Supplementary Information: Information sharing for coordination in fluctuating environments

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For notational convenience in this supplement, we abuse notation of strategy vectors s_i by assigning the numeric value $s_i(y_i) = 0$ if action A was assigned, and $s_i(y_i) = 1$ if B was assigned (y_i is either α_i or (α_i, β_i)). For instance, the strategy vector $s_{\text{FC}}^\top = [A, B]$ corresponds to the vector $s_{\text{FC}}^\top = [0, 1]$ whenever computations are needed. We will adopt the $0, 1$ notation for strategy vectors throughout this supplementary document.

A Explicit calculation of fitnesses

Here we calculate the fitnesses $f_p(s_1, s_2)$ and $f_{pq}(s_1, s_2)$ explicitly in terms of p, q, and the strategy vectors s_1 and s_2.

The fitness f_p

In the game $\mathcal{G}_p$, recall the average long-term fitness is (eq. (5) in the main text)

$$f_p(s_1, s_2) = \sum_{\alpha_1, \alpha_2, E} \pi_p(\alpha_1, \alpha_2, E) U(s_1(\alpha_1), s_2(\alpha_2), E). \tag{A.1}$$

It is a sum of eight terms since $(\alpha_1, \alpha_2, E) \in \{E_A, E_B\}^3$. We define $[Q_A^p]_{\alpha_1,\alpha_2} = P(\alpha_1, \alpha_2 | E_A)$ as the matrix of conditional probabilities on $E = E_A$, and similarly for Q_B for $E = E_B$,

$$Q_A^p \equiv \begin{bmatrix} p^2 & p\bar{p} \\ p\bar{p} & \bar{p}^2 \end{bmatrix}, \quad Q_B^p \equiv \begin{bmatrix} \bar{p}^2 & p\bar{p} \\ p\bar{p} & p^2 \end{bmatrix} \tag{A.2}$$

where $\bar{p} = 1 - p$.

The only nonzero terms in (A.1) occur when $U(s_1(\alpha_i), s_2(\alpha_j), E_A) = b_A$, which happens if and only if $s_1(\alpha_i) = s_2(\alpha_j) = 0$ (similarly, b_B for $E = E_B$). Therefore, the fitness can be written

$$f_p(s_1, s_2) = c_A(\mathbf{1} - s_1)^\top Q_A^p (\mathbf{1} - s_2) + c_B s_1^\top Q_B^p s_2 \tag{A.3}$$

where recall that $c_A = b_A \frac{v_{BA}}{v_{BA} + v_{AB}}$, $c_B = b_B \frac{v_{AB}}{v_{BA} + v_{AB}}$, and we denote $\mathbf{1}$ as the vector $[1, 1]^\top$. The expression can alternatively be written in a bilinear form,

$$f_p(s_1, s_2) = s_1^\top Q^p s_2 - c_A L_p^\top (s_1 + s_2) + c_A. \tag{A.4}$$

where

$$\begin{aligned} Q^p &\equiv c_A Q_A^p + c_B Q_B^p \\ L_p^\top &\equiv [p, \bar{p}]. \end{aligned} \tag{A.5}$$

The fitness f_{pq}

In the game $\mathcal{G}_{pq}$, recall the average long-term fitness is (eq. (11) in the main text)

$$f_{pq}(s_1, s_2) \equiv \sum_{y_1, y_2, E} \pi_{pq}(y_1, y_2, E) U(s_1(y_1), s_2(y_2), E). \tag{A.6}$$

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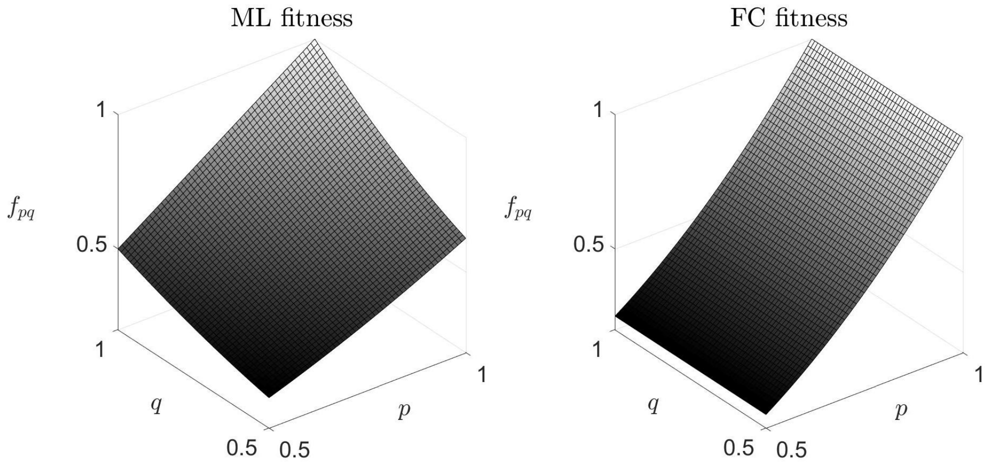

Figure S1: Fitnesses of the ML (Left) and FC (Right) strategy profiles for $\kappa = 1$ ($f_{pq}(\text{ML}_A) = f_{pq}(\text{ML}_B)$). In these contours, $b_A = b_B = 1$ and $v_{AB} = v_{BA}$, so $c_A = c_B = 0.5$ and the environments spend equal amounts of time in the long run. Hence, the fitness values on the z axis portray the fraction of time the players coordinate on the correct action. The fitnesses are $f_{pq}(\text{ML}) = (1 - 2p\bar{p})(q^2 + 1) + 2\bar{p}(2p - 1)q$ and $f_{pq}(\text{FC}) = p^2$.

The expressions for f_{pq} are derived in the same manner as f_p (eq. (A.4)). However, we need to define the new matrices Q_A and Q_B whose entries are the conditional probabilities $P(y_1, y_2|E_k)$, $k = A, B$, and where $y_i = (\alpha_i, \beta_i) \in \{E_A, E_B\}^2$.

$$Q_A^{pq} = \left[\begin{array}{cc|cc} p^2q^2 & p^2q\bar{q} & p\bar{p}q\bar{q} & p\bar{p}\bar{q}^2 \\ p^2q\bar{q} & p^2\bar{q}^2 & p\bar{p}q^2 & p\bar{p}q\bar{q} \\ \hline p\bar{p}q\bar{q} & p\bar{p}q^2 & \bar{p}^2\bar{q}^2 & \bar{p}^2q\bar{q} \\ p\bar{p}\bar{q}^2 & p\bar{p}q\bar{q} & \bar{p}^2q\bar{q} & \bar{p}^2q^2 \end{array}\right], \quad Q_B^{pq} = \left[\begin{array}{cc|cc} \bar{p}^2q^2 & \bar{p}^2q\bar{q} & p\bar{p}q\bar{q} & p\bar{p}\bar{q}^2 \\ \bar{p}^2q\bar{q} & \bar{p}^2\bar{q}^2 & p\bar{p}q^2 & p\bar{p}q\bar{q} \\ \hline p\bar{p}q\bar{q} & p\bar{p}q^2 & p^2\bar{q}^2 & p^2q\bar{q} \\ p\bar{p}\bar{q}^2 & p\bar{p}q\bar{q} & p^2q\bar{q} & p^2q^2 \end{array}\right] \tag{A.7}$$

Then $f_{pq}(s_1, s_2)$, where s_1, s_2 are four-vectors of ones and zeros, can be written

$$f_{pq}(s_1, s_2) = c_A(\mathbf{1} - s_1)^\top Q_A^{pq}(\mathbf{1} - s_2) + c_B s_1^\top Q_B^{pq} s_2 \tag{A.8}$$

where $\mathbf{1} = [1, 1, 1, 1]^\top$. An alternative bilinear form is

$$f_{pq}(s_1, s_2) = s_1^\top Q^{pq} s_2 - c_A L_{pq}^\top (s_1 + s_2) + c_A. \tag{A.9}$$

where

$$\begin{aligned} Q^{pq} &\equiv c_A Q_A^{pq} + c_B Q_B^{pq} \\ L_{pq}^\top &\equiv [p(pq + \bar{p}\bar{q}), p(p\bar{q} + \bar{p}q), \bar{p}(pq + \bar{p}\bar{q}), \bar{p}(p\bar{q} + \bar{p}q)] \end{aligned} \tag{A.10}$$

The fitness of FC is

$$f_{pq}(\text{FC}) = (c_A + c_B)p^2 \tag{A.11}$$

The fitnesses of ML_A and ML_B are

$$\begin{aligned} f_{pq}(\text{ML}_A) = f_{pq}(s_{\text{ML}_A}, s_{\text{ML}_A}) &= (c_A\bar{p}^2 + c_Bp^2)q^2 + 2c_A\bar{p}(2p - 1)q + c_A(1 - 2p\bar{p}) \\ f_{pq}(\text{ML}_B) = f_{pq}(s_{\text{ML}_B}, s_{\text{ML}_B}) &= (c_Ap^2 + c_B\bar{p}^2)q^2 + 2c_B\bar{p}(2p - 1)q + c_B(1 - 2p\bar{p}). \end{aligned} \tag{A.12}$$

They increase quadratically in q. Figure S1 plots the fitness surfaces of FC and ML.

B Proof: Nash equilibria and fitness maximizers in $\mathcal{G}_p$ are symmetric strategy profiles

We reproduce the statement from the main text in Section 3.1.

For $p \in [1/2, 1]$, all Nash equilibria and fitness maximizers of $\mathcal{G}_p$ are necessarily symmetric strategy profiles, i.e. $s_1^* = s_2^*$.

Proof. This fact is proven by exhaustion. For each of the six asymmetric strategy profiles, we show there is at least one player that can imitate the other's strategy to gain fitness. This implies Nash equilibria and fitness maximizers are necessarily symmetric strategy profiles. The fitnesses of these six profiles are given by the entries below (or above) the diagonal of the payoff matrix (6), reproduced here.

$$\begin{array}{c} \\ s_{\mathrm{OA}} \\ s_{\mathrm{FC}} \\ s_{\overline{\mathrm{FC}}} \\ s_{\mathrm{OB}} \end{array} \begin{array}{c} \begin{array}{cccc} s_{\mathrm{OA}} & s_{\mathrm{FC}} & s_{\overline{\mathrm{FC}}} & s_{\mathrm{OB}} \end{array} \\ \begin{bmatrix} c_A & c_A p & c_A \bar{p} & 0 \\ c_A p & (c_A + c_B)p^2 & (c_A + c_B)p\bar{p} & c_B p \\ c_A \bar{p} & (c_A + c_B)p\bar{p} & (c_A + c_B)\bar{p}^2 & c_B \bar{p} \\ 0 & c_B p & c_B \bar{p} & c_B \end{bmatrix} \end{array}$$

1. $f_p(s_{\mathrm{OA}}, s_{\mathrm{FC}}) = c_A p$. If the s_{FC} player switches to s_{OA}, the fitness becomes $f_p(s_{\mathrm{OA}}, s_{\mathrm{OA}}) = c_A > c_A p$ iff $p \neq 1$. If the s_{OA} player switches to s_{FC}, the fitness becomes $f_p(s_{\mathrm{FC}}, s_{\mathrm{FC}}) = (c_A + c_B)p^2 > c_A p$ iff $p > c_A/(c_A + c_B)$, which is satisfied at $p = 1$.
2. $f_p(s_{\mathrm{OA}}, s_{\overline{\mathrm{FC}}}) = c_A \bar{p}$. If the $s_{\overline{\mathrm{FC}}}$ player switches to s_{OA}, the fitness becomes $f_p(s_{\mathrm{OA}}, s_{\mathrm{OA}}) = c_A > c_B \bar{p}$.
3. $f_p(s_{\mathrm{OA}}, s_{\mathrm{OB}}) = 0$. If the s_{OB} player switches to s_{OA}, the fitness becomes $f_p(s_{\mathrm{OA}}, s_{\mathrm{OA}}) = c_A > 0$.
4. $f_p(s_{\mathrm{FC}}, s_{\overline{\mathrm{FC}}}) = (c_A + c_B)p\bar{p}$. If the $s_{\overline{\mathrm{FC}}}$ player switches to s_{FC}, the fitness becomes $f_p(s_{\mathrm{FC}}, s_{\mathrm{FC}}) = (c_A + c_B)p^2 > (c_A + c_B)p\bar{p}$.
5. $f_p(s_{\mathrm{FC}}, s_{\mathrm{OB}}) = c_B p$. If the s_{FC} player switches to s_{OB}, the fitness becomes $f_p(s_{\mathrm{OB}}, s_{\mathrm{OB}}) = c_B > c_B p$ if and only if $p \neq 1$. If the s_{OB} player switches to s_{FC}, the fitness becomes $f_p(s_{\mathrm{FC}}, s_{\mathrm{FC}}) = (c_A + c_B)p^2 > c_B p$ iff $p > c_B/(c_A + c_B)$, which is satisfied at $p = 1$.
6. $f_p(s_{\overline{\mathrm{FC}}}, s_{\mathrm{OB}}) = c_B \bar{p}$. If the $s_{\overline{\mathrm{FC}}}$ player switches to s_{OB}, the fitness becomes $f_p(s_{\mathrm{OB}}, s_{\mathrm{OB}}) = c_B > c_B \bar{p}$.

□

C Nash equilibrium conditions

The game $\mathcal{G}_p$

Due to the statement of Appendix B, we only need to consider the four symmetric strategy profiles to find the Nash equilibria of the game $\mathcal{G}_p$. We give an expression to check whether a symmetric strategy profile is a Nash equilibrium.

Theorem C.1. The strategy profile (s^*, s^*) is a (strict) Nash equilibrium in $\mathcal{G}_p$ if

$$\mathrm{diag}(2s^* - \mathbf{1})(Q^p s^* - c_A L_p) \succ 0 \tag{C.1}$$

where $\mathbf{1} = [1, 1]^\top$ and $\succ$ denotes element-wise strict inequality.

Proof. We define player i's best response s_i^* to player j's strategy s_j as

$$s_i^* = \mathrm{BR}_i(s_j) \equiv \arg\max_{s_i} f_p(s_i, s_j) \tag{C.2}$$

Since $f_p(s_i, s_j)$ is bilinear in s_i and s_j, the best-response can be written

$$s_i^*(E_k) = \begin{cases} 0 \text{ if } [Q^p s_j - c_A L_p]_{E_k} < 0 \\ 1 \text{ if } [Q^p s_j - c_A L_p]_{E_k} \geq 0 \end{cases}, \quad k = A, B \tag{C.3}$$

When the condition above is met with equality, player i is indifferent to choosing 0 or 1 at $s_i^*(E_k)$. For convention, we define the best-response to choose 1 in this case. The definition of a Nash equilibrium (s_1, s_2) states

$$s_1 \in \mathrm{BR}_1(s_2) \text{ and } s_2 \in \mathrm{BR}_2(s_1) \tag{C.4}$$

Since we can consider only symmetric strategy profiles, the condition simplifies to $s \in \mathrm{BR}(s)$ □

The Nash equilibrium regions for each of the four strategy profiles are derived by applying Theorem C.1, which gives two inequality conditions in terms of p and κ (after normalizing by dividing c_B). The fitness maximizer regions are determined by comparing fitnesses for each strategy. For instance, the region where OA maximizes fitness is derived by setting $f_p(\mathrm{OA}) > f_p(\mathrm{X})$ for each X = FC, $\overline{\mathrm{FC}}$, and OB. Table C.1 lists the properties of all symmetric strategy profiles in $\mathcal{G}_p$.

Profile (s,s)	vector form	$f_p(s,s)$	NE region (p,κ)	fitness max region
OA	$[0,0]^\top$	c_A	everywhere	$\kappa > \{p^2/(1-p^2), 1\}$
FC	$[0,1]^\top$	$(c_A + c_B)p^2$	$\kappa > \bar{p}/p,\ \kappa < p/\bar{p}$	$\kappa < p^2/(1-p^2), \kappa > 1/p^2 - 1$
$\overline{\mathrm{FC}}$	$[1,0]^\top$	$(c_A + c_B)\bar{p}^2$	nowhere	nowhere
OB	$[1,1]^\top$	c_B	everywhere	$\kappa < \{1/p^2 - 1, 1\}$

Table C.1: Properties for the symmetric strategy profiles of $\mathcal{G}_p$.

The game with information sharing $\mathcal{G}_{pq}$

The Nash equilibrium condition for $\mathcal{G}_{pq}$ is stated here.

Theorem C.2. The strategy profile (s_1^*, s_2^*) is a (strict) Nash equilibrium in $\mathcal{G}_{pq}$ if

$$\begin{aligned} \mathrm{diag}(2s_1^* - \mathbf{1})(Q^{pq}s_2^* - c_A L_{pq}) \succ 0 \\ \mathrm{diag}(2s_2^* - \mathbf{1})(Q^{pq}s_1^* - c_A L_{pq}) \succ 0 \end{aligned} \tag{C.5}$$

Proof. Similar arguments as in Theorem C.1. □

Corollary C.1. In $\mathcal{G}_{pq}$, FC is a Nash equilibrium for

$$\begin{aligned} q &< \frac{\kappa p}{\bar{p} + \kappa p}, && \text{if } \kappa < 1 \\ q &< \frac{p}{p + \kappa \bar{p}}, && \text{if } \kappa > 1 \\ q &< p, && \text{if } \kappa = 1 \end{aligned} \tag{C.6}$$

Proof. Recall $s_{\mathrm{FC}} = [0,0,1,1]^\top$. From Theorem C.2, the NE condition for $(s_{\mathrm{FC}}, s_{\mathrm{FC}})$ is

$$\mathrm{diag}(2s_{\mathrm{FC}} - \mathbf{1})(Q^{pq}s_{\mathrm{FC}} - c_A L_{pq}) \succ 0 \tag{C.7}$$

After dividing by c_B and p, we obtain four inequality conditions

$$\begin{aligned} -\kappa pq + \bar{p}\bar{q} < 0 \\ -\kappa p\bar{q} + \bar{p}q < 0 \\ -\kappa \bar{p}q + p\bar{q} > 0 \\ -\kappa \bar{p}\bar{q} + pq > 0 \end{aligned} \tag{C.8}$$

If $\kappa = 1$, the first and fourth inequality conditions above are the same, and the second and third are the same. This reduces (C.8) to two conditions - 1) $pq > \bar{p}\bar{q}$ and 2) $p\bar{q} > \bar{p}q$. Here, 1) is always true, and 2) reduces to $p > q$. For general $\kappa \neq 1$, we obtain

$$\max\left\{\frac{\bar{p}}{\bar{p} + p\kappa}, \frac{\kappa \bar{p}}{p + \bar{p}\kappa}\right\} < q < \min\left\{\frac{\kappa p}{\bar{p} + p\kappa}, \frac{p}{p + \bar{p}\kappa}\right\} \tag{C.9}$$

If $\kappa < 1$, it is $\frac{\bar{p}}{\bar{p}+p\kappa} < q < \frac{\kappa p}{\bar{p}+p\kappa}$. If $\kappa > 1$, it is $\frac{\kappa \bar{p}}{p+\bar{p}\kappa} < q <, \frac{p}{p+\bar{p}\kappa}$ □

The region is shown in Figure S3.

Corollary C.2. In $\mathcal{G}_{pq}$, ML_A is a Nash equilibrium for

$$\begin{aligned} p((1+\kappa)q\bar{q}-\kappa)+(\bar{q}-\kappa q) &< 0 \\ p(\kappa(2q-1)-(\kappa+1)q\bar{q})+(1-(1+\kappa)q) &< 0 \\ \kappa\bar{p}(\bar{p}q\bar{q}-pq-\bar{p}\bar{q})+p^2q\bar{q} &< 0 \\ \kappa\bar{p}(\bar{p}q^2-pq-\bar{p}\bar{q})+p^2q^2 &> 0 \end{aligned} \tag{C.10}$$

Proof. Apply Theorem C.2 □

The Nash Equilibrium region can be derived in a similar manner for ML_B, which coincides with ML_A when $\kappa = 1$. The region is shown in Figure S4.

Corollary C.3. In $\mathcal{G}_{pq}$, the asymmetric strategy profile (s_1, s_2) with $s_1 = s_{\mathrm{FC}} = [0, 0, 1, 1]$ and $s_2 = [0, 1, 0, 1]$ is a Nash equilibrium for

$$\begin{aligned} q &> \frac{p}{p+\kappa\bar{p}}, && \text{if } \kappa < 1 \\ q &> \frac{\kappa p}{\bar{p}+\kappa p}, && \text{if } \kappa > 1 \\ q &> p, && \text{if } \kappa = 1 \end{aligned} \tag{C.11}$$

The s_2 strategy "follows" the social cue sent by player 1. Hence we refer to this strategy profile as "FC-F" (Follow Cue - Follower).

Proof. Apply Theorem C.2 □

The region is shown in Figure S5.

D Fitness value of information sharing

Parameterizations of q_c

We derive the parameterizations for the critical threshold value $q_c(p)$ in the six cases that determine the fitness maximizer in $\mathcal{G}_{pq}$. These curves are the boundaries of the region where the ML strategies are fitness maximizers (see Figure S2).

Proposition D.1. The critical threshold sensing fidelity q_c is given by the following six parameterizations.

(a) $q_c = \frac{-2\kappa\bar{p}(2p-1)+\sqrt{(2\kappa\bar{p}(2p-1))^2+8\kappa p\bar{p}(\kappa\bar{p}^2+p^2)}}{2(\kappa\bar{p}^2+p^2)}$ for $\frac{\kappa-\sqrt{\kappa}}{\kappa-1} \le p \le \sqrt{\frac{\kappa}{\kappa+1}}$, $\kappa > 1$.

(b) $q_c = \frac{-2\kappa\bar{p}(2p-1)+\sqrt{(2\kappa\bar{p}(2p-1))^2-4(\kappa(1-2p\bar{p})-(\kappa+1)p^2)(\kappa\bar{p}^2+p^2)}}{2(\kappa\bar{p}^2+p^2)}$ for $\sqrt{\frac{\kappa}{\kappa+1}} \le p \le 1$, $\kappa > 1$.

(c) $q_c = \frac{-2\bar{p}(2p-1)+\sqrt{(2\bar{p}(2p-1))^2+8p\bar{p}(\kappa p^2+\bar{p}^2)}}{2(\kappa p^2+\bar{p}^2)}$ for $\frac{\sqrt{\kappa^{-1}}-1}{\kappa^{-1}-1} \le p \le \sqrt{\frac{1}{\kappa+1}}$, $\kappa < 1$.

(d) $q_c = \frac{-2\bar{p}(2p-1)+\sqrt{(2\bar{p}(2p-1))^2-4((1-2p\bar{p})-(\kappa+1)p^2)(\kappa p^2+\bar{p}^2)}}{2(\kappa p^2+\bar{p}^2)}$ for $\frac{\sqrt{\kappa^{-1}}-1}{\kappa^{-1}-1} \le p \le 1$, $\kappa < 1$.

(e) $q_c = \frac{-2\bar{p}(2p-1)+\sqrt{(2\bar{p}(2p-1))^2+8p\bar{p}(\bar{p}^2+p^2)}}{2(\bar{p}^2+p^2)}$ for $\frac{1}{2} \le p \le \sqrt{\frac{1}{2}}$, $\kappa = 1$.

(f) $q_c = \frac{-2\bar{p}(2p-1)+\sqrt{(2\bar{p}(2p-1))^2-4((1-2p\bar{p})-2p^2)(\bar{p}^2+p^2)}}{2(\bar{p}^2+p^2)}$ for $\sqrt{\frac{1}{2}} \le p \le 1$, $\kappa = 1$.

Proof. Depending on which of the six cases in the main text is given, $f_{pq}(\mathrm{ML}_A)$ or $f_{pq}(\mathrm{ML}_B)$ is equated to the fitness of OA, OB, or FC to derive the expression for q_c. Since q_c must be between 1/2 and 1, there is a lower cutoff for p in cases (a-d) below which the value of q_c is greater than one, and hence not a valid threshold. □

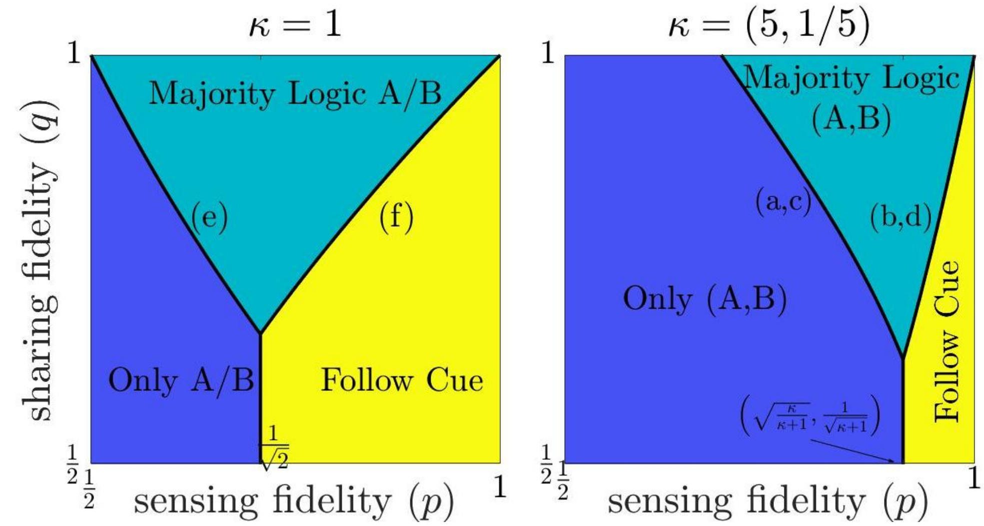

Figure S2: Boundary curves labeled (a-f) that define the critical threshold fidelity q_c that separates fitness maximizing regions. The labeled curves are parameterized according to Proposition D.1, which separate the region where the Majority Logic strategies are fitness maximizers.

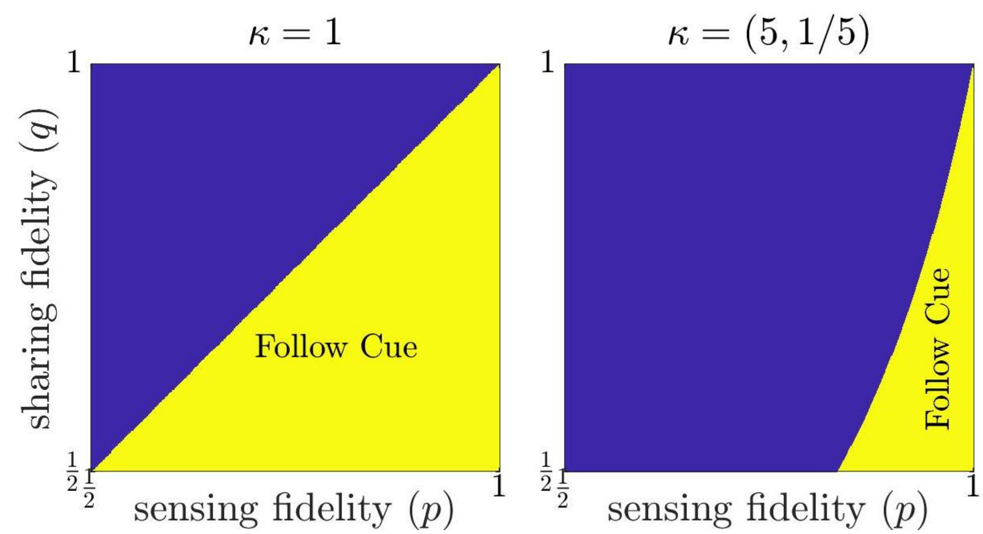

Figure S3: Nash equilibrium region for the Follow Cue (FC) strategy profile in the parameter space $(p, q) \in [\frac{1}{2}, 1]^2$, as given by Corollary C.1.

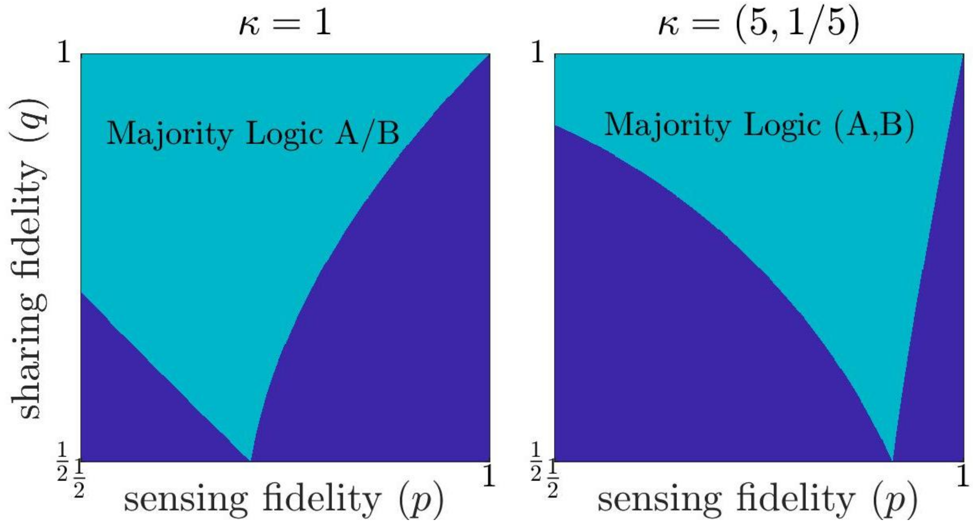

Figure S4: Nash equilibrium region for the Majority Logic (ML) strategy profiles in the parameter space $(p, q) \in [\frac{1}{2}, 1]^2$, as given by Corollary C.2.

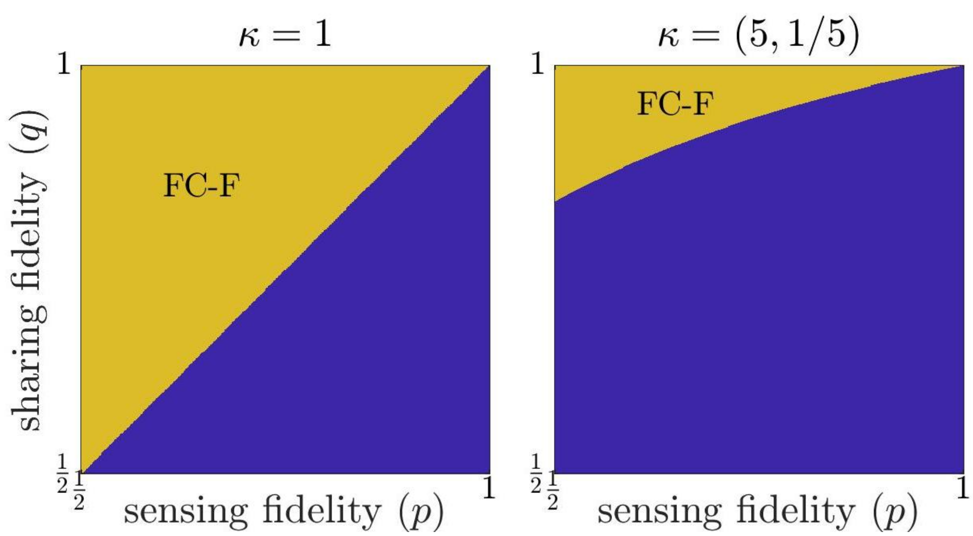

Figure S5: Nash equilibrium region for the FC-F strategy profile in the parameter space $(p, q) \in [\frac{1}{2}, 1]^2$, as given by Corollary C.3.