SUPPLEMENTARY INFORMATION

A rational theory of set size effects in working memory and attention

Ronald van den Berg & Wei Ji Ma

Contents

MODEL DETAILS

Relation between J and κ

We measure encoding precision as Fisher Information, denoted J. As derived in earlier work(1), the mapping between J and the concentration parameter κ of a Von Mises encoding noise distribution is $J(\kappa)=\kappa\frac{I_1(\kappa)}{I_0(\kappa)}$, where I_1 is the modified Bessel function of the first kind of order 1. Larger values of J map to larger values of κ, corresponding to narrower noise distributions.

Variable precision

In all our models, we incorporated variability in precision (2, 3) by drawing the precision for each encoded item independently from a Gamma distribution with mean $\bar{J}$ and scale parameter τ. We denote the distribution of a single precision value by $p(J\mid\bar{J},\tau)$ and the joint distribution of the precision values of all N items in a display by $p(\mathbf{J}\mid\bar{J},\tau)=\prod_{i=1}^{N}p(J_i\mid\bar{J},\tau)$.

Expected behavioral loss function by task

As a consequence of variability in precision, computation of expected behavioral loss requires integration over both the behavioral error, ε, and the vector with precision values, $\mathbf{J}$,

$$\bar{L}_{\text{behavioral}}\left(\bar{J},N\right)=\begin{cases}\sum_{\varepsilon}\int_{0}^{\infty}L_{\text{behavioral}}\left(\varepsilon\right)p\left(\varepsilon\mid\mathbf{J},N\right)p\left(\mathbf{J}\mid\bar{J},\tau\right)d\mathbf{J} & \text{if }\varepsilon\text{ is discrete}\\ \int_{0}^{\infty}\int_{0}^{\infty}L_{\text{behavioral}}\left(\varepsilon\right)p\left(\varepsilon\mid\mathbf{J},N\right)p\left(\mathbf{J}\mid\bar{J},\tau\right)d\mathbf{J}d\varepsilon & \text{if }\varepsilon\text{ is continuous}\end{cases}$$

The distribution of precision, $p\left(\mathbf{J}\mid\bar{J},\tau\right)$, is the same in all models, but $L_{\text{behavioral}}(\varepsilon)$ and $p(\varepsilon|\mathbf{J},N)$ are task-specific. We next specify these two components separately for each task.

Delayed estimation. In delayed estimation, the behavioral error only depends on the memory representation of the target item. We assume that this representation is corrupted by Von Mises noise,

$$p\left(\varepsilon\mid\mathbf{J},N\right)=p\left(\varepsilon\mid J_{\text{T}}\right)=\frac{1}{2\pi I_0\left(F\left(J_{\text{T}}\right)\right)}e^{F\left(J_{\text{T}}\right)\cos\left(\varepsilon\right)},$$

where J_{T} is the precision of the target item and $F(.)$ maps Fisher Information to a concentration parameter κ; we implement this mapping by numerically inverting the mapping specified above. Furthermore, the behavioral loss function is assumed to be a power-law function of the absolute estimation error, $L_{\text{behavioral}}=|\varepsilon|^{\beta}$, where $\beta>0$ is a free parameter.

Change detection. We assume that subjects report "change present" whenever the posterior ratio for a change exceeds 1,

$$\frac{p\left(\text{change present}\mid\mathbf{x},\mathbf{y}\right)}{p\left(\text{change absent}\mid\mathbf{x},\mathbf{y}\right)}>1,$$

where $\mathbf{x}$ and $\mathbf{y}$ denote the vectors of noisy measurements of the stimuli in the first and second displays, respectively. Under the Von Mises assumption, this rule evaluates to (4)

$$\frac{p_{\text{change}}}{1-p_{\text{change}}}\frac{1}{N}\sum_{i=1}^{N}\frac{I_0\left(\kappa_{\text{x},i}\right)I_0\left(\kappa_{\text{y},i}\right)}{I_0\left(\sqrt{\kappa_{\text{x},i}^2+\kappa_{\text{x},i}^2+2\kappa_{\text{x},i}\kappa_{\text{y},i}\cos\left(y_i-x_i\right)}\right)}>1,$$

where p_{change} is a free parameter representing the subject's prior belief that a change will occur, and $\kappa_{\text{x},i}$ and $\kappa_{\text{y},i}$ denote the concentration parameters of the Von Mises distributions associated with the observations of the stimuli at the i^{th} location in the first and second displays, respectively.

The behavioral error, ε, takes only two values in this task: correct and incorrect. We assume that observers map each of these values to a loss value,

$$L_{\text{behavioral}}(\varepsilon) = \begin{cases} L_{\text{incorrect}} & \text{if } \varepsilon \text{ is "incorrect"} \\ L_{\text{correct}} & \text{if } \varepsilon \text{ is "correct".} \end{cases}$$

For example, an observer might assign a loss of 0 to any correct decision and a loss of 1 to any incorrect decision. The expected behavioral loss is a weighted sum of $L_{\text{incorrect}}$ and L_{correct},

$$\bar{L}_{\text{behavioral}}(\bar{J}, N) = p_{\text{correct}}(\bar{J}, N) L_{\text{correct}} + \left(1 - p_{\text{correct}}(\bar{J}, N)\right) L_{\text{incorrect}},$$

where $p_{\text{correct}}(\bar{J}, N)$ is the probability of a correct decision. This probability is not analytic, but can be easily be approximated using Monte Carlo simulations.

Change localization. Expected behavioral loss is computed in the same way as in the change-detection task, except that a different decision rule must be used to compute $p_{\text{correct}}(\bar{J}, N)$. As shown in earlier work (3), the Bayes-optimal rule for the change-localization task is to report the location that maximizes $\frac{I_0(\kappa_{\text{x},i}) I_0(\kappa_{\text{y},i})}{I_0\left(\sqrt{\kappa_{\text{x},i}^2 + \kappa_{\text{x},i}^2 + 2\kappa_{\text{x},i}\kappa_{\text{y},i}\cos(y_i - x_i)}\right)}$, where all terms are defined in the same way as in the model for the change-detection task.

Visual search. The expected behavioral loss in the model for visual search is also computed in the same way as in the model for change detection, again with the only difference being the decision rule used to compute $p_{\text{correct}}(\bar{J}, N)$. The Bayes-optimal rule for this task is to report "target present" when $\frac{p_{\text{present}}}{1 - p_{\text{present}}} \frac{I_0(\kappa_{\text{D}}) e^{\kappa_i \cos(x_i - s_{\text{T}})}}{I_0\left(\sqrt{\kappa_i^2 + \kappa_{\text{D}}^2 + 2\kappa_i\kappa_{\text{D}}\cos(x_i - s_{\text{T}})}\right)}$, where p_{present} is the subject's prior belief that the target will be present, κ_{D} the concentration parameter of the distribution from which the distractors are drawn, κ_{i} the concentration parameter of the noise distribution associated to the stimulus at location i, x_{i} the noisy observation of the stimulus at location i, and s_{T} the value of the target (see (5) for a derivation).

The behavioral loss function drops out when the behavioral error is binary

When the behavioral error ε takes only two values, the behavioral loss can also take only two values. The integral in the expected behavioral loss (Eq (2) in the main text) then simplifies to a sum of two terms,

$$\begin{aligned}\bar{L}_{\text{behavioral}}\left(\bar{J},N\right) &= p_{\text{correct}}\left(\bar{J},N\right)L_{\text{correct}}+\left(1-p_{\text{correct}}\left(\bar{J},N\right)\right)L_{\text{incorrect}}\\ &= p_{\text{correct}}\left(\bar{J},N\right)\left(L_{\text{correct}}-L_{\text{incorrect}}\right)+L_{\text{incorrect}}.\end{aligned}$$

The optimal (loss-minimizing) value of $\bar{J}$ is then

$$\begin{aligned}\bar{J}_{\text{optimal}}\left(N\right) &= \underset{\bar{J}}{\operatorname{argmin}}\left[p_{\text{correct}}\left(\bar{J},N\right)\left(L_{\text{correct}}-L_{\text{incorrect}}\right)+L_{\text{incorrect}}+\tilde{\lambda}\bar{L}_{\text{neural}}\left(\bar{J},N\right)\right]\\ &= \underset{\bar{J}}{\operatorname{argmin}}\left[p_{\text{correct}}\left(\bar{J},N\right)\Delta_{\text{L}}+\tilde{\lambda}\bar{L}_{\text{neural}}\left(\bar{J},N\right)\right],\end{aligned}$$

where $\Delta_{\text{L}} \equiv L_{\text{correct}} - L_{\text{incorrect}}$. Since Δ_{L} and $\tilde{\lambda}$ have interchangeable effects on $\bar{J}_{\text{optimal}}$, we fix Δ_{L} to 1 and fit only $\tilde{\lambda}$ as a free parameter.

Conditions under which optimal precision declines with set size

In this section, we show that when the expected behavioral loss is independent of set size (as in single-probe delayed estimation and change detection), the rational model predicts optimal precision to decline with set size whenever the following four conditions are satisfied:

1) The expected behavioral loss is a strictly decreasing function of encoding precision, i.e., an increase in precision results in an increase in behavioral performance.
2) The expected behavioral loss is subject to a law of diminishing returns (6): the behavioral benefit obtained from a unit increase in precision decreases with precision. This law will hold when condition 1 holds and the loss function is bounded from below, which is generally the case as errors cannot be negative.
3) The expected neural loss is an increasing function of encoding precision.

4) The expected neural loss per unit of precision is a non-decreasing function of precision. On the premise that precision is proportional to spike rate (7, 8), this condition is satisfied if loss per spike increases with spike rate, which has been found to be the case (9).

These conditions translate to the following constraints on the first and second derivatives of the expected loss functions,

$$\begin{aligned}
&1.\ \bar{L}'_{\text{behavioral}}\left(\bar{J}\right)<0 \\
&2.\ \bar{L}''_{\text{behavioral}}\left(\bar{J}\right)>0 \\
&3.\ \bar{L}'_{\text{neural}}\left(\bar{J}\right)>0 \\
&4.\ \bar{L}''_{\text{neural}}\left(\bar{J}\right)\geq 0.
\end{aligned}$$

The loss-minimizing value of precision is found by setting the derivative of the expected total loss function to 0,

$$0=\bar{L}'_{\text{total}}\left(\bar{J}\right)=\bar{L}'_{\text{behavioral}}\left(\bar{J}\right)+\tilde{\lambda}N\bar{L}'_{\text{neural}}\left(\bar{J}\right),$$

which is equivalent to

$$-\frac{\bar{L}'_{\text{behavioral}}\left(\bar{J}\right)}{\bar{L}'_{\text{neural}}\left(\bar{J}\right)}=\tilde{\lambda}N. \qquad \text{(S1)}$$

The left-hand side is strictly positive for any $\bar{J}$, because of constraints 1 and 3 above. In addition, it is a strictly decreasing function of $\bar{J}$, because

$$-\frac{d}{d\bar{J}}\frac{\bar{L}'_{\text{behavioral}}\left(\bar{J}\right)}{\bar{L}'_{\text{neural}}\left(\bar{J}\right)}=\frac{\bar{L}''_{\text{behavioral}}\left(\bar{J}\right)\bar{L}'_{\text{neural}}\left(\bar{J}\right)-\bar{L}'_{\text{behavioral}}\left(\bar{J}\right)\bar{L}''_{\text{neural}}\left(\bar{J}\right)}{\left(\bar{L}'_{\text{neural}}\left(\bar{J}\right)\right)^2}$$

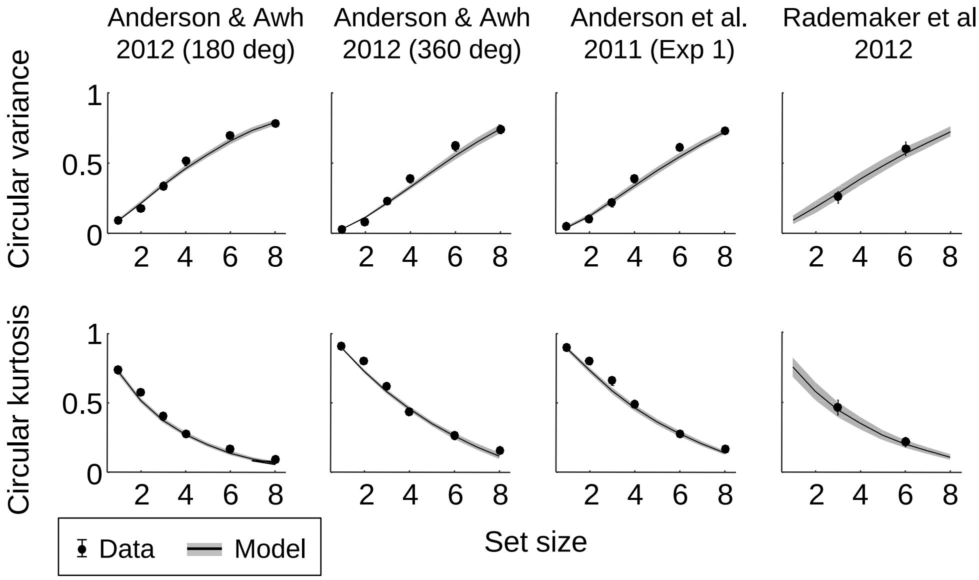
is necessarily greater than 0 due to the four constraints specified above. As illustrated in Supplementary Figure S1, Eq. (S1) can be interpreted as the intersection point between the function specified by the left-hand side (solid curve) and a flat line at a value $\tilde{\lambda}N$ (dashed lines). The value of $\bar{J}$ at which this intersection occurs (i.e., $\bar{J}_{\text{optimal}}$) necessarily decreases with N.

Hence, in tasks where the expected behavioral loss is independent of set size, our model predicts a decline of precision with set size whenever the above four, rather general conditions hold. When expected behavioral loss does depend on set size (such as in whole-array change detection or change localization), the proof above does not apply and we were not able to extend the proof to this domain.

REFERENCES

1. Keshvari S, van den Berg R, Ma WJ (2012) Probabilistic computation in human perception under variability in encoding precision. *PLoS One* 7(6).
2. Fougnie D, Suchow JW, Alvarez GA (2012) Variability in the quality of visual working memory. *Nat Commun* 3:1229.
3. van den Berg R, Shin H, Chou W-C, George R, Ma WJ (2012) Variability in encoding precision accounts for visual short-term memory limitations. *Proc Natl Acad Sci* 109:8780–8785.
4. Keshvari S, van den Berg R, Ma WJ (2013) No Evidence for an Item Limit in Change Detection. *PLoS Comput Biol* 9(2).
5. Mazyar H, Van den Berg R, Seilheimer RL, Ma WJ (2013) Independence is elusive : Set size effects on encoding precision in visual search. *J Vis* 13(5):1–14.
6. Mankiw NG (2004) *Principles of economics* doi:10.1017/CBO9780511511455.
7. Paradiso M a (1988) A theory for the use of visual orientation information which exploits the columnar structure of striate cortex. *Biol Cybern* 58(1):35–49.
8. Seung HS, Sompolinsky H (1993) Simple models for reading neuronal population codes. *ProcNatlAcadSci* 90(22):10749–10753.
9. Sterling P, Laughlin S (2015) *Principles of neural design.* (MIT Press).
10. Anderson DE, Awh E (2012) The plateau in mnemonic resolution across large set sizes indicates discrete resource limits in visual working memory. *Atten Percept Psychophys* 74(5):891–910.
11. Anderson DE, Vogel EK, Awh E (2011) Precision in visual working memory reaches a stable plateau when individual item limits are exceeded. *J Neurosci* 31(3):1128–38.
12. Rademaker RL, Tredway CH, Tong F (2012) Introspective judgments predict the precision and likelihood of successful maintenance of visual working memory. *J Vis* 12(13):21.

SUPPLEMENTARY FIGURES



Supplementary figure S1. Fits to the three delayed-estimation benchmark data sets that were excluded from the main analyses. Circular variance (top) and circular kurtosis (bottom) of the estimation error distributions as a function of set size, split by experiment. Error bars and shaded areas represent 1 s.e.m. of the mean across subjects. The first three datasets were excluded from the main analyses on the ground that they were published in papers that were later retracted (Anderson & Awh, 2012; Anderson et al. 2012). The Rademaker et al. (2012) dataset was excluded from the main analyses because it contains only two set sizes, which makes it less suitable for a fine-grained study of the relationship between encoding precision and set size.

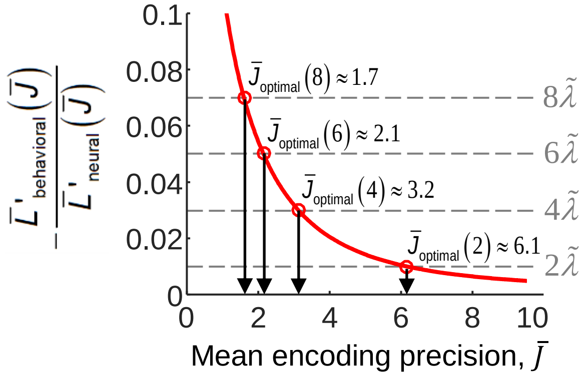


Figure S2. Graphical illustration of Eq. (S1). The value of $\bar{J}$ at which the equality described by Eq. (S1) holds is the intersection point between the function specified by the left-hand side (red curve) and a flat line at a value $N\lambda$. Since the left-hand side is strictly positive and also a strictly decreasing function of $\bar{J}$, the value at which this intersection occurs (i.e., $\bar{J}_{\text{optimal}}$) necessarily decreases with N.