Supporting Information: Multidimensional stimulus-response correlation

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1 Hybrid encoding and decoding

To implement the required optimization problems (Eq. 5-7 in the main text) we now formulate those expressions in matrix-vector notation. Let the stimulus be represented as a L-length row vector $\mathbf{s}$, where L is the duration of the stimulus and the neural response as a $D \times L$ matrix $\mathbf{R}$. Note that we have assumed that $\mathbf{s}$ and $\mathbf{R}$ have equivalent sampling rates – in practice, either the neural response or the stimulus must be resampled to the lower of the two native sampling rates. To implement the convolution, it will be convenient to define a Toeplitz matrix ${}^*\mathbf{s}$ with column and row indices τ and t: $({}^*\mathbf{s})_{\tau,t} = s(t-\tau)$, where $s(t)$, are the elements of vector $\mathbf{s}$ and, $\tau = 1 \dots Q$, denotes the taps of the applied temporal filter. Elements prior to the first sample can be set to zero assuming that there is no stimulation prior to the start. This matrix has dimensions $Q \times L$. With this we can now define the temporally filtered stimulus and spatially filtered response:

$$\begin{aligned} \mathbf{u} &= \mathbf{h}^T \, {}^*\mathbf{s}, \\ \mathbf{v} &= \mathbf{w}^T \mathbf{R}. \end{aligned} \tag{1}$$

The temporal and spatial filters that maximize the correlation between $\mathbf{u}$ and $\mathbf{v}$ are given CCA, which provides a set of components encompassing multiple filter pairs $\{\mathbf{h}_k, \mathbf{w}_k, k = 1 \dots K\}$ as the following eigenvectors [1, 3]:

$$\begin{aligned} \left({}^*\mathbf{s}\,{}^*\mathbf{s}^T\right)^{-1} {}^*\mathbf{s}\mathbf{R}^T \left(\mathbf{R}\mathbf{R}^T\right)^{-1} \mathbf{R}\,{}^*\mathbf{s}^T \mathbf{h}_k &= \rho_k \mathbf{h}_k, \\ \left(\mathbf{R}\mathbf{R}^T\right)^{-1} \mathbf{R}\,{}^*\mathbf{s}^T \left({}^*\mathbf{s}\,{}^*\mathbf{s}^T\right)^{-1} {}^*\mathbf{s}\mathbf{R}^T \mathbf{w}_k &= \rho_k \mathbf{w}_k. \end{aligned} \tag{2}$$

The number of components K is limited by the rank of the data: $K = min(rank({}^*\mathbf{s}), rank(\mathbf{R}))$. The maximal correlation of ρ_1 is achieved by projecting the stimulus onto $\mathbf{h_1}$ and the neural response onto $\mathbf{w_1}$. Subsequent components $(\mathbf{h_k}, \mathbf{w_k}), k = 2, \dots, K$, yield projections temporally uncorrelated with previous ones and progressively lower correlations, such that $\rho_1 > \rho_2 > \dots \rho_K$.

For practical purposes, it is worth noting that equation (2) is the conventional CCA solution which has been implemented in many toolboxes. In particular, one can simply execute the `canoncorr` function in MATLAB with matrices $\mathbf{R}$ and ${}^*\mathbf{s}$ as inputs. However, in most cases these toolboxes will not have implemented regularization, as discussed next.

2 Regularization

While the hybrid encoding-decoding model will generally possess fewer parameters than conventional encoding or decoding, it may still be beneficial to regularize the solutions. In particular, CCA inverts the covariance matrices of the neural response and the stimulus ($\mathbf{R}\mathbf{R}^T$ and ${}^*\mathbf{s}\,{}^*\mathbf{s}^T$ respectively). Prior to inversion of these matrices, it is important to limit dimensions with small eigenvalues that are dominated by noise. To this end one can substitute the inverse of the covariance matrix $\mathbf{C}$ with:

$$\mathbf{C}^{-1} \leftarrow \mathbf{B} \left[\boldsymbol{\Lambda}^{-1}\right]_J \mathbf{B}^T, \tag{3}$$

where $\mathbf{B}$ is a matrix of eigenvectors of $\mathbf{C}$ sorted in descending order of associated eigenvalues, and $\left[\mathbf{\Lambda}^{-1}\right]_J$ a diagonal matrix with the corresponding eigenvalues inverted and set to zero for all dimensions beyond J. Decreasing J increases the strength of regularization. Here we selected the value of J (i.e., 10) as the knee point of the eigenvalue spectrum for both neural response and stimulus. Importantly, none of the results reported in the main text depended critically on this choice.

3 Spatial and temporal response

To visualize the spatial distribution of neural activity associated with each component, it is conventional to use the "forward model" formalism [4, 2]. The forward model is defined as the linear mapping that best recovers the neural response $\mathbf{R}$ from the decoded response $\mathbf{V}$ in a least-squares sense, namely

$$\mathbf{A}_r = \left(\mathbf{V}\mathbf{V}^T\right)^{-1}\mathbf{V}\mathbf{R}^T. \tag{4}$$

where $\mathbf{V} = \mathbf{W}^T\mathbf{R}$ is the matrix of decoded neural responses, $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2 \ldots \mathbf{w}_K]$ is a matrix of J CCA-derived spatial filters, and the corresponding forward models are the columns of matrix $\mathbf{A}_r = [\mathbf{a}_1, \mathbf{a}_2 \ldots \mathbf{a}_K]$. The kth column $\mathbf{a}_k$ reflects the spatial mapping from the stimulus-tracking source of neural activity $\mathbf{v}_k$ to the scalp sensors.

This forward model is equal (up to a scaling of each component) to the "spatial response" defined here as the linear mapping that best recovers (in a least-squares sense) the neural response $\mathbf{R}$ from the temporally filtered stimulus $\mathbf{U}$:

$$\mathbf{A}_s = \left(\mathbf{U}\mathbf{U}^T\right)^{-1}\mathbf{U}\mathbf{R}^T. \tag{5}$$

where $\mathbf{H}$ and $\mathbf{V}$ denote matrices composed of the vectors $\mathbf{h}_k$, and $\mathbf{v}_k$, respectively. The proportionality of the forward model and this spatial response follows from the fact that both $\mathbf{U}\mathbf{U}^T$ and $\mathbf{V}\mathbf{V}^T$ are diagonal matrices, and that $\mathbf{U}\mathbf{R}^T \propto \mathbf{V}\mathbf{R}^T$, where $\propto$ indicates that both sides are equal up to a diagonal scaling matrix (see Eqs. 4.31 and 4.26 in [1] respectively). Therefore, $\mathbf{A}_r \propto \mathbf{V}\mathbf{R}^T \propto \mathbf{U}\mathbf{R}^T \propto \mathbf{A}_s$.

In total, $\mathbf{H}$ is the "temporal response" and $\mathbf{A}$ is the "spatial response" which together best recover the neural response from the stimulus with the following rank-K linear estimate:

$$\hat{\mathbf{R}} = \mathbf{A}\mathbf{H}^T\,{}^*\mathbf{s}. \tag{6}$$

4 Encoding and decoding

For comparison and reference we provide here the equations for the encoding and decoding approaches. To implement the temporal correlation (filtering) in equation (6) of the main text we define a Hankel matrix with column and row indices τ and t: $({}^\star\mathbf{r})_{\tau,t} = r(t+\tau)$, and a block-Hankel matrix that concatenates the responses for all sensors, ${}^\star\mathbf{R} = \left[{}^\star\mathbf{r}_1^T, \ldots, {}^\star\mathbf{r}_D^T\right]^T$. With that we can write the encoded response and decoded stimulus as:

$$\hat{\mathbf{r}}_i = \mathbf{h}_i^T\,{}^*\mathbf{s}, \tag{7}$$

$$\hat{\mathbf{s}} = \mathbf{w}^T\,{}^\star\mathbf{R}. \tag{8}$$

The best encoding and decoding filters (Eqs. 5 and 6 in the main text) are unique and are given by the conventional least-squares estimates:

$$\mathbf{h}_i = \left({}^*\mathbf{s}\,{}^*\mathbf{s}^T\right)^{-1}{}^*\mathbf{s}\mathbf{r}_i^T, \tag{9}$$

$$\mathbf{w} = \left({}^*\mathbf{R}\,{}^*\mathbf{R}^T\right)^{-1}{}^\star\mathbf{R}\mathbf{s}^T. \tag{10}$$

Note the similarities of these equations to the CCA equations (2).

References

[1] Magnus Borga. *Learning multidimensional signal processing*. PhD thesis, Linköping University, 1998.

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[3] Harold Hotelling. Relations between two sets of variates. *Biometrika*, 28(3/4):321–377, 1936.

[4] Lucas C Parra, Clay D Spence, Adam D Gerson, and Paul Sajda. Recipes for the linear analysis of eeg. *Neuroimage*, 28(2):326–341, 2005.