

**SUPPLEMENTARY MATERIAL FOR ‘CO-OPERATION, COMPETITION AND  
CROWDING: A DISCRETE FRAMEWORK LINKING ALLEE KINETICS,  
NONLINEAR DIFFUSION, SHOCKS AND SHARP-FRONTED TRAVELLING WAVES’**

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S1. Equal motility rates, no grouped agent death.	2
S2. Different motility rates, no grouped agent death.	3
S2.1. Strictly positive nonlinear diffusivity function	3
S2.2. Extinction-degenerate non-negative nonlinear diffusivity function	4
S2.3. Positive-negative-positive nonlinear diffusivity function.	5
S2.4. Capacity-degenerate positive-negative nonlinear diffusivity function	6
S3. Equal motility rates, different death rates.	7
S4. Different motility rates, different death rates.	8
S4.1. Positive-negative-positive nonlinear diffusivity function	8
S4.2. Capacity-degenerate positive-negative nonlinear diffusivity function	9
S4.3. Positive-negative nonlinear diffusivity function	10

The travelling wave behaviour presented here represents the same competitive and/or co-operative mechanisms described in the main document. As such, the discussion about the implications of the travelling wave behaviour and subsequent persistence of the population is not repeated. Note that the results here are Cases 5-8 with different Allee kinetics. Additional details can be found in the corresponding cases in the main document.

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S1. EQUAL MOTILITY RATES, NO GROUPED AGENT DEATH.

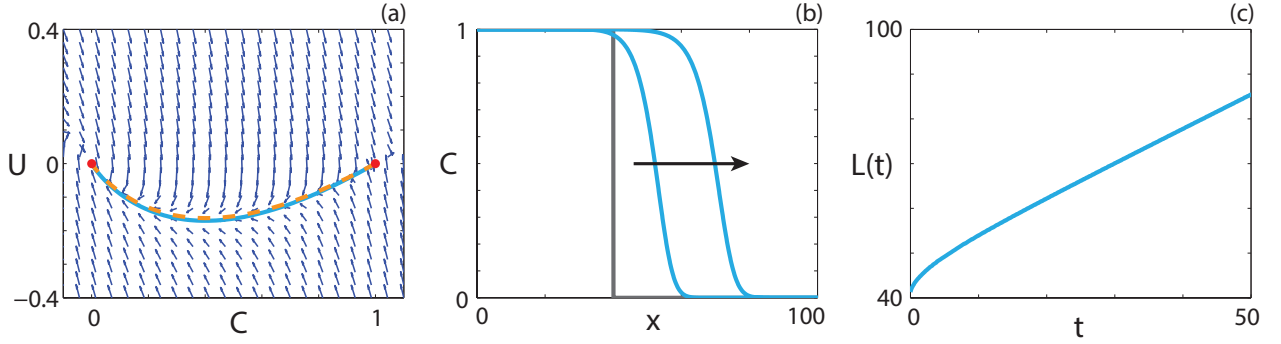


FIGURE S1. **Travelling wave behaviour for Equation (46) with the reverse Allee effect and constant  $F(C)$  (Case 5).** (a) Phase plane for the system (48)-(49) with the numerical solution to Equations (46) (cyan, solid) and (47) (orange, dashed), in  $(C, U)$  co-ordinates, superimposed. Red circles correspond to equilibrium points. (b) Numerical solution to Equation (46) calculated at  $t = 25$  and  $t = 50$ . The grey lines indicate the initial condition and the arrow indicates the direction of increasing time. (c) The time evolution of the position of the leading edge of the wave front. All results are obtained with  $\delta x = 0.1$ ,  $\delta t = 0.01$ ,  $\epsilon = 10^{-6}$ ,  $P_m^i = 1.0$ ,  $P_m^g = 1.0$ ,  $P_p^i = 0.6$ ,  $P_p^g = 0.2$ ,  $P_d^i = 0.3$ ,  $P_d^g = 0$ ,  $v = 0.756$ .

S2.1. Strictly positive nonlinear diffusivity function.

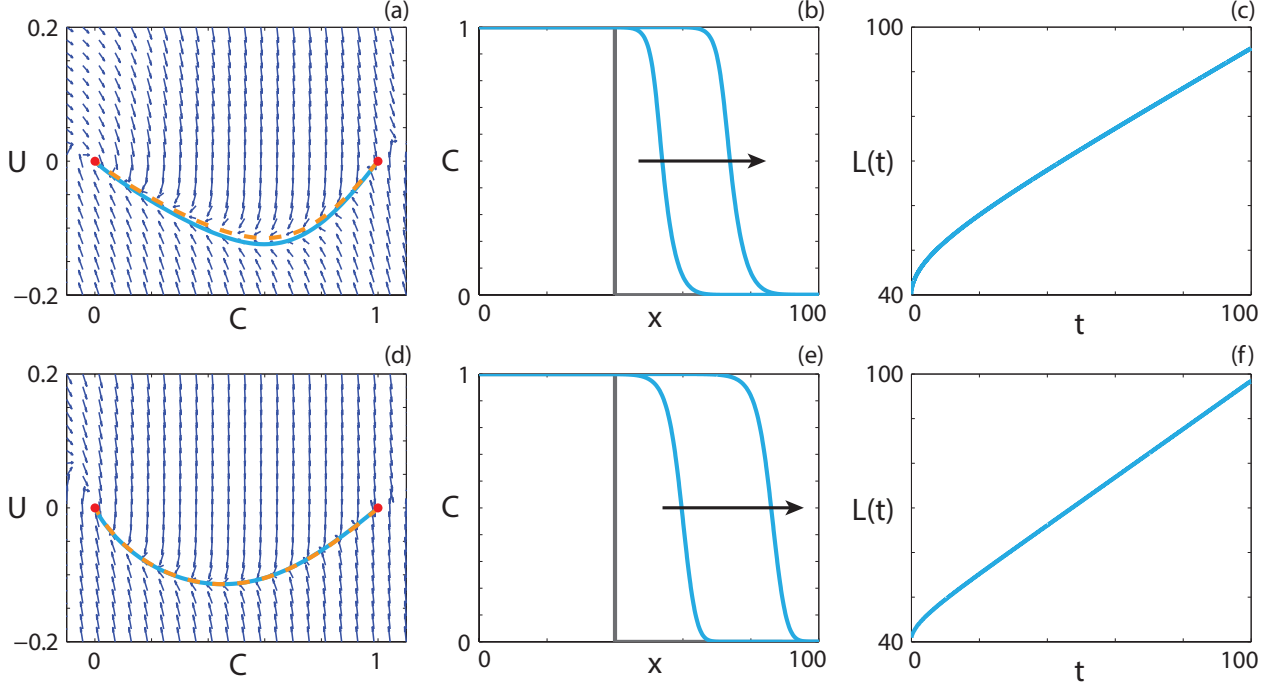


FIGURE S2. Travelling wave behaviour for Equation (50) with the (a)-(c) weak Allee effect and the (d)-(f) reverse Allee effect and strictly positive  $F(C)$  (Case 6.1). (a),(d) Phase plane for the system (52)-(53) with the numerical solution to Equations (50) (cyan, solid) and (51) (orange, dashed), in  $(C, U)$  co-ordinates, superimposed. Red circles correspond to equilibrium points. (b),(e) Numerical solution to Equation (50) calculated at  $t = 50$  and  $t = 100$ . The grey lines indicate the initial condition and the arrows indicate the direction of increasing time. (c),(f) The time evolution of the position of the leading edge of the wave front. All results are obtained with  $\delta x = 0.1$ ,  $\delta t = 0.01$ ,  $\epsilon = 10^{-6}$ ,  $P_d^g = 0$ , (a)-(c)  $P_m^i = 1.0$ ,  $P_m^g = 0.5$ ,  $P_p^i = 0.4$ ,  $P_p^g = 0.3$ ,  $P_d^i = 0.3$ ,  $v = 0.448$ , (d)-(f)  $P_m^i = 0.5$ ,  $P_m^g = 0.25$ ,  $P_p^i = 0.6$ ,  $P_p^g = 0.2$ ,  $P_d^i = 0.3$ ,  $v = 0.536$ .

## S2.2. Extinction-degenerate non-negative nonlinear diffusivity function.

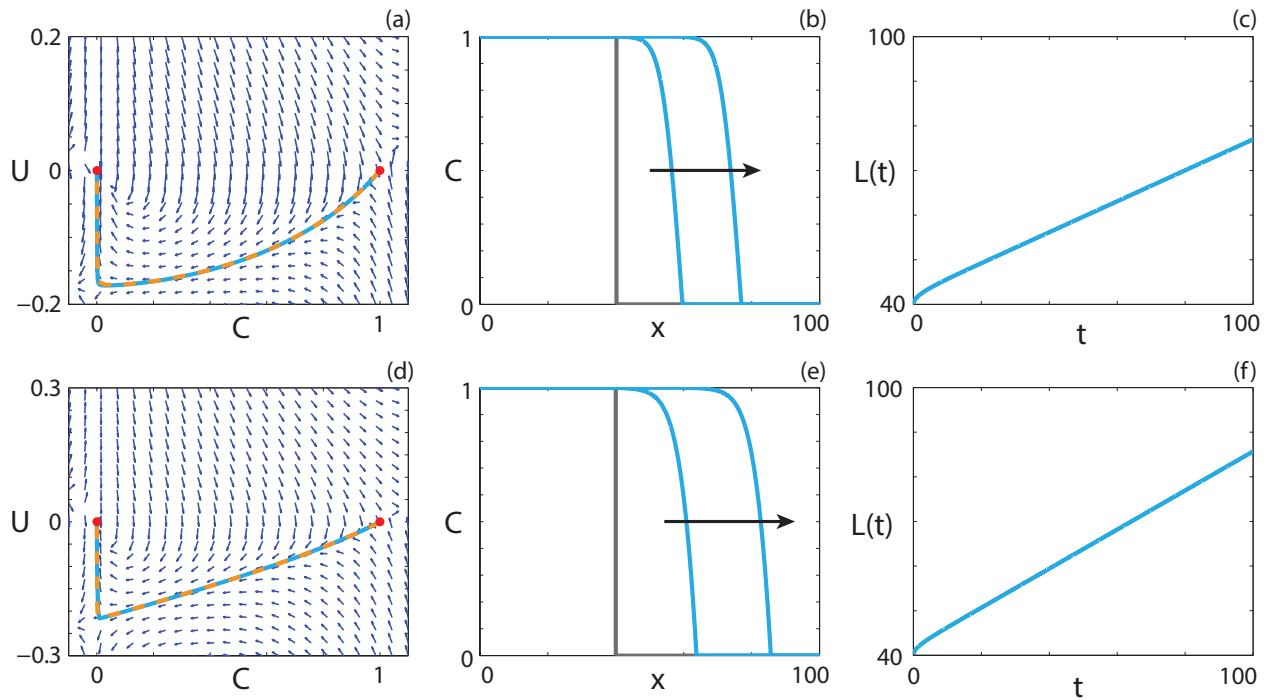


FIGURE S3. Travelling wave behaviour for Equation (50) with the (a)-(c) weak Allee effect and the (d)-(f) reverse Allee effect and extinction-degenerate non-negative  $F(C)$  (Case 6.2). (a),(d) Phase plane for the system (52)-(53) with the numerical solution to Equations (50) (cyan, solid) and (51) (orange, dashed), in  $(C, U)$  co-ordinates, superimposed. Red circles correspond to equilibrium points. (b),(e) Numerical solution to Equation (50) calculated at  $t = 50$  and  $t = 100$ . The grey lines indicate the initial condition and the arrows indicate the direction of increasing time. (c),(f) The time evolution of the position of the leading edge of the wave front. All results are obtained with  $\delta x = 0.1$ ,  $\delta t = 0.01$ ,  $\epsilon = 10^{-6}$ ,  $P_m^i = 0$ ,  $P_m^g = 1.0$ ,  $P_d^g = 0$ , (a)-(c)  $P_p^i = 0.4$ ,  $P_p^g = 0.3$ ,  $P_d^i = 0.3$ ,  $v = 0.347$ , (d)-(f)  $P_p^i = 0.6$ ,  $P_p^g = 0.2$ ,  $P_d^i = 0.3$ ,  $v = 0.438$ .

### S2.3. Positive-negative-positive nonlinear diffusivity function.

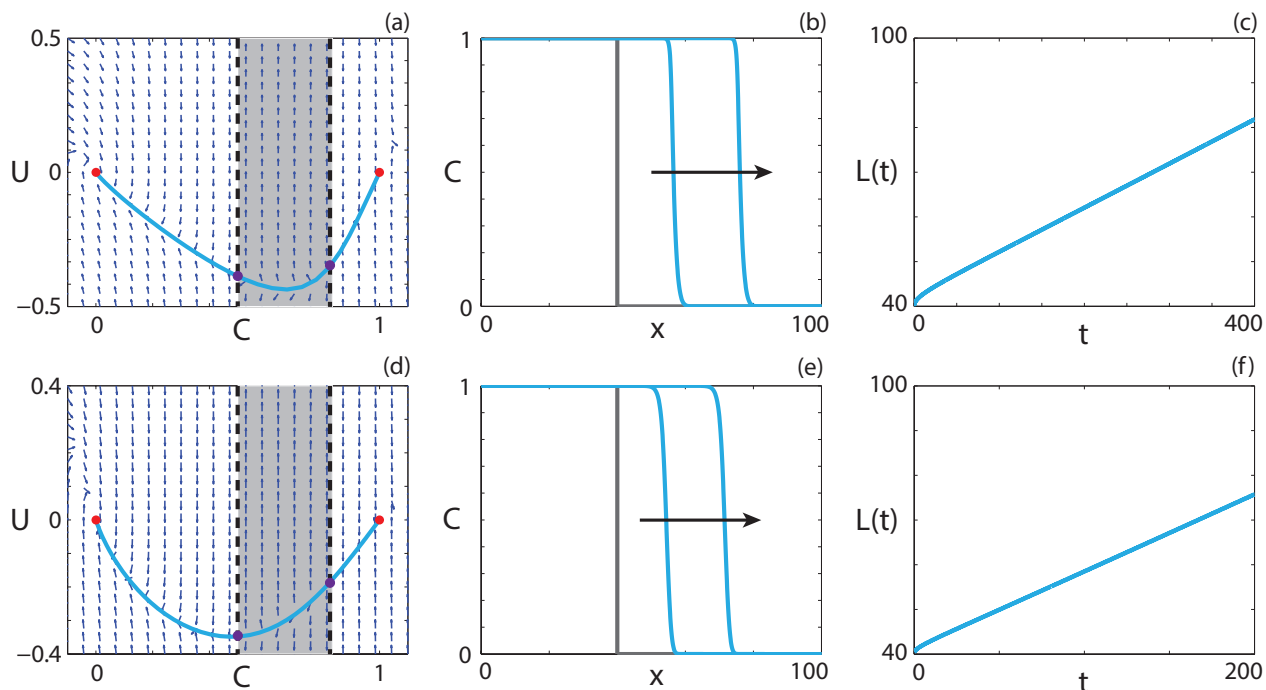


FIGURE S4. Travelling wave behaviour for Equation (50) with the (a)-(c) weak Allee effect and the (d)-(f) reverse Allee effect and positive-negative-positive  $F(C)$  (Case 6.3). (a),(d) Phase plane for the system (52)-(53) with the numerical solution to Equation (50) (cyan, solid), in  $(C, U)$  co-ordinates, superimposed. The dashed black lines denote a wall of singularities. Red circles correspond to equilibrium points and purple circles correspond to holes in the wall. (b),(e) Numerical solution to Equation (50) calculated at (b)  $t = 200$  and  $t = 400$ , (e)  $t = 100$  and  $t = 200$ . The grey lines indicate the initial condition and the arrows indicate the direction of increasing time. (c),(f) The time evolution of the position of the leading edge of the wave front. All results are obtained with  $\delta x = 0.1$ ,  $\delta t = 0.01$ ,  $\epsilon = 10^{-6}$ ,  $P_d^g = 0$ , (a)-(c)  $P_m^i = 0.05$ ,  $P_m^g = 0.01$ ,  $P_p^i = 0.4$ ,  $P_p^g = 0.3$ ,  $P_d^i = 0.3$ ,  $v = 0.098$ , (d)-(f)  $P_m^i = 0.5$ ,  $P_m^g = 0.1$ ,  $P_p^i = 0.6$ ,  $P_p^g = 0.2$ ,  $P_d^i = 0.3$ ,  $v = 0.172$ .

S2.4. Capacity-degenerate positive-negative nonlinear diffusivity function.

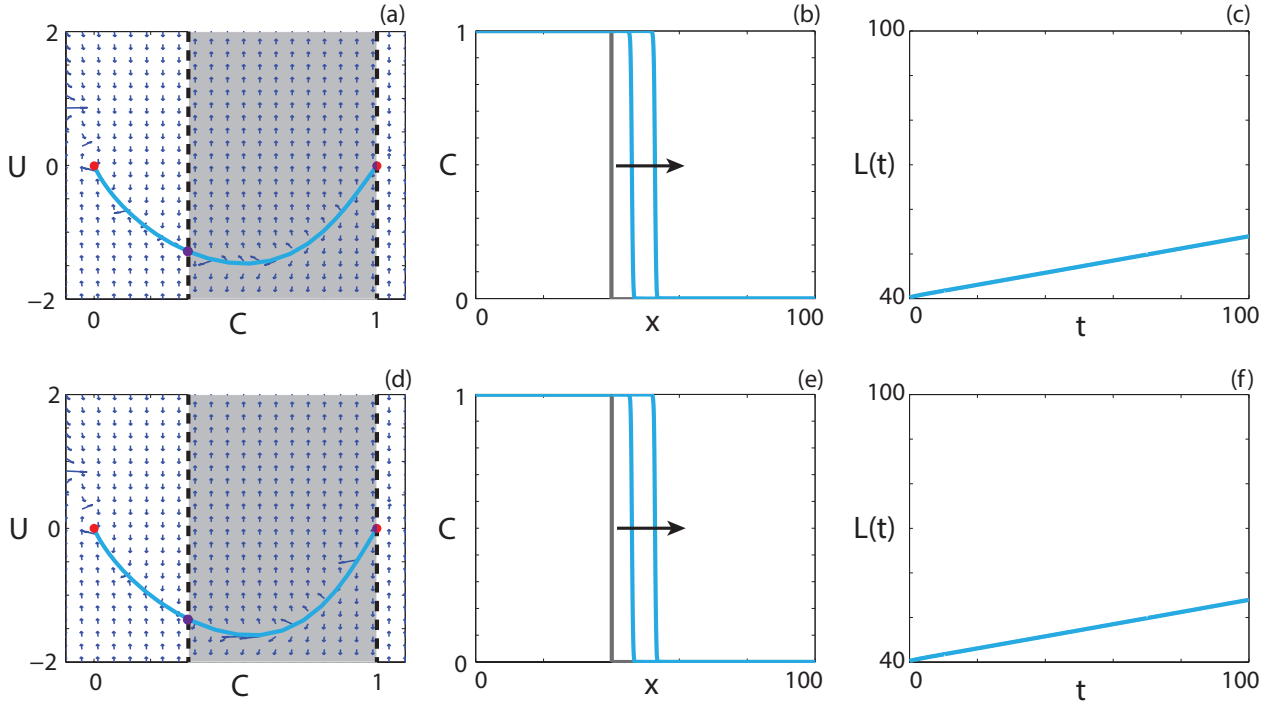


FIGURE S5. Travelling wave behaviour for the (a)-(c) weak Allee effect and the (d)-(f) reverse Allee effect with capacity-degenerate  $F(C)$  (Case 6.4). (a), (d) Phase plane for the system (52)-(53) with the numerical solution of Equation (50) (cyan, solid), in  $(C, U)$  co-ordinates, superimposed. The dashed black lines denote a wall of singularities. Red circles correspond to equilibrium points and purple circles correspond to holes in the wall. (b), (e) Numerical solution of Equation (50) calculated at  $t = 50$  and  $t = 100$ . The grey lines indicate the initial condition and the arrows indicate the direction of increasing time. (c), (f) The time evolution of  $L(t)$ . All results are obtained with  $\delta x = 0.1$ ,  $\delta t = 0.01$ ,  $\epsilon = 10^{-6}$ ,  $P_m^i = 0.01$ ,  $P_m^g = 0$ ,  $P_p^i = 1.0$ ,  $P_d^i = 0.1$ ,  $P_d^g = 0$ , (a)-(c)  $P_p^g = 0.8$ ,  $v = 0.098$ , (d)-(f)  $P_p^g = 0.95$ ,  $v = 0.136$ .

S3. EQUAL MOTILITY RATES, DIFFERENT DEATH RATES.

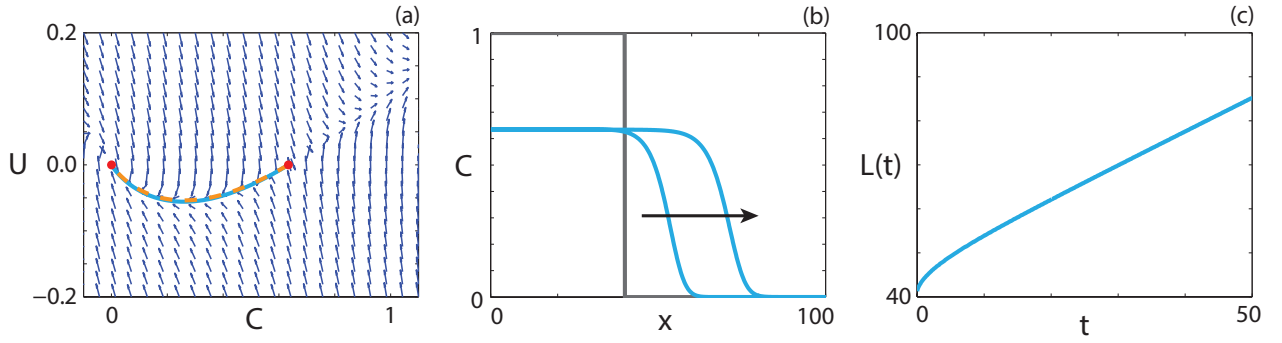


FIGURE S6. **Travelling wave behaviour for Equation (61) with the reverse Allee effect and constant  $F_s(\bar{C})$  (Case 7).** (a) Phase plane for the system (64)-(65) with the numerical solution to Equations (61) (cyan, solid) and (63) (orange, dashed), in  $(C, U)$  co-ordinates, superimposed. Red circles correspond to equilibrium points. (b) Numerical solution to Equation (61) calculated at  $t = 25$  and  $t = 50$ . The grey lines indicate the initial condition and the arrow indicates the direction of increasing time. (c) The time evolution of the position of the leading edge of the wave front. All results are obtained with  $\delta x = 0.1$ ,  $\delta t = 0.01$ ,  $\epsilon = 10^{-6}$ ,  $P_m^i = 1.0$ ,  $P_m^g = 1.0$ ,  $P_p^i = 0.6$ ,  $P_p^g = 0.2$ ,  $P_d^i = 0.3$ ,  $P_d^g = 0.1$ ,  $v = 0.760$ .

S4.1. Positive-negative-positive nonlinear diffusivity function.

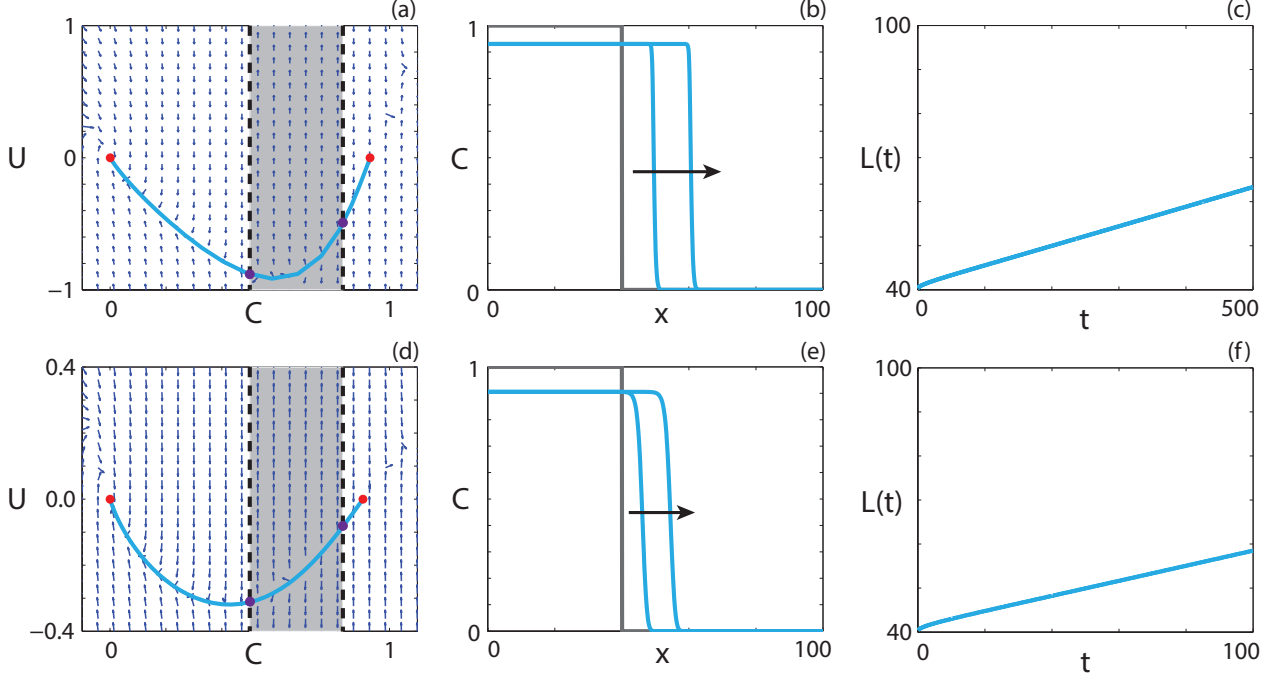


FIGURE S7. Travelling wave behaviour for Equation (68) with the (a)-(c) weak Allee effect and the (d)-(f) reverse Allee effect with positive-negative-positive  $F_s(\bar{C})$  (Case 8.3). (a), (d) Phase plane for the system (70)-(71) with the numerical solution to Equations (68) (cyan, solid) and (69) (orange, dashed), in  $(C, U)$  co-ordinates, superimposed. Red circles correspond to equilibrium points. (b), (e) Numerical solution to Equation (68) calculated at (b)  $t = 250$  and  $t = 500$ , (e)  $t = 50$  and  $t = 100$ . The grey lines indicate the initial condition and the arrow indicates the direction of increasing time. (c), (f) The time evolution of the position of the leading edge of the wave front. All results are obtained with  $\delta x = 0.1$ ,  $\delta t = 0.01$ ,  $\epsilon = 10^{-6}$ , (a)-(c)  $P_m^i = 0.01$ ,  $P_m^g = 0.002$ ,  $P_p^i = 0.3$ ,  $P_p^g = 0.4$ ,  $P_d^i = 0.3$ ,  $P_d^g = 0.02$ ,  $v = 0.045$ , (d)-(f)  $P_m^i = 0.05$ ,  $P_m^g = 0.01$ ,  $P_p^i = 0.6$ ,  $P_p^g = 0.2$ ,  $P_d^i = 0.3$ ,  $P_d^g = 0.02$ ,  $v = 0.172$ .



S4.2. Capacity-degenerate positive-negative nonlinear diffusivity function.

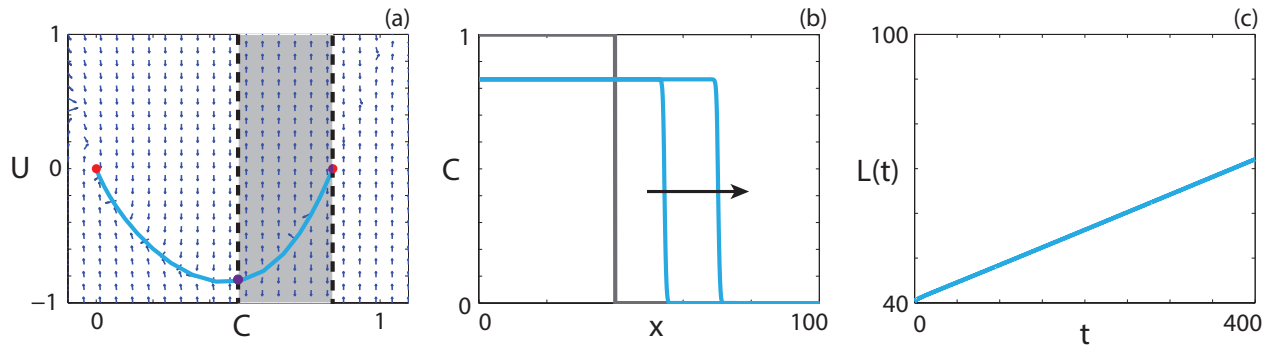


FIGURE S8. Travelling wave behaviour for Equation (68) with the weak Allee effect and positive-negative capacity-degenerate  $F_s(\bar{C})$  (Case 8.5). (a) Phase plane for the system (70)-(71) with the numerical solution to Equations (68) (cyan, solid) and (69) (orange, dashed), in  $(C, U)$  co-ordinates, superimposed. Red circles correspond to equilibrium points. (b) Numerical solution to Equation (68) calculated at  $t = 200$  and  $t = 400$ . The grey lines indicate the initial condition and the arrow indicates the direction of increasing time. (c) The time evolution of the position of the leading edge of the wave front. All results are obtained with  $\delta x = 0.1$ ,  $\delta t = 0.01$ ,  $\epsilon = 10^{-6}$ ,  $P_m^i = 0.01$ ,  $P_m^g = 0.002$ ,  $P_p^i = 0.6$ ,  $P_p^g = 0.5$ ,  $P_d^i = 0.3$ ,  $P_d^g = 0.08$ ,  $v = 0.079$ .

### S4.3. Positive-negative nonlinear diffusivity function.

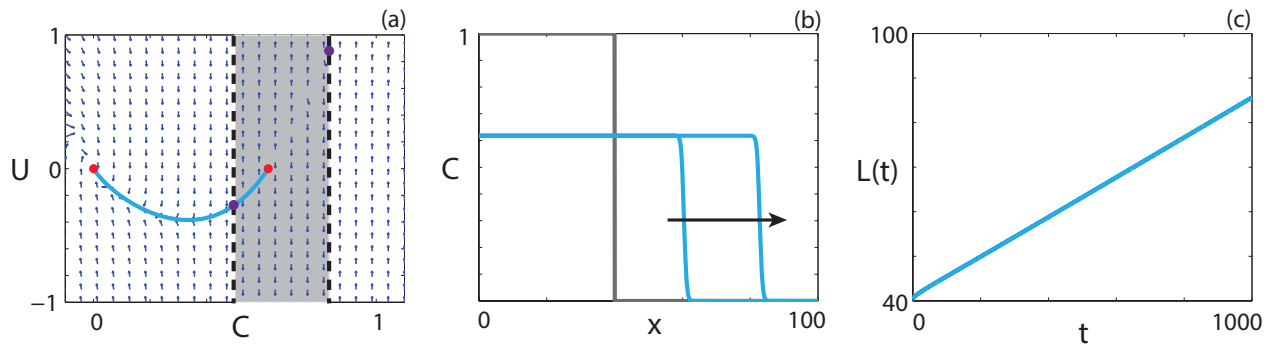


FIGURE S9. **Travelling wave behaviour for Equation (68) with the reverse Allee effect and positive-negative  $F_s(\bar{C})$  (Case 8.4).** (a) Phase plane for the system (70)-(71) with the numerical solution to Equations (68) (cyan, solid) and (69) (orange, dashed), in  $(C, U)$  co-ordinates, superimposed. Red circles correspond to equilibrium points. (b) Numerical solution to Equation (68) calculated at  $t = 50$  and  $t = 100$ . The grey lines indicate the initial condition and the arrow indicates the direction of increasing time. (c) The time evolution of the position of the leading edge of the wave front. All results are obtained with  $\delta x = 0.1$ ,  $\delta t = 0.01$ ,  $\epsilon = 10^{-6}$ ,  $P_m^i = 0.01$ ,  $P_m^g = 0.002$ ,  $P_p^i = 0.4$ ,  $P_p^g = 0.3$ ,  $P_d^i = 0.3$ ,  $P_d^g = 0.1$ ,  $v = 0.045$ .