585}}{850}, \frac{(61+\sqrt{37585})(1345+7\sqrt{37585})}{3060(197+\sqrt{37585})}, \frac{18259+97\sqrt{37585}}{153(197+\sqrt{37585})}) \\ &\triangleq (a_0, d_0, c_0, \Lambda_0, \sigma_0) \\ &\doteq (1, 0.108676, 5.478144, 0.575787, 0.619774), \end{aligned}$$

which satisfies $c = c^g(a)$, i.e. Theorem 11 (b) occur.

Theorem 12 When $(a, d, c, \Lambda, \sigma) = (a_0, d_0, c_0, \Lambda_0, \sigma_0)$, then E_1 is a Bogdanov-Takens point of codimension 3, and the system (3) localized at E_1 is topologically equivalent to

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = x^2 + Gx^3y + \mathcal{O}(|(x, y)|^3), \end{cases}$$

where $G < 0$.

Proof First of all, applying a linear transformation $T_1 : (x, y) \rightarrow (u, v)$, defined by $u = x - x_1, v = y - y_1$, we can reduce system (3) further to the form

$$\begin{cases} \frac{du}{dt} = p_{10}u + p_{01}v + \sum_{2 \leq i+j \leq 4} p_{ij}u^i v^j + \mathcal{O}(|(u, v)|^5), \\ \frac{dv}{dt} = q_{10}u + q_{01}v + \sum_{2 \leq i+j \leq 4} q_{ij}u^i v^j + \mathcal{O}(|(u, v)|^5), \end{cases} \quad (11)$$

where

$$\begin{aligned} p_{10} &= -d - \frac{y_1(ay_1+1)}{y_1^2+c}, \quad p_{01} = \sigma - \frac{-y_1^2+2acy_1+c}{(y_1^2+c)^2}x_1, \quad p_{11} = -\frac{-y_1^2+2acy_1+c}{(y_1^2+c)^2}, \\ p_{02} &= -\frac{x_1(ac^2-3acy_1^2-3cy_1+y_1^3)}{(c+y_1^2)^3}, \quad p_{12} = \frac{-ac^2+3acy_1^2+3cy_1-y_1^3}{(c+y_1^2)^3}, \quad p_{03} = \frac{x_1(4ac^2y_1-4acy_1^3+c^2-6cy_1^2+y_1^4)}{(c+y_1^2)^4}, \\ p_{13} &= \frac{v^3(4ac^2y_1-4acy_1^3+c^2-6cy_1^2+y_1^4)}{(c+y_1^2)^4}, \quad p_{04} = \frac{x_1(ac^3-10ac^2y_1^2+5acy_1^4-5c^2y_1+10cy_1^3-y_1^5)}{(c+y_1^2)^5}, \\ q_{10} &= \frac{y_1(ay_1+1)}{y_1^2+c}, \quad q_{01} = -1 + \frac{-y_1^2+2acy_1+c}{(y_1^2+c)^2}x_1, \quad q_{11} = \frac{2acy_1+c-y_1^2}{(c+y_1^2)^2}, \\ q_{02} &= \frac{x_1(ac^2-3acy_1^2-3cy_1+y_1^3)}{(c+y_1^2)^3}, \quad q_{12} = \frac{(ac^2-3acy_1^2-3cy_1+y_1^3)}{(c+y_1^2)^3}, \quad q_{03} = \frac{x_1(4ac^2y_1-4acy_1^3+c^2-6cy_1^2+y_1^4)}{(c+y_1^2)^4}, \\ q_{13} &= \frac{(-4ac^2y_1+4acy_1^3-c^2+6cy_1^2-y_1^4)}{(c+y_1^2)^4}, \quad q_{04} = \frac{x_1(ac^3-10ac^2y_1^2+5acy_1^4-5c^2y_1+10cy_1^3-y_1^5)}{(c+y_1^2)^5}, \\ p_{20} &= p_{21} = p_{22} = p_{30} = p_{31} = p_{40} = q_{20} = q_{21} = q_{22} = q_{30} = q_{31} = q_{40} = 0. \end{aligned}$$

Another transformation $T_2 : (u, v) \rightarrow (x, y)$, defined by $x = v$, $y = q_{10}u - p_{10}v$, reduces system (11) to

$$\begin{cases} \frac{dx}{dt} = y + \sum_{2 \leq i+j \leq 4} a_{ij}x^i y^j + \mathcal{O}(|(x, y)|^5), \\ \frac{dy}{dt} = \sum_{2 \leq i+j \leq 4} b_{ij}x^i y^j + \mathcal{O}(|(x, y)|^5), \end{cases}$$

where

$$\begin{aligned} a_{11} &= \frac{q_{11}}{q_{10}}, \quad a_{20} = q_{02} + \frac{p_{10}q_{11}}{q_{10}}, \quad a_{21} = \frac{q_{12}}{q_{10}}, \quad a_{30} = \frac{q_{03}q_{10} + p_{10}q_{12}}{q_{10}^2}, \quad a_{40} = q_{04}, \\ b_{11} &= p_{11} - \frac{p_{10}q_{11}}{q_{10}}, \quad b_{20} = p_{10}p_{11} - p_{10}q_{02} + p_{02}q_{10} - \frac{p_{10}^2q_{11}}{q_{10}^2}, \quad b_{21} = p_{12} - \frac{p_{10}q_{12}}{q_{10}}, \\ b_{30} &= p_{10}(p_{12} - q_{03}) + p_{03}q_{10} - \frac{p_{10}^2q_{12}}{q_{10}^2}, \quad b_{40} = -p_{10}q_{04} + p_{04}q_{10} \end{aligned}$$

and other coefficients are equal to zero.

Then using the near-identity transformation $T_3 : (x, y) \rightarrow (u, v)$, defined by $u = x + \frac{1}{2}(a_{11} + b_{02})x^2 - a_{02}xy$, $v = y + a_{20}x^2 - b_{20}xy$, and parameters $(a_0, d_0, c_0, \Lambda_0, \sigma_0)$ make $b_{11} + 2a_{20} = 0$, so we obtain

$$\begin{cases} \frac{du}{dt} = v + \sum_{3 \leq i+j \leq 4} d_{ij}u^i v^j + \mathcal{O}(|(u, v)|^5), \\ \frac{dv}{dt} = b_{20}u^2 + \sum_{3 \leq i+j \leq 4} e_{ij}x^i y^j + \mathcal{O}(|(u, v)|^5), \end{cases} \quad (12)$$

where

$$\begin{aligned} d_{30} &= a_{30} + a_{20}(a_{11} + b_{02}) - a_{02}b_{20}, \\ d_{21} &= -a_{02}(a_{20} - b_{11}) + a_{21} + a_{11}(a_{11} + b_{02}), \\ d_{12} &= 2a_{12}, \quad d_{03} = -a_{02}^2 + a_{03}, \\ d_{13} &= a_{13} + a_{03}(a_{11} + 4b_{02}) - a_{02}(3a_{02}b_{02} + b_{03}), \\ d_{40} &= a_{40} - (a_{11} + b_{02})(a_{11}a_{20} + \frac{1}{2}a_{30} - \frac{3}{2}a_{02}b_{20}) \\ &\quad - \frac{3}{2}a_{20}(a_{11} + b_{02})^2 - a_{20}a_{21} - a_{02}b_{30}, \\ d_{31} &= a_{31} - a_{11}^3 + a_{02}(2a_{30} + 3a_{20}b_{02} + a_{11}(4a_{20} + b_{11}) - b_{21} - 3a_{02}b_{20}) \\ &\quad + b_{02}(a_{21} - a_{11}^2) - 2a_{12}a_{20}, \\ d_{22} &= a_{22} - 3a_{03}a_{20} + \frac{1}{2}a_{12}(a_{11} + 5b_{02}) + a_{02}^2(a_{20} - 2b_{11}) \\ &\quad + a_{02}(a_{21} + 2a_{11}(a_{11} + b_{02}) - b_{12}), \\ d_{04} &= -a_{02}a_{03} + a_{04}, \\ e_{30} &= b_{30} - a_{20}b_{11} - (a_{11} + 2b_{02})b_{20}, \\ e_{21} &= a_{11}a_{20} - \frac{1}{2}(a_{11} + b_{02})b_{11} + 2a_{02}b_{20} + b_{21}, \\ e_{12} &= -b_{02}(a_{11} + b_{02}) + a_{02}(4a_{20} + b_{11}) + b_{12}, \\ e_{03} &= a_{02}a_{20} + b_{03}, \quad e_{04} = b_{04} + a_{03}a_{20}, \\ e_{40} &= a_{20}(2(a_{30} - a_{11}a_{20} - a_{02}b_{20}) - a_{20}b_{02} - b_{21}) \\ &\quad + \frac{3}{2}(a_{11} + b_{02})(a_{20}b_{11} + b_{02}b_{20} - b_{30}) \\ &\quad + \frac{5}{4}(a_{11} + b_{02})^2b_{20} - b_{02}b_{30} + b_{40}, \end{aligned}$$

$$\begin{aligned}
 e_{31} &= a_{20}(2a_{21} - 2b_{12} + 3b_{02}^2) - b_{02}(a_{30} + b_{21}) + \frac{b_{02}^2 b_{11}}{2} \\
 &\quad + \frac{1}{2}a_{11}^2(-2a_{20} + b_{11}) + a_{11}(b_{02}(3a_{20} + b_{11}) \\
 &\quad - 4a_{02}b_{20} - b_{21}) - a_{02}(a_{20}(6a_{20} + 4b_{11}) + 5b_{02}b_{20} - 3b_{30}) + b_{31}, \\
 e_{22} &= \frac{1}{2}(a_{20}(4a_{12} - 6b_{03}) + (a_{11}^2 - 2a_{21} + 14a_{02}a_{20} + b_{12} - 3b_{02}^2)b_{02} \\
 &\quad + (6a_{02}b_{20} + 4b_{21} - b_{02}b_{11})a_{02} + 2b_{22} - a_{11}(2b_{02}^2 + 3a_{02}b_{11} + b_{12})), \\
 e_{13} &= 2a_{03}a_{20} - a_{12}b_{02} + 2b_{02}b_{03} + a_{02}^2(4a_{20} + b_{11}) \\
 &\quad + a_{02}(-b_{02}(a_{11} + 4b_{02}) + b_{12}) + b_{13}.
 \end{aligned}$$

We perform a near-identity smooth change of coordinates

$$\begin{aligned}
 x &= u - (\frac{1}{3}d_{21} + \frac{1}{6}e_{12})u^3 - (\frac{1}{2}d_{12} + \frac{1}{2}e_{03})u^2v, \\
 y &= v + d_{30}u^3 + d_{03}v^3 - \frac{1}{2}e_{12}u^2v - e_{03}uv^2.
 \end{aligned}$$

Then system (12) becomes

$$\begin{cases} \frac{dx}{dt} = y + d_{40}x^4 + d_{04}y^4 + g_{31}x^3y + d_{13}xy^3 + d_{22}x^2y^2 + \mathcal{O}(|(x, y)|^5), \\ \frac{dy}{dt} = b_{20}x^2 + e_{30}x^3 + f_{40}x^4 + y(f_{21}x^2 + f_{31}x^3) + f_{13}xy^3 + e_{04}y^4 + f_{22}x^2y^2 + \mathcal{O}(|(x, y)|^5), \end{cases} \quad (13)$$

where

$$\begin{aligned}
 g_{31} &= (d_{31} - \frac{e_{11}}{2}(d_{12} + e_{03})), \quad f_{21} = 3d_{30} + e_{21}, \quad f_{40} = \frac{1}{6}(b_{20}(4d_{21} - e_{12}) + 6e_{40}), \\
 f_{31} &= b_{20}d_{12} - b_{20}e_{03} + e_{31}, \quad f_{13} = e_{13}, \quad f_{22} = 3b_{20}d_{03} + e_{22}.
 \end{aligned}$$

In order to kill the non-resonant cubic terms of system (13), we let

$$\begin{aligned}
 u &= x + \frac{1}{12}(d_{04} + d_{13} + e_{04} + f_{13} - f_{22} - 3g_{31})x^4 + d_{04}y^4 \\
 &\quad + \frac{1}{6}(-2d_{22} + d_{04} + d_{13} + e_{04} + f_{13})x^3y, \\
 v &= y + d_{40}x^4 + \frac{1}{3}(d_{04} + d_{13} + e_{04} + f_{13} - f_{22})x^3y \\
 &\quad + (d_{04} + d_{13})xy^3 - \frac{1}{2}(d_{04} + d_{13} + e_{04} + f_{13})x^2y^2,
 \end{aligned}$$

system (13) becomes

$$\begin{cases} \frac{du}{dt} = v + \mathcal{O}(|(u, v)|^5), \\ \frac{dv}{dt} = b_{20}u^2 + e_{30}u^3 + f_{40}u^4 + (f_{21}u^2 + (4d_{40} + f_{31})u^3)v + \mathcal{O}(|(u, v)|^5). \end{cases} \quad (14)$$

Finally, we set $x = b_{20}u$, $y = b_{20}(v + \mathcal{O}(|(u, v)|^5))$, system (14) becomes

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = x^2 + \frac{e_{30}}{b_{20}^2}x^3 + \frac{f_{40}}{b_{20}^3}x^4 + y(\frac{f_{21}}{b_{20}^2}x^2 + \frac{4d_{40} + f_{31}}{b_{20}^3}x^3) + \mathcal{O}(|(x, y)|^5). \end{cases}$$

According to Lemma 4 the above system is equivalent to the system

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = x^2 + Gx^3y + \mathcal{O}(|(x, y)|^3), \end{cases}$$

where $G = \frac{b_{20}(4d_{40} + f_{31}) - e_{30}f_{21}}{b_{20}^4}$. Compute coefficients $b_{20}, d_{40}, f_{31}, e_{30}, f_{21}$ with the condition $(a, d, c, \Lambda, \sigma) = (a_0, d_0, c_0, \Lambda_0, \sigma_0)$, and straightforward calculation leads to $G = -1.061119 \times 10^6 < 0$. Thus, E_1 is a cusp type of Bogdanov-Takens singular with codimension 3.

Remark 9 Authors in [8, 20] proved their epidemic model with saturated incidence rate, undergo a Bogdanov-Takens bifurcation of codimension 2. When we consider the incidence of a combination of the saturated incidence rate and a non-monotonic incidence, the codimension of Bogdanov-Takens bifurcation can be up to 3.

Remark 10 Xiao and Ruan (see [10]) showed that either the number of infective individuals tends to zero as time evolves or the disease persists. Authors in [8, 20] proved that their epidemic model with saturated incidence rate, undergo a Bogdanov-Takens bifurcation of codimension 2. When we consider the incidence of a combination of the saturated incidence rate and a non-monotonic incidence, the codimension of Bogdanov-Takens bifurcation can be up to 3.

Conclusions

In this paper, by combining qualitative and bifurcation analyses, we study an SIS epidemic model with the incidence rate $\frac{aI^2}{c+I^2} + \frac{bI}{c+I^2}$, which can describe the inhibition effect from the behavioral change and interpret the “psychological” effect. In Section 2, we give a full-scale analysis for the types and stability of equilibria $E_i (i = 0, 1, 2, 3, 4, 5)$. We prove that system (2) can occur backward bifurcation and the backward bifurcation will disappear if $a = 0$. At equilibrium $E_i (i = 3, 5)$, degenerate Hopf bifurcation arises under certain conditions. When the critical condition $\Psi_i (i = 3, 5)$ satisfied, we calculate the Liapunov value of the weak focus and obtain the value 2 for the maximal multiplicity of weak focus, indicating that there exist at most two limit cycles around $E_i (i = 3, 5)$. In Fig. 6, and Fig. 8, we give the phase portrait corresponding to equilibrium E_3 and E_5 about exhibiting a unique limit cycle and adding a new limit cycle after small perturbation of parameter Λ and c . In Subsection 3.3, we proved that the model exhibits Bogdanov-Takens bifurcation of codimension 2 and codimension 3, under certain conditions. If parameter $a = 0$, the model can just have Bogdanov-Takens bifurcation of codimension 2, shown in [20].

In fact, we show that the model exhibits multi-stable states. This interesting phenomenon indicates that the initial states of an epidemic can determine the final states of an epidemic to extinct or not. Moreover, the periodical oscillation signify that the trend of the disease may be affected by the behavior of susceptible and the effect of psychology of the disease.

References

1. Driessche PV, Watmough J. A simple SIS epidemic model with a backward bifurcation. J Math Biol. 2000 Sep;40(6):525–540.
2. Anderson RM, May RM. Infectious diseases of humans: dynamics and control. Epidemiology & Infection. 1992 Feb;108(1):211–211.
3. Liu W. Levin SA. Iwasa Y. Influence of nonlinear incidence rate upon the behavior of SIRS epidemiological models. J Math Biol. 1986 May;23(2):187–204.
4. Hethcote HW, Driessche P. Some epidemiological models with nonlinear incidence. J Math Biol. 1991 Apr;29(3):271–287.
5. Moghadas SM. Analysis of an epidemic model with bistable equilibria using the Poincare index. Appl Math Comput. 2004 Apr;149(3):689–702.
6. Alexander ME, Moghadas SM. Periodicity in an epidemic model with a generalized non-linear incidence. Math Biosci. 2004 Jun;189(1):75–96.

7. Alexander ME, Moghadas SM. Bifurcation analysis of an SIRS epidemic model with generalized incidence rate. *SIAM J Appl Math.* 2005 May;65(5):1794–1816.
8. Ruan SG, Wang WD. Dynamical behavior of an epidemic model with a nonlinear incidence rate. *J Differential Equations.* 2003 Apr;188(1):135–163.
9. Tang YL, Huang DQ, Ruan SG, Zhang WN. Coexistence of limit cycle in a SIRS model with a nonlinear incidence rate. *SIAM J Appl Math.* 2008 Mar;69(2):621–639.
10. Xiao DM, Ruan SG. Global analysis of an epidemic model with nonmonotone incidence rate. *Mathematical Biosciences.* 2007 Nov;208(2):419–429.
11. Leung G, Lam T, Ho L, Ho S, Chan B, Wong I, and A Hedley Leung GM. et al. The impact of community psychological response on outbreak control for severe acute respiratory syndrome in Hong Kong. *J Epidemiol Commun. Health.* 2003 Nov;57(11): 857–863.
12. Xiao DM, Zhou YG. Qualitative analysis of an epidemic model. *Canadian Applied Math Quarterly.* 2006 Jul;14(4):469–492.
13. Zhou YG, Xiao DM, Li YL. Bifurcation of an epidemic model with non-monotone incidence rate of saturated mass action. *Chaos, Soliton and Fractal.* 2007 Jun;32(5):1903–1915.
14. Li GH, Zhang YX. Dynamic behaviors of a modified SIR model in epidemic diseases using nonlinear incidence and recovery rates. *PLoS One.* 2017 Apr;12(4):e0175789.
15. Zhang ZF, Ding TR, Huang WZ, Dong ZX. *Qualitative Theoriey of Differential Equations.* American Mathamatical Society, Province, RI. 1992.
16. Guckenheimer J, Holmes PJ. *Nonlinear oscillations, dynamical system, and bifurcation of vector field.* Springer-Verlag, New York. 1996.
17. Perko L. *Differential Equations and Dynamical Systems.* Second edition, Texts in Applied Mathematics, Springer-Verlag, New York. 1996.
18. Lamontagne Y, Coutu V, Rousseau C. Bifurcation analysis of a predator-prey system with generalized Holling type III functional response. *J Dynam Differential Equations.* 2008 Sep;20(3):535–571.
19. Dushoff J, Huang W, Castillochavez C, Backwards bifurcations and catastrophe in simple models of fatal diseases. *J Math Biol.* 1998 May;36(3):227–248.
20. Li J, Zhao Y, Zhu HP. Bifurcation of an SIS model with nonlinear contact rate. *J Math Anal Appl.* 2015 Oct;432(2):1119–1138.
21. Arnold VI. *Geometrical methods in the theory of ordinary differential equations.* Springer-Verlag, New York. 1983.