Appendix of Cylcop: An R package for Circular-linear Copulae with Angular Symmetry

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Contents

A1 Wrapped Cauchy Distribution

The implementation of the wrapped Cauchy distribution in the `circular`-package does not allow for the calculation of the distribution function or the quantile function. Therefore, we use a numerical approximation impelmented in the `Wrapped`-package. However, this package uses a different parametrization than then `circular`-package. The equation for the wrapped Cauchy probability density with the parametrization of the `Wrapped`-package is

$$
\begin{aligned}
f(\theta;\mu,\gamma) &= \frac{1}{2\pi}\frac{\sinh(\gamma)}{\cosh(\gamma)-\cos(\theta-\mu)} \\
&= \frac{1}{2\pi}\frac{0.5(\exp(\gamma)-\exp(-\gamma))}{0.5(\exp(\gamma)+\exp(-\gamma))-\cos(\theta-\mu)} \\
&= \frac{1}{2\pi}\frac{\exp(\gamma)-\exp(-\gamma)}{\exp(\gamma)+\exp(-\gamma)-2\cos(\theta-\mu)} \\
&= \frac{1}{2\pi}\frac{[\exp(-\gamma)]^{-1}-\exp(-\gamma)}{[\exp(-\gamma)]^{-1}+\exp(-\gamma)-2\cos(\theta-\mu)} \\
&= \frac{1}{2\pi}\frac{[\exp(-\gamma)]^{-1}\Big(1-[\exp(-\gamma)]^2\Big)}{[\exp(-\gamma)]^{-1}\Big(1+[\exp(-\gamma)]^2-2\exp(-\gamma)\cos(\theta-\mu)\Big)} \\
&= \frac{1}{2\pi}\frac{[1-[\exp(-\gamma)]^2}{1+[\exp(-\gamma)]^2-2\exp(-\gamma)\cos(\theta-\mu)} \\
&= \frac{1}{2\pi}\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\theta-\mu)} \\
&= f(\theta;\mu,\rho)
\end{aligned}
\tag{1}
$$

The last equality is in the parametrization of the wrapped Cauchy density of the `circular`-package. The relationship between the two parametrizations is $\rho = exp(-\gamma)$.

A2 von Mises Copula

A2.1 PDF

We need to show that $c(u,v) = 2\pi g(2\pi[u-v])$ is indeed a circular-linear copula. As in the main text, we will assume the vonMises density g to have as support the entire real line. First of all, note that the function g is not exactly the vonMises density f. It is the same formula, but its support is $[2\pi(0-1), 2\pi(1-0)] = [-2\pi, 2\pi]$. The equation for the PDF of the vonMises copula is

$$
\begin{aligned}
f_{\Theta,X}(\theta,x) &= c(F_\Theta(\theta),F_X(x))f_\Theta(\theta)f_X(x) \\
&= 2\pi g(2\pi[F_\Theta(\theta)-F_X(x)])f_\Theta(\theta)f_X(x) \quad \theta\in[-\pi,\pi),\ x\in\mathbb{R}\,.
\end{aligned}
\tag{2}
$$

When integrating with respect to either θ or x we must get back the corresponding (other) marginal distribution

$$\begin{aligned}
\int_{-\infty}^{\infty} f_{\Theta,X}(\theta,x) &= 2\pi \int_{-\infty}^{\infty} g(2\pi[F_\Theta(\theta) - F_X(x)] \mid \mu = \mu_0) f_\Theta(\theta) f_X(x)\ dx \\
&= f_\Theta(\theta) \int_0^{2\pi} g(2\pi[F_\Theta(\theta) - q \mid \mu = \mu_0)\ dq \\
&= f_\Theta(\theta) \int_0^{2\pi} g(-q \mid \mu = \mu_0 - 2\pi F_\Theta(\theta))\ dq \\
&= f_\Theta(\theta) \int_0^{2\pi} g(q \mid \mu = 2\pi F_\Theta(\theta) - \mu_0)\ dq \\
&= f_\Theta(\theta)\,.
\end{aligned} \tag{3}$$

In the second step, we made the substitution $q = 2\pi F_X(x)), \quad dx = \frac{dq}{2\pi f_X(x)}$, and the second-to-last step is for symmetry reasons. Next, it is trivial to show with the properties of g that the copula and the joint density function are periodic in the first argument.

$$\begin{aligned}
\lim_{\theta\to\pi} f_{\Theta,X}(\theta,x) &= 2\pi g(2\pi[1 - F_X(x)] \mid \mu = \mu_0) \lim_{\theta\to\pi} f(\theta) f_X(x) \\
&= 2\pi g(2\pi - 2\pi F_X(x) \mid \mu = \mu_0) f(-\pi) f_X(x) \\
&= 2\pi g(2\pi[0 - F_X(x)] \mid \mu = \mu_0 - 2\pi) f(-\pi) f_X(x) \\
&= 2\pi g(2\pi[0 - F_X(x)] \mid \mu = \mu_0) f(-\pi) f_X(x) \\
&= f_{\Theta,X}(-\pi, x)
\end{aligned} \tag{4}$$

A2.2 Derivation of the CDF with Bessel Functions

With the two substitutions $q = 2\pi F_X(x)$, $dx = \frac{dq}{2\pi f_X(x)}$ and $r = 2\pi F_\Theta(\theta)$, $d\theta = \frac{dq}{2\pi f_\Theta(\theta)}$ we can write

$$\begin{aligned}
F_{\Theta,X}(a,b) &= \int_{-\infty}^{b}\int_{-\infty}^{a} f_{\Theta,X}(\theta,x)\ d\theta dx \\
&= 2\pi \int_{-\infty}^{a}\int_{-\infty}^{b} g(2\pi[F_\Theta(\theta) - F_X(x)] \mid \mu = \mu_0) f_\Theta(\theta) f_X(x)\ dx d\theta \\
&= \frac{1}{2\pi}\int_{0}^{2\pi F_\Theta(a)}\int_{0}^{2\pi F_X(b)} g(q \mid \mu = r - \mu_0)\ dq dr \\
&= \frac{1}{2\pi}\int_{0}^{2\pi F_\Theta(a)}\int_{0}^{2\pi F_X(b)} \frac{1}{2\pi}\left(1 + \frac{2}{I_0(\kappa)}\sum_{j=1}^{\infty} I_j(\kappa)\cos[j(q+\mu_0 - r)]\right) dq dr \\
&= \frac{1}{2\pi}\int_{0}^{2\pi F_\Theta(a)} \frac{1}{2\pi}\left(2\pi F_X(b) + \frac{2}{I_0(\kappa)}\sum_{j=1}^{\infty} I_j(\kappa)\frac{\sin[j(2\pi F_X(b) + \mu_0 - r)]}{j}\right. \\
&\qquad \left. - \frac{2}{I_0(\kappa)}\sum_{j=1}^{\infty} I_j(\kappa)\frac{\sin[j(\mu_0 - r)]}{j}\right) dr \\
&= \frac{1}{2\pi}\int_{0}^{2\pi F_\Theta(a)}\left(F_X(b) + \frac{1}{\pi I_0(\kappa)}\sum_{j=1}^{\infty} I_j(\kappa)\frac{\sin[j(2\pi F_X(b) + \mu_0 - r)] - \sin[j(\mu_0 - r)]}{j}\right) dr \\
&= \frac{1}{2\pi}\left(2\pi F_\Theta(a) F_X(b) + \frac{1}{\pi I_0(\kappa)}\sum_{j=1}^{\infty} I_j(\kappa)\frac{\cos[j(2\pi F_\Theta(a) - 2\pi F_X(b) - \mu_0)] - \cos[j(2\pi F_\Theta(a) - \mu_0)]}{j^2}\right. \\
&\qquad \left. - \frac{1}{\pi I_0(\kappa)}\sum_{j=1}^{\infty} I_j(\kappa)\frac{\cos[j(-2\pi F_X(b) - \mu_0)] - \cos[-j\mu_0]}{j^2}\right) \\
&= F_\Theta(a) F_X(b) + \frac{1}{2\pi^2 I_0(\kappa)}\sum_{j=1}^{\infty} I_j(\kappa)\left(\frac{\cos[j(2\pi F_\Theta(a) - 2\pi F_X(b) - \mu_0)] - \cos[j(2\pi F_\Theta(a) - \mu_0)]}{j^2}\right. \\
&\qquad \left. + \frac{-\cos[j(-2\pi F_X(b) - \mu_0)] + \cos[-j\mu_0]}{j^2}\right)
\end{aligned} \tag{5}$$

A3 Copula with Quadratic Sections

As stated in the main text, the 4 necessary and sufficient conditions for a function

$$C(u,v) = uv + \psi(u)v(1-v)\,. \tag{6}$$

to be a copula with a density that is periodic in u are

1. $\psi(u)$ is absolutely continuous on $[0,1]$.
2. $|\psi'(u)| \leq 1$ a.e. on $[0,1]$.
3. $|\psi(u)| \leq \min(u, 1-u) \quad \forall u \in [0,1]$.
4. $\psi'(u)$ must be periodic on $[0,1]$.

We have chosen $\psi'(u) = a2\pi\cos(2\pi u)$, which fulfills property 2 and 3 only for certain values of a: $|\psi'(u)| \leq 1$ holds for $-1/2\pi \leq a \leq 1/2\pi$. The third property can be rewritten:

$$\begin{aligned} \frac{d}{du} a\sin(2\pi u) &\leq \begin{cases} \frac{d}{du}\min(u, 1-u), & 0 \leq u < 0.5 \\ -\frac{d}{du}\min(u, 1-u), & 0.5 < u \leq 1 \end{cases} \\ = a2\pi\cos(2\pi u) &\leq \begin{cases} 1, & 0 \leq u < 0.5 \\ 1, & 0.5 < u \leq 1 \end{cases} \end{aligned} \tag{7}$$

and is therfore also satisfied when

$$-1/2\pi \leq a \leq 1/2\pi \tag{8}$$

A4 Copula with Cubic Sections

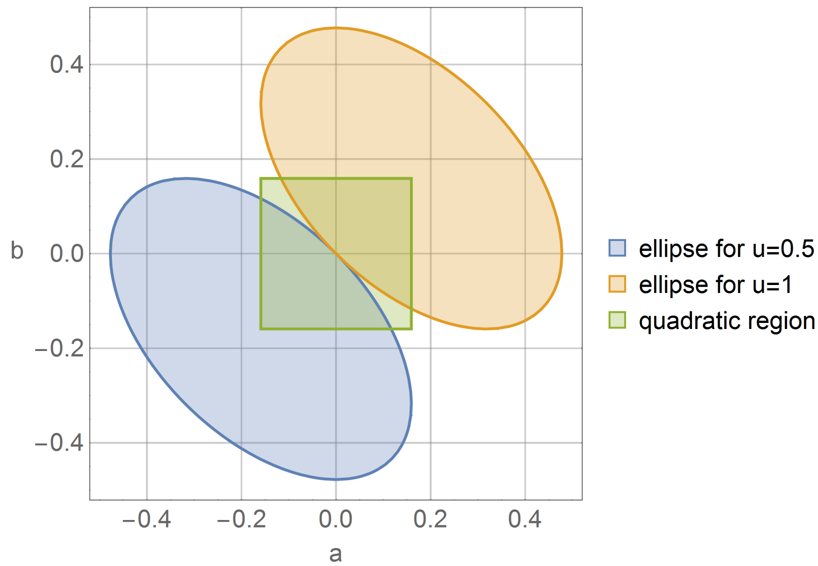

Figure A1: The set of permissible values of parameters a and b is the union of the square and the intersection of the ellipses (i.e. the square).

A5 Rectangular Patchwork Copula

We can derive the density by taking the derivatives of the CDF with respect to u and v. First for $(u, v) \in R_1$:

$$\begin{aligned} c_{patch}(u,v) &= \frac{\partial^2}{\partial u\,\partial v} C_{patch}(u,v) \\ &= \frac{\partial^2}{\partial u\,\partial v} \lambda_1 C_1(\psi_u(u), \psi_v(v)) + C_{bg}(u_1, v) \\ &= \lambda_1 \psi_u'(u)\psi_v'(v) c_1(\psi_u(u), \psi_v(v)) \end{aligned} \tag{9}$$

where we defined $\psi_u(u) = \frac{V_{C_{bg}}([u_1,u]\times[0,1])}{\lambda_1}$ and $\psi_v(v) = \frac{V_{C_{bg}}([u_1,u_2]\times[0,v])}{\lambda_1}$. We now have to find the derivatives of these two functions:

$$\begin{aligned}
\psi_u'(u) &= \frac{d}{du}\psi_u(u) \\
&= \frac{1}{\lambda_1}\frac{d}{du}\Big(C_{bg}(u_1,0) + C_{bg}(u,1) - C_{bg}(u_1,1) - C_{bg}(u,0)\Big) \\
&= \frac{1}{\lambda_1}\frac{d}{du}u \\
&= \frac{1}{\lambda_1}
\end{aligned} \tag{10}$$

$$\begin{aligned}
\psi_v'(v) &= \frac{d}{dv}\psi_v(v) \\
&= \frac{1}{\lambda_1}\frac{d}{dv}\Big(C_{bg}(u_1,0) + C_{bg}(u_2,v) - C_{bg}(u_1,v) - C_{bg}(u_2,0)\Big) \\
&= \frac{1}{\lambda_1}\frac{d}{dv}\Big(C_{bg}(u_2,v) - C_{bg}(u_1,v)\Big) \\
&= \frac{1}{\lambda_1}\Big(\int_0^{u_2} c_{bg}(x,v)\,dx - \int_0^{u_1} c_{bg}(x,v)\,dx\Big) \\
&= \frac{1}{\lambda_1}\int_{u_1}^{u_2} c_{bg}(x,v)\,dx
\end{aligned} \tag{11}$$

With similar calcualtions for R_2 we finally arrive at

$$c_{patch}(u,v) = \begin{cases}
\frac{1}{\lambda_1}\Big(\int_{u_1}^{u_2} c_{bg}(x,v)\,dx\Big)\, c_1\Big(\frac{V_{C_{bg}}([u_1,u]\times[0,1])}{\lambda_1}, \frac{V_{C_{bg}}([u_1,u_2]\times[0,v])}{\lambda_1}\Big) & \text{if } (u,v)\in R_1 \\
\frac{1}{\lambda_2}\Big(\int_{1-u_2}^{1-u_1} c_{bg}(x,v)\,dx\Big) & \\
\times c_1\Big(1 - \frac{V_{C_{bg}}([1-u_2,u]\times[0,1])}{\lambda_2}, \frac{V_{C_{bg}}([1-u_2,1-u_1]\times[0,v])}{\lambda_2}\Big) & \text{if } (u,v)\in R_2 \\
c_{bg}(u,v)) & \text{otherwise}
\end{cases} \tag{12}$$

A6 Perfect correlation

We obtained samples from a circular-linear, symmetric copula with perfect correlation using

```r
sample <- rCopula(1000000, cyl_rect_combine(normalCopula(1)))
```

These draws are shown in panel a of figure A2. When we calculate the circular-linear correlation coefficient of this sample (which consists of points along the lines $u = v/2$ and $u = 1 - v/2$), we will obtain a value close to 1. The same is, however, not true for the normalized mutual information. Therefore, we have included the argument `symmetrize` in the function `mi_cyl`. If `symmetrize=TRUE`, the empirical copula is calculated from the data and all u-values larger than 0.5 are set to $1 - 0.5$. The result of `mi_cyl` is then approximately 1 (for computational reasons it is actually only exactly 1 with very large sample sizes), in the case of perfect correlation (see panel b of figure A2).

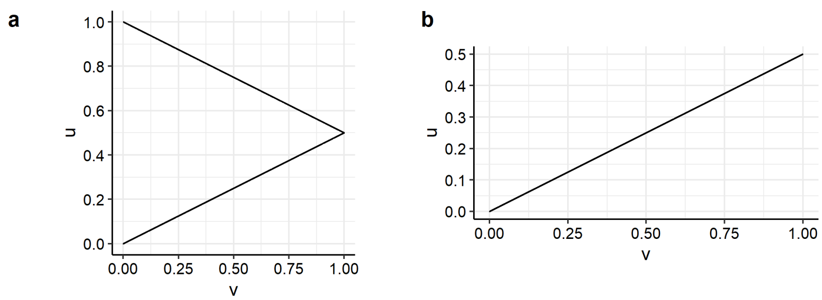

Figure A2: Left: draws from a circular-linear, symmetric copula with perfect correlation. Right: u-values larger than 0.5 of those draws are set to 1-0.5.

A7 Copulae in `opt_auto`

The copulae that are fit to the data when using the `opt_auto` function are

- `cyl_vonmises(flip = F)`
- `cyl_vonmises(flip = T)`
- `cyl_quadsec()`
- `cyl_cubsec()`
- `cyl_rot_combine(frankCopula(), shift = F)`
- `cyl_rot_combine(claytonCopula(), shift = F)`
- `cyl_rot_combine(gumbelCopula(), shift = F)`
- `cyl_rot_combine(frankCopula(), shift = T)`
- `cyl_rot_combine(claytonCopula(), shift = T)`

- `cyl_rot_combine(gumbelCopula(), shift = T)`
- `cyl_rect_combine(copula = frankCopula(), low_rect = c(0, 0.5), up_rect = "symmetric"), flip_up = T`
- `cyl_rect_combine(copula = claytonCopula(), low_rect = c(0, 0.5), up_rect = "symmetric"), flip_up = T`
- `cyl_rect_combine(copula = gumbelCopula(), low_rect = c(0, 0.5), up_rect = "symmetric"), flip_up = T`
- `cyl_rect_combine(copula = claytonCopula(), low_rect = c(0, 0.5), up_rect = "symmetric"), flip_up = F`
- `cyl_rect_combine(copula = gumbelCopula(), low_rect = c(0, 0.5), up_rect = "symmetric"),flip_up = F`

The rectangular patchwork copula based on the Frank copula with parameter α and `flip_up = F` is the same as a rectangular patchwork copula based on the Frank copula with parameter $-\alpha$ and `flip_up = T` and is therefore omitted from the list.

A8 Non-parameteric Density Estimates

We calculated non-parametric density estimates of angles drawn from a mixed von Mises distribution with parameters $\mu_1 = 0$. $\mu_2 = \pi$, $\kappa_1 = 2$, $\kappa_2 = 1$, and prop $= 0.7$. As bandwidth, we used 13.6 (result of`cylcop::opt_circ_bw(theta = traj$angle, loss="adhoc", kappa.est="ML")` and 43.2 (result of `cylcop::opt_circ_bw(theta = traj$angle, loss="adhoc", kappa.est="trigmoments")`.

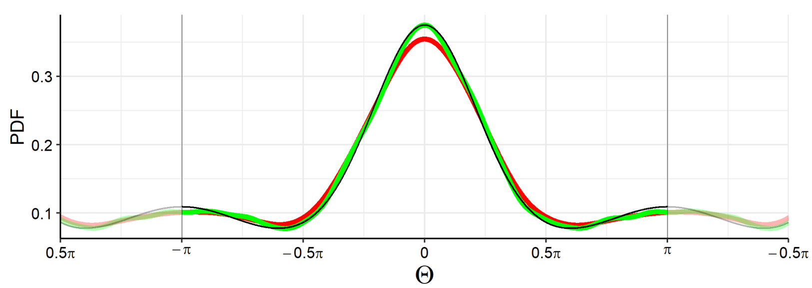

Figure A3: Black: true mixed von Mises density, red: non-parametric density estimate obtained with bandwidth 13.6, green: non-parametric density estimate obtained with bandwidth 43.2.