

Supporting Information for “Resilience, invariability, and ecological stability across levels of organization”

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S1 Definition of stability measures

In this section we define the stability measures used in the main text. We consider a continuous-time dynamical system,

$$\frac{dX_i}{dt} = F_i(X_1, \dots, X_n) \quad \text{for } i = 1, \dots, n,$$

or in vector form,

$$\frac{dX}{dt} = F(X), \tag{1}$$

where X is a vector containing the dynamical variables X_i and F is a non-linear map consisting of the right-hand sides F_i of the dynamical equations.

We assume that the stationary state of dynamical system (1) is an equilibrium point (that is, we exclude limit cycles and chaotic attractors). Denoting the equilibrium point by X^* , we have $F(X^*) = 0$. Linearizing around this equilibrium point, we get the linear system

$$\frac{d\tilde{X}}{dt} = A\tilde{X}, \tag{2}$$

with $\tilde{X} = X - X^*$ and A the Jacobian matrix of F evaluated at X^* ,

$$A_{ij} = \left. \frac{\partial F_i(X)}{\partial X_j} \right|_{X=X^*}.$$

The condition for stability of the linear dynamical system (2), and thus the condition for local stability of the equilibrium point X^* , reads that all eigenvalues of A have negative real part,

$$\Re(\lambda_i) < 0 \quad \text{for all eigenvalues } \lambda_i \text{ of } A.$$

Equivalently, the stability condition can be expressed in terms of the dominant eigenvalue λ_{dom} of dynamical system (2), that is, the eigenvalue of A with the largest real part,

$$\Re(\lambda_{\text{dom}}) < 0. \tag{3}$$

From here on we assume X^* to be stable. We are interested in quantifying how stable the equilibrium point X^* is. We define several such stability measures.

Asymptotic resilience \mathcal{R}

Looking at condition (3), a straightforward choice for a stability measure is

$$\mathcal{R} = -\Re(\lambda_{\text{dom}}). \tag{4}$$

Clearly, \mathcal{R} is positive for stable systems. It is close to zero when the dynamical system is close to the stability boundary (3).

Measure \mathcal{R} is often called *asymptotic resilience*, because, generically, it equals the asymptotic rate of return to equilibrium after a pulse perturbation. This property can be easily formulated in the case of a single, real dominant eigenvalue (other cases can be dealt with similarly, but with additional technical difficulties). Denote by U the initial displacement of the system, and assume that a linear combination $\sum_i W_i \tilde{X}_i(t)$ of the dynamical variables is observed. Then, the observed return to equilibrium is

$$D(t) = W^T \tilde{X}(t) = W^T e^{At} U.$$

We impose a genericity condition: displacement vector U should not be orthogonal to the left eigenvector associated with λ_{dom} , and observation vector W should not be orthogonal to the (right) eigenvector associated with λ_{dom} . Then,

$$\mathcal{R} = \lim_{t \rightarrow \infty} -\frac{1}{|D(t)|} \frac{d|D(t)|}{dt} = \lim_{t \rightarrow \infty} -\frac{\ln |D(t)|}{t},$$

that is, the long-term rate of return to equilibrium is given by asymptotic resilience. Note that the genericity condition is almost always satisfied, so that the long-term return rate is essentially independent of the applied perturbation and of the observed system variable. In the main text this property is referred to as the rigidity of asymptotic resilience.

Invariability measures

Alternatively, stability can be quantified by looking at the response of dynamical system (1) to perturbations. We introduce a stochastic forcing term that we assume to be sufficiently small, so that we can study its effects using the linear dynamical system (2). The resulting dynamical system is

$$\frac{d\tilde{X}_i}{dt} = \sum_j A_{ij} \tilde{X}_j + \gamma_i(t) \quad \text{for } i = 1, \dots, n, \quad (5)$$

with $\gamma_i(t)$ the stochastic forcing. We make the following assumptions:

- $\gamma_i(t)$ is a Gaussian variable with zero mean. This guarantees that the deterministic behavior of dynamical system (5) is given by dynamical system (2). Furthermore, we assume that the variance of $\gamma_i(t)$ is proportional to $(X_i^*)^2$. This corresponds to the assumption that in the nonlinear dynamical system the stochastic forcing acts in a multiplicative way. Alternatively, one could assume, for example, that the variance of $\gamma_i(t)$ is proportional to X_i^* , which would lead to quite different invariability measures.
- $\gamma_i(t)$ as a function of t (and for fixed i) is white noise. This means that the $\gamma_i(t)$ at different t are mutually independent and have the same Gaussian distribution. Equivalently, this means that the frequency spectrum of the noise is colorless, that is, all frequency components have the same (expected) amplitude.
- $\gamma_i(t)$ for different i (and for fixed t) are independent. This means that there are no correlations between the noise terms acting on different system variables. Without detailed information about the perturbations, correlations between $\gamma_i(t)$ and $\gamma_j(t)$ can be both positive and negative. Hence, the assumption of no correlations can be considered as a mean-case scenario (worst-case scenarios are considered by Arnoldi *et al.* 2016).

Based on these assumptions, we can write dynamical system (5) using the formalism of stochastic differential equations (Van Kampen, 2007),

$$d\tilde{X}_i = \sum_j A_{ij} \tilde{X}_j dt + \sigma X_i^* dW_i(t) \quad \text{for } i = 1, \dots, n,$$

or in vector form,

$$d\tilde{X} = A\tilde{X}dt + \sigma \text{diag}(X^*) dW(t), \quad (6)$$

with $W(t)$ a vector of n independent Wiener processes $W_i(t)$. Dynamical system (6) can be simulated using the iterative scheme

$$\tilde{X}(t + \Delta t) = \tilde{X}(t) + A\tilde{X}(t)\Delta t + \sigma \text{diag}(X^*) \xi(t)\sqrt{\Delta t},$$

for a small time step Δt . Here the vector $\xi(t)$ contains n independent Gaussian variables with zero mean and unit standard deviation. The constant σ is a measure of the intensity of the stochastic forcing. For small σ the linearization provides an accurate approximation of the nonlinear dynamics (Fig. S1). Note that σ^2 can be interpreted as the variance per unit of time of the applied stochasticity, implying that σ^2 has units of 1/time.

For stable systems, dynamics (6) approach a stationary state after a sufficiently long time. In the stationary state dynamical variables $\tilde{X}_i(t)$ fluctuate around zero (and dynamical variables $X_i(t)$ fluctuate around X_i^*). The covariance matrix C of the stationary fluctuations can be obtained from the Lyapunov equation (Van Kampen, 2007),

$$AC + CA^\top + \sigma^2 \text{diag}(X^*)^2 = 0. \quad (7)$$

The covariance matrix C allows us to compute the stationary fluctuations of any system variable $Q(X)$. Recall that the linearization is based on the assumption that the deviations \tilde{X} from the equilibrium X^* are small (which is satisfied for small σ). We use the same assumption to restrict attention to variables $Q(X)$ that are of first order in \tilde{X} ,

$$Q(X) = Q(X^*) + V_Q^\top \tilde{X},$$

where V_Q is the gradient vector of Q evaluated at X^* ,

$$(V_Q)_i = \left. \frac{\partial Q(X)}{\partial X_i} \right|_{X=X^*}.$$

Then, the variance of the stationary fluctuations are given by

$$\text{Var}(Q) = V_Q^\top C V_Q.$$

By normalizing this variance with respect to the mean $Q(X^*)$ of variable $Q(X)$, and with respect to the intensity σ of the applied perturbation, we obtain the variability measure $\text{Var}(Q)/Q(X^*)^2/\sigma^2$. The invariability \mathcal{I}_Q of system variable Q is then defined as

$$\mathcal{I}_Q = \frac{\sigma^2}{2} \frac{1}{\text{CV}(Q)^2} = \frac{\sigma^2}{2} \frac{Q(X^*)^2}{\text{Var}(Q)}, \quad (8)$$

where CV denotes the coefficient of variation. The factor 1/2 can be justified by looking at one-dimensional dynamical systems. Setting $n = 1$, the linearized system (2) becomes

$$\frac{d\tilde{X}}{dt} = -a\tilde{X},$$

where a is a positive number (assuming that the system is stable). Clearly, $\mathcal{R} = a$. We compute the stationary fluctuations of a linear system variable $Q(X) = qX$. Solving the Lyapunov equation (7), we find that

$$C = \frac{\sigma^2 X^{*2}}{2a} \quad \text{and} \quad \text{Var}(Q) = \frac{q^2 \sigma^2 X^{*2}}{2a},$$

so that,

$$\mathcal{I}_Q = \frac{\sigma^2}{2} \frac{Q(X^*)^2}{\text{Var}(Q)} = \frac{\sigma^2}{2} \frac{q^2 X^{*2}}{\frac{q^2 \sigma^2 X^{*2}}{2a}} = a = \mathcal{R}.$$

Hence, the factor $1/2$ in definition (8) guarantees that invariability \mathcal{I}_Q and resilience \mathcal{R} coincide for one-dimensional dynamical systems. This equality indicates that, more generally, the values of invariability and resilience can be compared, and in particular, that they have the same units, namely, units of $1/\text{time}$. Note that, while invariability is closely related to the coefficient of variation (which is unitless), the comparability with asymptotic resilience is due to its normalization with respect to the applied stochasticity σ^2 (which has units of $1/\text{time}$).

Ecosystem invariability \mathcal{I}_{eco}

The examples of the paper deal with dynamical systems of which dynamical variables X_i represent biomass. To quantify the stability at the ecosystem level, we consider the invariability of the total biomass present in the ecosystem. As system variable Q we take total biomass, that is, $Q(X) = X_{\text{tot}} = \sum_i X_i$. Then, the corresponding invariability \mathcal{I}_{eco} is

$$\mathcal{I}_{\text{eco}} = \frac{\sigma^2}{2} \frac{1}{\text{CV}(X_{\text{tot}})^2} = \frac{\sigma^2}{2} \frac{(X_{\text{tot}}^*)^2}{\text{Var}(X_{\text{tot}})} \quad (9)$$

We call this measure *ecosystem invariability*. By construction, it captures (an aspect of) stability at the ecosystem level. Note that, in contrast to \mathcal{R} , \mathcal{I}_{eco} need not be close to zero when the dynamical system is close to the stability boundary (3).

Population invariability \mathcal{I}_{pop}

Invariability can also be used to address stability at the population level. Taking as system variable Q the biomass of a single population, $Q(X) = X_i$, we get

$$\mathcal{I}_{X_i} = \frac{\sigma^2}{2} \frac{(X_i^*)^2}{\text{Var}(X_i)} \quad \text{where } X_i \text{ is a biomass variable.}$$

We combine these invariabilities into a single, ecosystem-wide measure of population-level stability. There are different possibilities to do so. One could start from the idea that the most unstable population determines the population-level stability of the entire ecosystem. This leads to the definition of *minimum invariability*,

$$\mathcal{I}_{\text{min}} = \min_i \mathcal{I}_{X_i} = \frac{\sigma^2}{2} \min_i \frac{(X_i^*)^2}{\text{Var}(X_i)}. \quad (10)$$

By construction, measure (10) potentially depends on the invariability of rare species. In fact, as we show in the paper, minimum invariability is typically determined by rare species. This might be problematic, especially from an estimation point of view. To estimate a measure that depends strongly on rare species, we need accurate data for these rare species, which is typically difficult to obtain, precisely because they are rare.

Alternatively, one could consider the average invariability of all species. For example (Lehman & Tilman, 2000),

$$\frac{1}{n} \sum_i \mathcal{I}_{X_i} = \frac{\sigma^2}{2n} \sum_i \frac{(X_i^*)^2}{\text{Var}(X_i)}, \quad (11)$$

where n denotes the number of species (that is, the number of terms in the sum). However, also for measure (11) rare species can have a relatively large effect. There are two reasons

for this. First, \mathcal{I}_{X_i} need not be small when X_i^* is small. It is even possible that the contribution of a rare species dominates the sum in (11). Second, even if all rare species have small invariability, their presence will reduce the average invariability (11). If there are many rare species, this can have a large effect on the outcome of (11).

These problems can be solved by using a weighted average (weighted by population biomass). Here we propose to take a weighted average of the coefficients of variation (Thibaut & Connolly, 2013),

$$\mathcal{I}_{\text{pop}} = \frac{\sigma^2}{2} \frac{1}{\left(\sum_i \frac{X_i^*}{X_{\text{tot}}^*} \text{CV}(X_i)\right)^2} = \frac{\sigma^2}{2} \frac{(X_{\text{tot}}^*)^2}{\left(\sum_i \sqrt{\text{Var}(X_i)}\right)^2} \quad (12)$$

which we call *population invariability*. The last expression in (12) shows that rare species can only contribute to this stability measure if their summed standard deviation is comparable to those of common species, which is improbable. Moreover, \mathcal{I}_{pop} has the additional advantage that it directly connects to \mathcal{I}_{eco} . Indeed, their ratio equals

$$\frac{\mathcal{I}_{\text{pop}}}{\mathcal{I}_{\text{eco}}} = \frac{\text{Var}(X_{\text{tot}})}{\left(\sum_i \sqrt{\text{Var}(X_i)}\right)^2},$$

a quantity that has been proposed as a measure of ecosystem-wide synchrony (Loreau & de Mazancourt, 2008). It is always smaller than or equal to one, and reaches the maximal value of one for a perfectly synchronous ecosystem. Note that, just as \mathcal{I}_{eco} but contrary to \mathcal{R} , \mathcal{I}_{pop} need not be close to zero when the dynamical system is close to the stability boundary (3).

S2 Details of consumer-resource model (Figs. 2 and 3)

In Fig. 2 we analyze the stability properties of a consumer-resource model. This model has a pool of $S = 3$ consumer species competing for $R = 2$ resources. We use the variable N_i for the biomass of consumer species i and the variable R_k for the amount of resource k . Resource R_k is consumed by consumer N_i at rate c_{ik} and transformed into biomass with efficiency e_{ik} . All consumers N_i have mortality rate m . We consider abiotic resources with supply rate a_k and supply level A_k . This leads to the dynamical equations,

$$\begin{aligned} \frac{dN_i}{dt} &= \sum_k c_{ik} R_k N_i - m N_i, & i = 1, 2, 3, \\ \frac{dR_k}{dt} &= a(A_k - R_k) - \sum_i c_{ik} R_k N_i, & k = 1, 2. \end{aligned} \quad (13)$$

The coexistence diagram corresponding to Fig. 2 is shown in Fig. S1. The zero net growth isoclines of the consumer species intersect, defining coexistence regions in the plane of resource supply levels (A_1, A_2) . In Fig. 2 the resource supply levels A_k are varied such that regions of competitive exclusion alternate with regions of coexistence (Fig. S1, black thin solid line). The other parameter values of model (13) are $m = 1$, $a = 0.6$ and

$$\begin{array}{lll} c_{11} = 0.2790 & c_{21} = 0.7071 & c_{31} = 0.9409 \\ c_{12} = 0.9603 & c_{22} = 0.7071 & c_{32} = 0.3387. \end{array}$$

In Fig. 3 we study the stability properties of a reduced model obtained from model (13). To construct the reduced model, we combine the individual consumer species into a single

aggregate consumer variable N_{agg} . Denoting the aggregate functional response by $f_k(R_k)$, the dynamical equations are

$$\begin{aligned}\frac{dN_{\text{agg}}}{dt} &= \sum_k f_k(R_k)N_{\text{agg}} - mN_{\text{agg}}, \\ \frac{dR_k}{dt} &= a(A_k - R_k) - f_k(R_k)N_{\text{agg}}, \quad k = 1, 2.\end{aligned}\tag{14}$$

Whereas the original model (13) describes the population-level structure of the consumer guild, the reduced model (14) can be seen as an ecosystem-level model in which the total biomass of the consumer guild is lumped.

To construct the functional response $f_k(R_k)$ of the reduced model (14), note that the species' consumption rates in the original model satisfy a trade-off: a species with a large consumption rate of resource 1 has a small consumption rate of resource 2 (and vice versa). In fact, the consumption rates were chosen to lie on the trade-off curve $c_{i1}^2 + c_{i2}^2 = m$. Then, we consider the three species of the original model as members of a continuous pool of consumer species, spanning the entire trade-off curve. We assume that in the reduced model the aggregate consumer has the consumption characteristics of the best competitor (for given resource supply levels A_1 and A_2) from this continuous pool of consumer species. This assumption allows us to derive the reduced functional response $f_k(R_k)$.

Explicitly, we replace the three consumers N_i by a pool of consumers $N(x)$, indexed by the continuous parameter $x \in [0, \sqrt{m}]$. Consumer x has consumption rates $c_1(x) = x$ and $c_2(x) = \sqrt{m - x^2}$ for resource 1 and 2, respectively. The corresponding zero net growth isocline is given by $c_1(x)R_1 + c_2(x)R_2 = m$. There is a single resource point (R_1, R_2) for which consumer x is the best consumer; it is given by $R_1 = c_1(x)$ and $R_2 = c_2(x)$. Indeed, this point lies on the zero net growth isocline of consumer x and below (i.e., closer to the origin) the zero net growth isoclines of any other consumer. There is a half-line of resource supply points (A_1, A_2) leading to this resource point at equilibrium. In this equilibrium consumer x excludes the other consumers, so that $N_{\text{agg}} = N_i$. Its per capita consumption rates are $c_1(x)R_1 = c_1(x)^2 = R_1^2$ for resource 1 and $c_2(x)R_2 = c_2(x)^2 = R_2^2$ for resource 2. Hence, we find that $f_k(R_k) = R_k^2$.

We note that this model reduction is not intended as a generally applicable technique. Simply, in this specific example, it allows us to link ecosystem-level model (14) to population-level model (13).

The zero net growth isocline of the aggregate consumer is plotted as a black thick dashed line in Fig. S1. It lies below the zero net growth isoclines of the three consumer species, and is tangent to each of them. This graphically illustrates the construction of the aggregate functional response. The other parameter values of model (14) are $m = 1$ and $a = 1$. The resource supply levels A_k are varied along the black thin full line in Fig. S1.

S3 Effect of absent species on stability measures

We define an absent species as a species that belongs to the species pool, but that does not persist in the equilibrium community. Here we describe how the stability patterns of Fig. 2 dependent on the way absent species are dealt with. We show that asymptotic stability \mathcal{R} is strongly depend on absent species, while invariabilities \mathcal{I}_{pop} and \mathcal{I}_{eco} are independent of absent species.

Suppose species i is absent in the equilibrium community, $X_i^* = 0$. Then, dynamical equation (1) of species i can be written as

$$\frac{dX_i}{dt} = F_i(X) = X_i G_i(X).$$

Using this notation, we find that row i of Jacobian matrix A is equal to

$$A_{ij} = \begin{cases} G_i(X^*) & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

All components of row i are zero except the diagonal component. As a consequence, the eigenvalues of A are equal to this diagonal component together with the eigenvalues of a smaller matrix $A^{(i)}$. This matrix $A^{(i)}$ is obtained by eliminating row i and column i of matrix A . If matrix A has dimensions $n \times n$, matrix $A^{(i)}$ has dimensions $(n-1) \times (n-1)$.

As a result, the stability condition (3) decomposes into two parts: matrix $A^{(i)}$ should be stable (that is, all its eigenvalues should have negative real part) and diagonal component A_{ii} should be negative. The ecological interpretation is as follows. First, note that the matrix $A^{(i)}$ is obtained by eliminating species i from the species pool. Hence, the first part of the stability condition states that the other species should constitute a stable equilibrium, thereby discarding their interactions with species i . Second, note that the diagonal component $A_{ii} = G_i(X^*)$ is equal to the invasion fitness of species i in the equilibrium community of the other species. To see this, suppose that variables X_j for all $j \neq i$ are at equilibrium, $X_j = X_j^*$, and that variable X_i is very small but different from zero. Then, the relative rate of change $\frac{1}{X_i} \frac{dX_i}{dt}$ of variable X_i is equal to $G_i(X^*)$. Clearly, species i , which is absent in the equilibrium community, should be unable to invade for the equilibrium to be stable.

This implies that absent species can have a large effect on asymptotic resilience \mathcal{R} . Indeed, if an absent species is almost able to invade the equilibrium community (that is, a negative invasion fitness that is small in absolute value), its invasion fitness can determine the dominant eigenvalue λ_{dom} . Hence, the choice of whether to include absent species in the computation of asymptotic resilience can change the resulting stability pattern qualitatively. In particular, when approaching a bifurcation point in which an absent species becomes able to invade, asymptotic resilience drops to zero when accounting for that absent species, but stays positive otherwise (Fig. S2; compare panels a and d).

Similarly, we can analyze the effect of absent species on invariabilities \mathcal{I}_{pop} and \mathcal{I}_{eco} . To do so, we solve the Lyapunov equation (7) for an equilibrium in which species i is absent, $X_i^* = 0$. Then, as explained above, row i of Jacobian matrix A has zero components except on the diagonal. We compute component (i, i) of the Lyapunov equation,

$$0 = \sum_k A_{ik} C_{ki} + \sum_k C_{ik} A_{ik} + (\sigma X_i^*)^2 = 2A_{ii} C_{ii}.$$

Hence, because $A_{ii} \neq 0$, we find that $C_{ii} = 0$. Recalling that C is a covariance matrix, we know that $C_{jk}^2 \leq C_{jj} C_{kk}$ for all j and k (Cauchy-Schwarz inequality). This implies that $C_{ik} = C_{ki} = 0$ for all k . Hence, the non-zero components of the covariance matrix C can be obtained by solving the Lyapunov equation for the dynamical system in which the absent species have been removed from the species pool. This implies that invariabilities \mathcal{I}_{pop} and \mathcal{I}_{eco} are insensitive to the inclusion of absent species.

The different effects of absent species on the stability measures have important consequences for linking these stability measures (Fig. S2). When not accounting for absent species, that is, when restricting Jacobian matrix A to persistent species, there is a qualitative link between asymptotic resilience \mathcal{R} and minimum invariability \mathcal{I}_{\min} (Fig. S2; compare panels a and c). When accounting for absent species, that is, when computing Jacobian matrix A for the entire species pool, asymptotic resilience \mathcal{R} changes while single-population invariabilities \mathcal{I}_{X_i} do not. Hence, the qualitative link between them is lost. However, this link can be recovered by introducing the invasion fitness $G_i(X^*)$ of absent species i . Explicitly, we take the minimum of the set

$$\{\mathcal{I}_{X_i} \mid \text{species } i \text{ is present}\} \cup \{-G_i(X^*) \mid \text{species } i \text{ is absent}\}.$$

By doing so, we get again a qualitative link between asymptotic resilience and single-population invariabilities (Fig. S2; compare panels d and f).

It may seem artificial, when computing the stability of an ecosystem, to include species that are absent from the ecosystem. However, one should note that the previous models are based on the (unrealistic) assumption that the modelled ecosystem is closed. When introducing a weak immigration flow from the species pool, all species are present in the ecosystem. In particular, those that were absent in the closed system, are present at a small abundance in the open system. Then, the stability patterns for asymptotic resilience are dissimilar to those when accounting only for persistent species, but they are similar to those when accounting also for absent species (Fig. S2; compare panel g to panels a and d). This suggests that the patterns obtained for the entire species pool are the relevant ones, as only those are robust with respect to the addition of a small amount of immigration. Note that invariabilities \mathcal{I}_{pop} and \mathcal{I}_{eco} (which are insensitive to the inclusion of absent species) are also robust with respect to a weak immigration flow.

Finally, we remark that absent species can also affect the generic link between asymptotic resilience and the long-term rate of return to equilibrium. When asymptotic resilience is determined by an absent species i , the return rate for any other species $k \neq i$ is different from (and larger than) \mathcal{R} . Mathematically, this is due to the particular structure of the left eigenvector φ_{dom} associated with the dominant eigenvalue λ_{dom} . Indeed, all components $k \neq i$ of φ_{dom} are zero, as can be seen from the structure of the Jacobian A described above. More generally, when asymptotic resilience is determined by species i with non-zero but small equilibrium abundance, the associated left eigenvector φ_{dom} typically has small components $k \neq i$ compared to component i . This phenomenon, called localization, reduces the relevance of asymptotic resilience as a dynamical characteristic (Suweis *et al.*, 2015).

S4 Details of random interactions model (Fig. 4)

Here we describe the details of the model with random competitive interactions used in Fig. 4. It is based on the Lotka-Volterra dynamical equations,

$$\frac{dN_i}{dt} = N_i \left(a_i - \sum_{j=1}^{S_{\text{pool}}} b_{ij} N_j \right), \quad i = 1, \dots, S_{\text{pool}},$$

with $a_i = 1$ and $b_{ii} = 1$. The interaction coefficients b_{ij} , $i \neq j$ are obtained by randomly drawing numbers from the Gaussian distribution with mean 0 and standard deviation 0.2,

and taking the absolute value. In the resulting dynamical system, the equilibrium in which all species coexist is not necessarily feasible and stable. Therefore, we integrate the dynamical equations until reaching a new equilibrium, which is feasible and stable (note that all our simulations eventually converged to an equilibrium). For this equilibrium, which has $S \leq S_{\text{pool}}$ species, we compute the three stability measures \mathcal{R} , \mathcal{I}_{pop} and \mathcal{I}_{eco} . We repeat this procedure for values of S_{pool} in the range from 1 to 100 until our data set consists of 1000 equilibria for each value of S in the range from 1 to 30.

In Fig. S3 we illustrate the robustness of the patterns shown in Fig. 4 with respect to absent species. Absent species enter the patterns in two different ways. First, we can compute Jacobian matrices A with or without absent species. As explained in Appendix S3, this choice has no effect on invariabilities \mathcal{I}_{pop} and \mathcal{I}_{eco} , but it does affect asymptotic resilience \mathcal{R} . Second, we can plot the stability measures against the number of species present in the equilibrium community (that is, excluding absent species), or against the number of species in the species pool (that is, including absent species). This choice affects all stability measures. Fig. S3 shows that the stability patterns are robust with respect to both choices.

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Figure S1

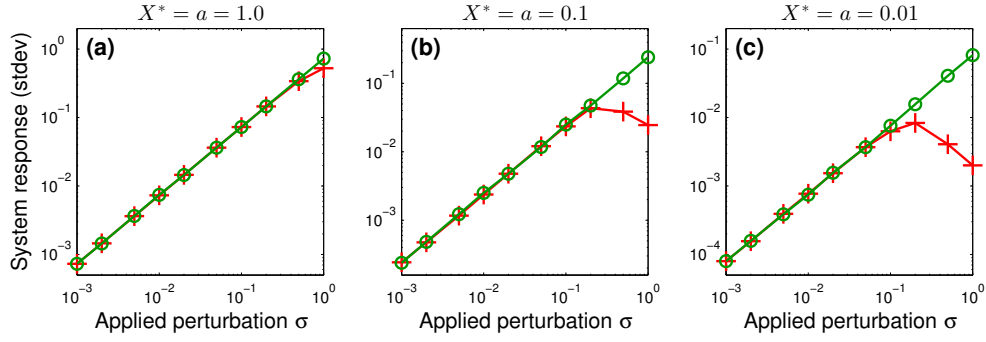


Figure S1: Invariability close to extinction threshold. Logistic population model $dX/dt = aX - bX^2$ with $b = 1.0$ fixed and (a) $a = 1.0$; (b) $a = 0.1$; (c) $a = 0.01$. The population becomes extinct (equilibrium biomass X^* tends to zero) when the bifurcation parameter a tends to zero. We apply a white-noise perturbation with intensity σ (on x -axis), and measure the fluctuations of the system response by the standard deviation $\sqrt{\text{Var}(X)}$ (on y -axis). Red line with crosses: nonlinear model; green line with circles: linearization around equilibrium. Results for nonlinear and linear model coincide for weak perturbations, but they start to differ at smaller intensity σ when approaching the bifurcation. Nevertheless, however close to the bifurcation, one can always take σ small enough such that the linearization provides an accurate approximation of the nonlinear dynamics.

Figure S2

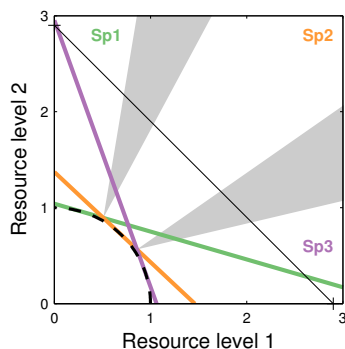


Figure S2: Details of resource competition model. Colored lines: zero net growth isoclines of the three consumer species (a species can persist at equilibrium if resource levels (R_1, R_2) lie on its zero net growth isocline). Grey-shaded triangles: coexistence regions (two species can coexist at equilibrium if resource supply levels (A_1, A_2) lie in their coexistence region). Black thin solid line: range of resource supply levels (A_1, A_2) scanned along the x -axis of Figs. 2 and 3. Black thick dashed line: zero net growth isocline of aggregate consumer in reduced model.

Figure S3

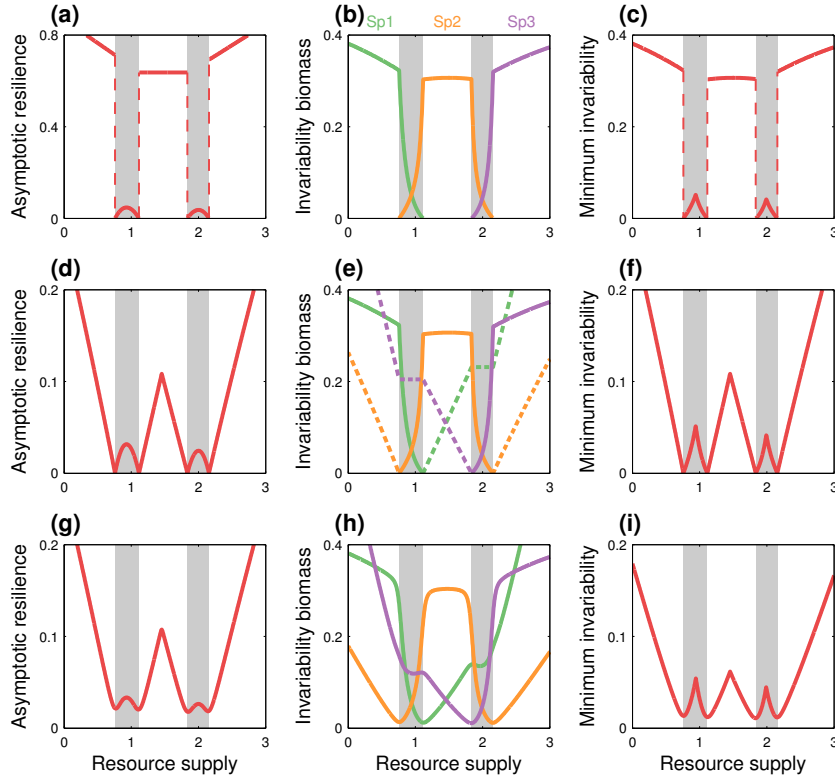


Figure S3: Effect of absent species in resource competition model. Same model as Fig. 2. (a) When we restrict the system to species present in the equilibrium community, asymptotic resilience is discontinuous each time a species is able to invade (or a species becomes extinct). (b & c) The same pattern is obtained by taking the minimum of single-population invariabilities (shown for the three species in panel b; their minimum is shown in panel c). (d) When we account for absent species, asymptotic resilience is continuous, but drops to zero at each bifurcation corresponding to an invasion or an extinction. (e & f) This pattern can be reproduced by taking the minimum of single-population invariability (for persistent species; full line in panel e) and (the opposite of) invasion fitness (for absent species; dashed line in panel e). (g) When adding a small immigration flow, asymptotic resilience has a behavior similar to the one obtained for the entire species pool. Strictly speaking, there are no longer invasion/extinction bifurcations, but these bifurcations (for the case without immigration) are still visible in the stability pattern. (h & i) The same pattern is obtained by taking the minimum of single-population invariabilities. Note that all species are present in the equilibrium community.

Figure S4

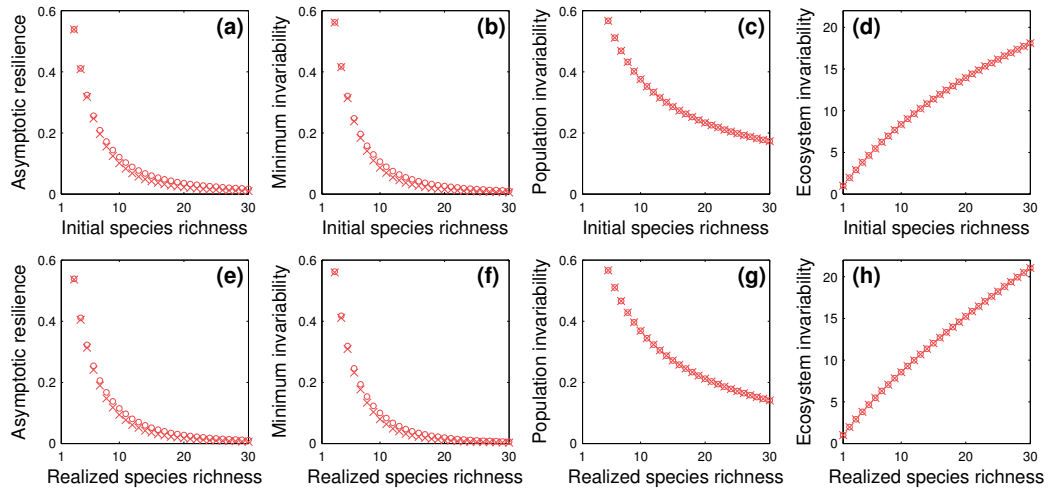


Figure S4: Effect of absent species in random interactions model. Stability is plotted against species richness for the model with random competitive interactions of Fig. 4. Results are shown for community matrices A with absent species (\times -marks) and without absent species (circles). For each species richness level stability values are averaged over 1000 simulations. (a–d) Initial species richness, i.e., the number of species in the species pool, is plotted on the x -axis. (e–h) Realized species richness, i.e., the number of species in the equilibrium community, is plotted on the x -axis.