

The Basic Reproductive Number for Disease Systems with Multiple Coupled Heterogeneities

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Abstract

In mathematical epidemiology, a well-known formula describes the impact of heterogeneity on the basic reproductive number for situations in which transmission is separable and for which there is one source of variation in susceptibility and one source of variation in infectiousness. This formula is written in terms of the magnitudes of the heterogeneities, as quantified by their coefficients of variation, and the correlation between them. A natural question to ask is whether analogous results apply when there are multiple sources of variation in susceptibility and/or infectiousness. In this paper we demonstrate that under three or more coupled heterogeneities, the basic reproductive number depends on details of the distribution of the heterogeneities in a way that is not seen in the well-known simpler situation. We provide explicit results for the cases of multivariate normal and multivariate log-normal distributions, showing that the basic reproductive number can again be expressed in terms of the magnitudes of the heterogeneities and the pairwise correlations between them. The results, however, differ between the two multivariate distributions, demonstrating that no formula of this type applies generally when there are three or more

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coupled heterogeneities. We see that the results are approximately equal when heterogeneities are relatively small and show that an earlier result in the literature (Koella, 1991) should be viewed in this light. We provide numerical illustrations of our results.

1. Introduction

The basic reproductive number, R_0 , plays a crucial role in determining both whether a pathogen is able to spread and the strength of control measures needed to halt its spread. The simplest descriptions of R_0 assume simple transmission scenarios, such as perfect mixing of a population and homogeneity of the individuals in the population, e.g. in terms of their susceptibility and infectiousness. The inadequacies of such descriptions have long been realized and much attention has been directed towards understanding the impact of heterogeneities in transmission on the basic reproductive number. Early efforts included accounting for differing activity levels amongst the population and various mixing patterns of the population (e.g. proportionate/random mixing, assortative and disassortative mixing). Much of this work was prompted by the heterogeneities known to exist for the spread of sexually transmitted infections, notably gonorrhoea and HIV (Nold (1980); Hethcote and Yorke (1984); Anderson et al. (1986); May and Anderson (1987); Jacquez et al. (1988); Gupta et al. (1989)). In the context of vector-borne diseases, it has long been realized that vectors' bites are not distributed uniformly across hosts; instead, there is a heterogeneity in hosts' attractiveness to vectors, with some individuals being disproportionately favored to receive bites (Carnevale et al. (1978); Dye and Hasibeder (1986); De Benedictis et al. (2003); Liebman et al. (2014)).

A now standard argument shows that the basic reproductive number for a multi-type transmission system can be calculated as the dominant eigenvalue of the next generation matrix (Diekmann and Heesterbeek (2000)). For an n -type setting, the next generation matrix is an n by n non-negative matrix whose (i, j) 'th entry gives the average number of secondary infections of type i caused

26 by a type j individual in an otherwise entirely susceptible population. Conse-
27 quently, much attention has been directed towards those special cases of het-
28 erogeneous transmission that lead to next generation matrices whose dominant
29 eigenvalue is analytically tractable and hence for which the basic reproductive
30 number can be calculated explicitly. In the context of spatial heterogeneity,
31 these include symmetric spatial configurations such as equally-sized patches
32 with all-to-all or nearest neighbor contacts (see, for example Lloyd and May
33 (1996)).

34 More generally, a commonly-studied situation involves separable transmis-
35 sion (Diekmann and Heesterbeek (2000)), where each group has a susceptibility,
36 a_i and an infectiousness, b_i . In this case, the next generation matrix is of rank
37 one, and, taking groups to be of equal sizes, has entries $a_i b_j / n$, and dominant
38 eigenvalue

$$R_0 = \frac{1}{n} \sum_{i=1}^n a_i b_i. \quad (1)$$

39 Using the result

$$E(XY) = E(X)E(Y) + \text{Cov}(X, Y) \quad (2)$$

40 for the expectation of a product of random variables, eqn (1) can be rearranged
41 into the following well-known formula (Dietz (1980); Dye and Hasibeder (1986))
42 that sheds insight into the impact of heterogeneity on R_0 in this separable
43 setting:

$$\begin{aligned} R_0 &= \bar{a}\bar{b}(1 + r_{ab}CV_aCV_b) \\ &= R_0^{\text{hom}}(1 + r_{ab}CV_aCV_b). \end{aligned} \quad (3)$$

44 Here, \bar{a} and \bar{b} denote the average values of a_i and b_i , r_{ab} denotes the Pearson
45 product-moment correlation coefficient between the a_i and b_i , CV_a and CV_b
46 denote the coefficients of variation (i.e. standard deviation divided by the mean)
47 of a_i and b_i and R_0^{hom} denotes the value of R_0 that would be predicted if the
48 heterogeneity was ignored, i.e. the average values of a_i and b_i were used. We

49 emphasize that these results are exact, holding for arbitrary distributions of the
50 a_i and b_i .

51 Heterogeneity can inflate or deflate the value of R_0 , depending on whether
52 there is positive or negative correlation between susceptibility and infectivity
53 across the groups (Dietz, 1980). In the special case where susceptibility and
54 infectivity are proportional, e.g. for a situation such as differing activity levels or
55 mosquito biting preferences where the heterogeneity impacts both susceptibility
56 and infectiousness in the same way, the formula reduces to

$$\begin{aligned} R_0 &= R_0^{\text{hom}} (1 + \text{CV}_a^2) \\ &= R_0^{\text{hom}} \left(1 + \frac{\text{Var}(a)}{\bar{a}^2} \right). \end{aligned} \quad (4)$$

57 This formula has appeared in the literature numerous times in a number of
58 different settings and guises (Dietz, 1980; Dye and Hasibeder, 1986; May and
59 Anderson, 1987).

60 Particularly with the increasing realization that many systems are subject
61 to multiple, often coupled, heterogeneities (Paull et al., 2012; Vazquez-Prokopec
62 et al., 2016), an important question is whether results such as eqns. (3) and (4)
63 generalize to situations in which there are more than two heterogeneities. In
64 this paper, we show that the answer to this question is no: the effect of multiple
65 interacting heterogeneities on the basic reproductive number depends on the
66 details of the distributions of the heterogeneities, in contrast to what occurs in
67 the two-heterogeneity setting. We provide results for both multivariate normal
68 and multivariate log-normal distributions of heterogeneities and demonstrate
69 that the two settings can give markedly different results.

70 **2. Results**

71 We assume that there are n types of individuals, resulting from N different
72 heterogeneities, N_1 of which impact susceptibility and N_2 of which impact in-
73fectiousness. We further assume that the susceptibility of a type i individual

74 can be written as the product $x_i^1 x_i^2 \cdots x_i^{N_1}$, taken over the heterogeneities that
 75 impact susceptibility, and that the infectivity of a type j individual can sim-
 76 ilarly be written as $x_j^1 \cdots x_j^{N_2}$. Taking group sizes to be equal and assuming
 77 separable transmission, the entries of the next generation matrix will have the
 78 form $x_i^1 x_i^2 \cdots x_i^{N_1} x_j^1 \cdots x_j^{N_2} / n$, where $N_1 + N_2 = N$. This matrix is of rank one
 79 and has dominant eigenvalue given by

$$R_0 = \frac{1}{n} \sum_{i=1}^n x_i^1 x_i^2 \cdots x_i^N. \quad (5)$$

80 As explained above, the well-known result arises from the ability to express
 81 the expectation of the product of a pair of random variables in terms of their
 82 two expectations and their covariance. Extension of the result requires corre-
 83 sponding manipulations of expectations of products of three or more random
 84 variables—the so-called product moments of the joint distribution.

85 2.1. Analytic Results

86 In the case of a set of random variables whose joint distribution is multivari-
 87 ate normal, numerous authors have obtained results for product moments (see,
 88 for example, Isserlis (1918), Bendat and Piersol (1966), Bär and Dittrich (1971)
 89 and Song and Lee (2015)). For instance, in the four dimensional case we have
 90 (Bendat and Piersol (1966) and Bär and Dittrich (1971))

$$\begin{aligned} E(X_1 X_2 X_3 X_4) &= E(X_1 X_2) E(X_3 X_4) + E(X_1 X_3) E(X_2 X_4) + \\ &E(X_1 X_4) E(X_2 X_3) - 2E(X_1) E(X_2) E(X_3) E(X_4). \end{aligned} \quad (6)$$

91 The expectations of pairwise products can be rewritten in the way described
 92 above to give

$$\begin{aligned} E(X_1 X_2 X_3 X_4) &= E(X_1) E(X_2) E(X_3) E(X_4) \times \\ &\{ (1 + r_{X_1, X_2} CV_{X_1} CV_{X_2}) (1 + r_{X_3, X_4} CV_{X_3} CV_{X_4}) + \\ &(1 + r_{X_1, X_3} CV_{X_1} CV_{X_3}) (1 + r_{X_2, X_4} CV_{X_2} CV_{X_4}) + \\ &(1 + r_{X_1, X_4} CV_{X_1} CV_{X_4}) (1 + r_{X_2, X_3} CV_{X_2} CV_{X_3}) - \\ &2 \}. \end{aligned} \quad (7)$$

93 We remark that the case of three random variables can be obtained by setting
 94 $X_4 = 1$.

95 For a set of N multivariate lognormally distributed random variables, prod-
 96 uct moments are given by the formula (Kotz et al. (2000))

$$\mathbb{E} \left(\prod_{j=1}^N X_j^{r_j} \right) = \exp \left(\mathbf{r}^T \boldsymbol{\xi} + \frac{1}{2} \mathbf{r}^T V \mathbf{r} \right), \quad (8)$$

97 where $\boldsymbol{\xi}$ and V are the mean and variance of the corresponding multivariate
 98 normal distribution. Some simple manipulation leads to

$$\begin{aligned} \mathbb{E}(X_1 X_2 X_3 X_4) &= \mathbb{E}(X_1) \mathbb{E}(X_2) \mathbb{E}(X_3) \mathbb{E}(X_4) \times \\ &\quad (1 + r_{X_1, X_2} \text{CV}_{X_1} \text{CV}_{X_2}) (1 + r_{X_1, X_3} \text{CV}_{X_1} \text{CV}_{X_3}) \times \\ &\quad (1 + r_{X_1, X_4} \text{CV}_{X_1} \text{CV}_{X_4}) (1 + r_{X_2, X_3} \text{CV}_{X_2} \text{CV}_{X_3}) \times \\ &\quad (1 + r_{X_2, X_4} \text{CV}_{X_2} \text{CV}_{X_4}) (1 + r_{X_3, X_4} \text{CV}_{X_3} \text{CV}_{X_4}). \end{aligned} \quad (9)$$

99 Given that eqns (7) and (9) differ, and that their reduced forms when $X_4 = 1$
 100 also differ, we have shown that there is no general formula of this type for the
 101 basic reproductive number when there are three or more coupled heterogeneities.
 102 We do notice, however, that the two formulae give approximately equal results
 103 in the limit of small coefficients of variation, *i.e.* when one can ignore products
 104 involving two or more pairs of coefficients of variation.

105 The majority of papers in the literature that provide analytic results for
 106 the basic reproductive number under heterogeneity focus on at most two cou-
 107 pled heterogeneities. One notable exception is the work of Koella (1991), which
 108 provides—without proof or qualification for its applicability—the following for-
 109 mula for a vector-borne pathogen subject to heterogeneities in mosquito biting
 110 rate, a , human susceptibility, b , and duration of human infection, ρ

$$R_0 = R_0^{\text{hom}} \left[1 + \frac{\text{Var}(a)}{\bar{a}^2} + 2 \frac{\text{Cov}(a, b)}{\bar{a}\bar{b}} + 2 \frac{\text{Cov}(a, \rho)}{\bar{a}\bar{\rho}} + \frac{\text{Cov}(\rho, b)}{\bar{\rho}\bar{b}} \right]. \quad (10)$$

111 Note that the single biological heterogeneity in biting rate impacts both in-
 112 fectiousness and susceptibility, resulting in it being treated as two perfectly
 113 correlated heterogeneities.

114 We note that equation (10) has no terms that involve products of pairs of
115 covariances (or, in the language of the earlier formulae, correlation coefficients).
116 As in the remark above comparing results between multivariate normal and
117 lognormal distributions, this formula should, in general, be seen as an approx-
118 imation that is likely most accurate when coefficients of variation are small
119 (*i.e.* the heterogeneities are relatively minor). As a comment that is germane
120 to a numerical example shown below, we remark that the Koella formula does
121 agree with the result for the multivariate normal distribution, eqn (7), if the
122 coefficient of variation describing either human susceptibility or the duration of
123 human infection is equal to zero.

124 2.2. Numerical Results

125 We illustrate the above results using numerical simulation, allowing us to
126 explore the differences between predictions made using the formulae for the two
127 distributions and also using the formula in the small coefficient of variation limit.
128 For concreteness, we place these simulations within the vector-host setting de-
129 scribed by Koella (1991), but for simplicity we hold one of the factors constant.
130 Specifically, hosts differ in their attractiveness to mosquitoes, impacting their
131 susceptibility and infectiousness (thus treated as two perfectly correlated het-
132 erogeneities, X_1 and X_2 , within our framework), and also in their durations of
133 infection, X_3 . Setting $X_4 = 1$ and taking $X_2 = X_1$, we obtain the following two
134 formulae:

$$R_0 = E(X_1)^2 E(X_3) \{1 + CV_{X_1}^2 + 2r_{X_1, X_3} CV_{X_1} CV_{X_3}\} \quad (11)$$

135 for bivariate normally distributed heterogeneities, and

$$R_0 = E(X_1)^2 E(X_3) \{1 + CV_{X_1}^2\} \{1 + r_{X_1, X_3} CV_{X_1} CV_{X_3}\}^2 \quad (12)$$

136 for bivariate lognormally distributed heterogeneities. We notice that in this
137 reduced setting of $X_4 = 1$, the first of these formulae coincides with the small
138 coefficient of variation limit of the two general formulae, and, as discussed above,

139 also agrees with the Koella formula. Furthermore, we see that when the cor-
140 relation coefficient, r_{X_1, X_3} , between the two heterogeneities is zero, the two
141 formulae are identical.

142 2.2.1. Bivariate Normal Distribution

143 Figure (1) illustrates the performance of eqn (11) when the group attributes
144 X_1 and X_3 are sampled from a bivariate normal distribution. Parameter values
145 (given in the figure caption) were chosen for illustrative purposes and are not
146 intended to represent a specific real-world infection. For each of a thousand
147 replicates, either ten (panel a) or a thousand (panel b) pairs of values of biting
148 rate and duration of human infection were sampled from a bivariate normal
149 distribution. The actual R_0 value for each replicate, calculated from eqn (5), is
150 plotted against the value of R_0 predicted for that replicate by the MVN formula,
151 eqn (11). In addition, we show (using a red square) the R_0 value that would be
152 predicted if there was no heterogeneity (*i.e.* X_1 and X_3 are set equal to their
153 respective average values) and the value predicted by the MVN formula for the
154 underlying MVN distribution (green star).

155 To aid comparison, the 45° diagonal line is shown on both plots: deviations
156 from these lines represent deviations from the values predicted by the MVN
157 formula. We quantitate these deviations by calculating the coefficient of deter-
158 mination, R^2 , in a way that is familiar from regression theory. In both cases,
159 the R^2 value is high, and approximately equal to 0.99. The numerically calcu-
160 lated values of R_0 fall in a cloud centered on the value predicted by the MVN
161 formula, with the size of the cloud being smaller for the panel resulting from the
162 larger number of groups. The deviations here result from sampling error, with
163 samples drawn from the bivariate normal not being perfectly representative of
164 the entire distribution. This effect is more pointed when there are fewer samples
165 (*i.e.* fewer groups), with the cloud of points shrinking as the number of groups
166 increases. In fact, since the basic reproductive number, eqn (5), is calculated
167 as the average of a sequence of independent, identically distributed quantities,
168 the central limit theorem can be used to quantify the variation seen about the

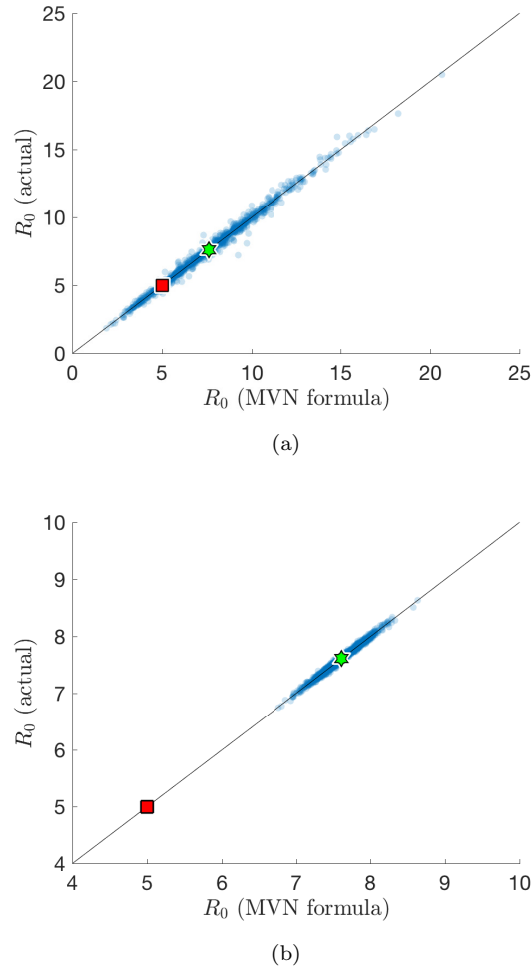


Figure 1: Comparison of R_0 values calculated numerically (using eqn 5) with those predicted by the MVN formula (eqn 11) for heterogeneities distributed according to an MVN distribution. Each blue circle represents the values of R_0 obtained for (panel a) a ten group model or (panel b) a thousand group model with pairs of values of biting rate, X_1 , and average duration of human infection X_3 drawn from a bivariate normal distribution with means 1 and 5, respectively, variances $\text{Var}(X_1) = 0.2$ and $\text{Var}(X_3) = 4$, and correlation $r_{X_1, X_2} = 0.9$. The value of the susceptibility parameter was fixed at 1 for each group. The red square denotes the value of R_0 if there was no heterogeneity, *i.e.* obtained using the average values, R_0^{hom} , while the green star denotes the value of R_0 calculated using the MVN formula using the means, variances and covariances of the underlying bivariate normal distribution. As described in the text, the predictive ability of the MVN formula is measured by the coefficient of determination, R^2 , and equals: (panel a) 0.991 and (panel b) 0.988. Note the different scales on the axes between the two panels.

169 central predicted value. We remark that a corresponding figure generated in the
170 case of two coupled heterogeneities would exhibit no deviation from the diago-
171 nal line as eqn (3) is exact: it does not rely on any distributional assumption of
172 heterogeneities across groups.

173 2.2.2. Bivariate Lognormal Distribution

174 Use of a multivariate lognormal distribution allows us to explore settings in
175 which the components of transmission exhibit more severe heterogeneity and
176 to assess the extent to which the impact of such heterogeneities are misrepre-
177 sented by either the small coefficient of variation formula or by the formula that
178 pertains in the multivariate normal case.

179 In figure (2), the biting rates and durations of infectiousness are drawn from
180 independent lognormal distributions, with means 1 and 3, and variances 1 and
181 8, respectively. We compare the performance of the MVN formula (panel a)
182 and MVLN formula (panel b) for a 1000-group setting (*i.e.* 1000 samples are
183 drawn from the distributions). In this case, because there is zero correlation
184 between the two heterogeneities, the population-level predictions of the MVN
185 and MVLN formulae (*i.e.* the values obtained using the moments of the un-
186 derlying distribution) are identical (shown by a yellow diamond). We see that
187 the MVLN formula provides a better description ($R^2 = 0.748$ using the MVN
188 formula, while $R^2 = 0.852$ using the MVLN formula), which is to be expected
189 given that samples were drawn from an MVLN distribution.

190 Figure (3) explores a situation in which there is a positive correlation ($r =$
191 0.6) between the two components, *i.e.* the biting rate and duration of infectious-
192 ness, of the bivariate lognormal distribution. We see that the MVLN formula
193 performs well (panel b, $R^2 = 0.852$). The positive correlation leads to the MVN
194 formula underestimating R_0 compared to the MVLN formula, for example as
195 witnessed by the central estimates of R_0 (yellow diamond: MVLN, green star:
196 MVN). The MVN formula consistently provides a large underestimate of the
197 true value of R_0 (panel a), and its predictive ability is poor (the negative value
198 of R^2 , -0.596, indicates that the formula performs worse on these points than a

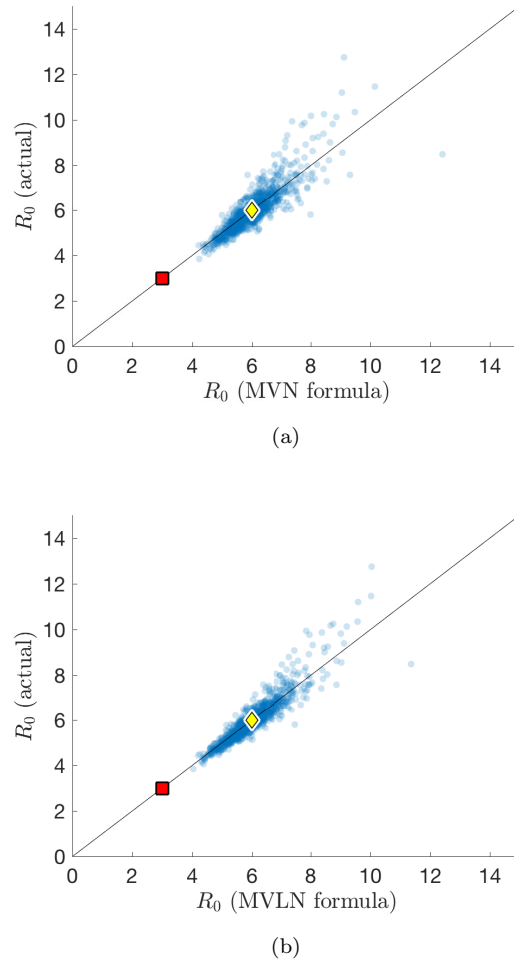


Figure 2: Comparison of R_0 values calculated numerically (using eqn 5) with those predicted by the formula derived from the MVN distribution (eqn 11) (panel a) and those predicted by the formula derived from the MVLN distribution (eqn 12) (panel b). Each blue circle represents the values of R_0 obtained for a thousand group model with values of biting rate, X_1 , and average duration of human infection, X_3 , drawn from independent lognormal distributions (means 1 and 3, variances 1 and 8, respectively). The value of the susceptibility parameter was fixed at 1 for each group. The red square denotes the value of R_0 obtained using the average values, R_0^{hom} , while the yellow diamond denotes the value of R_0 calculated using the MVLN formula using the means, variances and covariances of the bivariate lognormal distribution. Because the distributions of X_1 and X_3 are assumed to be independent in this figure, the population-level predictions of the MVLN and MVN formulae are identical. The predictive ability of the MVLN formula (panel b, $R^2 = 0.852$) is greater than that of the MVN formula (panel a, $R^2 = 0.748$), as should be expected given that draws were made from an MVLN distribution.

199 constant predictor).

200 Finally, we consider a setting in which the two heterogeneities are negatively
201 correlated, with $r = -0.2$. Figure (4) shows that in this case, the MVLN
202 formula correctly predicts lower values of R_0 than does the MVN formula (*e.g.*
203 compare the locations of the yellow diamond and the green star, obtained from
204 MVLN and MVN formulae, respectively, using the moments of the underlying
205 distribution). The MVLN formula provides reasonable predictions ($R^2 = 0.735$),
206 whereas the MVN formula ($R^2 = -4.34$) consistently overestimates the value
207 of R_0 , and often by a considerable amount.

208 3. Discussion

209 In this paper we have shown that the well-known result for the impact of two
210 coupled heterogeneities on the basic reproductive number of an epidemiological
211 system under separable transmission does not have a general counterpart when
212 there are three or more coupled heterogeneities. In the more general setting,
213 the formula for the basic reproductive number depends on details of the joint
214 distribution of the heterogeneities in a way that is quite different than in the
215 setting with two heterogeneities. We were able to derive formulae that related
216 the basic reproductive number to the magnitudes of the heterogeneities and
217 their pairwise correlations for the special cases of multivariate normal and mul-
218 tivariate lognormal distributions of heterogeneities. Under particular limiting
219 cases (typically in the limit of low levels of heterogeneity), the two formulae give
220 similar predictions. We showed that an earlier result in the literature (Koella,
221 1991) should be viewed as an approximate result, although we noted that in ap-
222 propriate limiting cases, the result agrees with our formula for the multivariate
223 normal distribution.

224 Given the reliance of the R_0 formulae obtained here on the joint distribution
225 of the heterogeneities, our results are only exact as the number of groups in
226 the multi-type model approaches infinity. For a finite number of groups, where
227 the levels of the heterogeneities across groups are sampled from the underlying

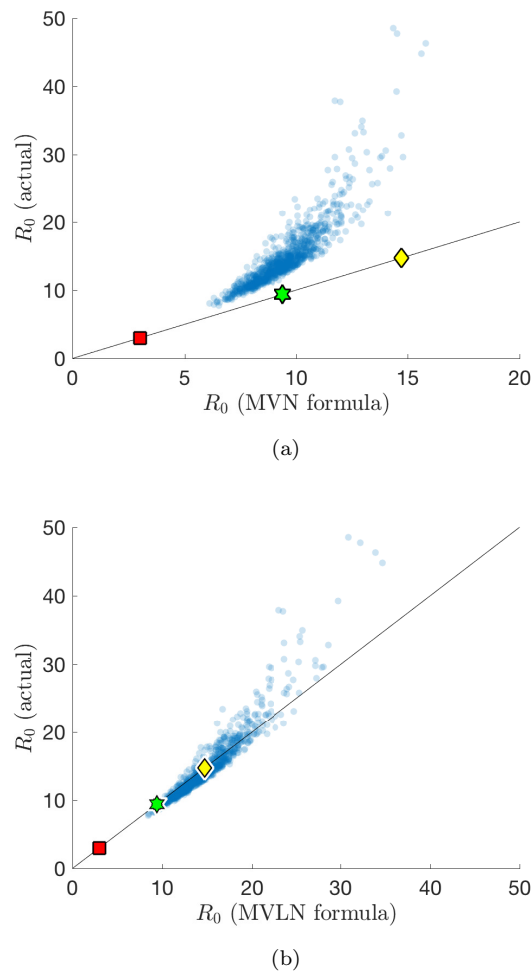


Figure 3: Comparison of R_0 values calculated numerically (using eqn 5) with those predicted by the formula derived from the MVN distribution (eqn 11) (panel a) and those predicted by the formula derived from the MVLN distribution (eqn 12) (panel b). Details are as in Figure (2), except that here there is a positive correlation, $r = 0.6$, between the two components of the bivariate log-normal distribution. In this case, the population-level predictions of the MVN and MVLN formulae (green star and yellow diamond, respectively) differ. The predictive ability of the MVLN formula (panel b, $R^2 = 0.828$) is greater than that of the MVN formula (panel a, $R^2 = -0.596$). The MVN formula performs worse than a constant predictor and consistently provides a large underestimate of the true value of R_0 .

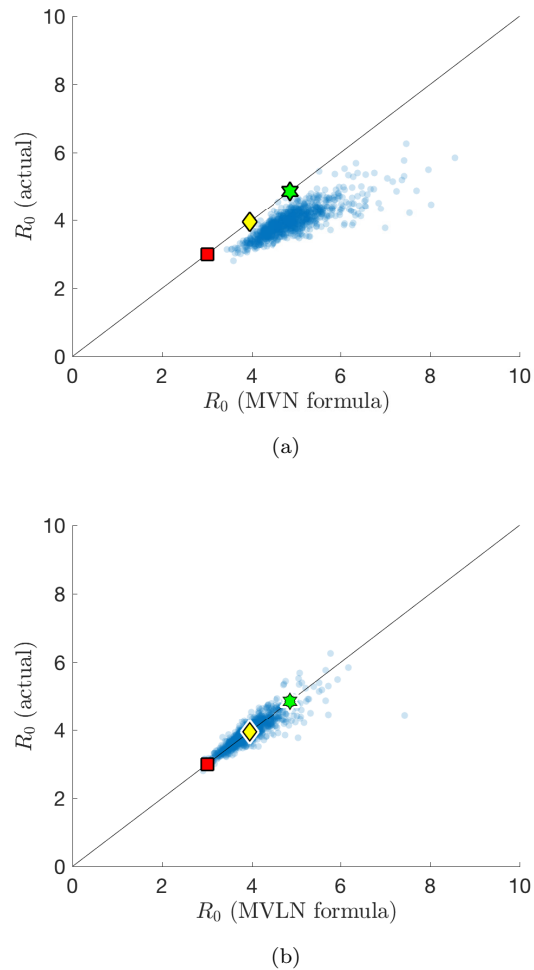


Figure 4: Comparison of R_0 values calculated numerically (using eqn 5) with those predicted by the formula derived from the MVN distribution (eqn 11) (panel a) and those predicted by the formula derived from the MVLN distribution (eqn 12) (panel b). Details are as in Figures (2) and (3), except that here there is a negative correlation, $r = -0.2$, between the two components of the bivariate lognormal distribution. Again, the predictive ability of the MVLN formula (panel b, $R^2 = 0.721$) is greater than that of the MVN formula (panel a, $R^2 = -4.91$), with the latter performing worse than a constant predictor and consistently providing a large overestimate of the true value of R_0 .

228 distribution, the predictions made by these formulae are not perfect. This again
229 is in contrast to the two heterogeneity setting, in which the well-known result
230 is exact.

231 Although theoretical attention has typically focused on the two heterogeneity
232 case, and this has provided much insight, heterogeneous transmission in the real
233 world typically involves more than two factors (Paull et al. (2012); Vazquez-
234 Prokopec et al. (2016)). As such, it is important to gain understanding of
235 how multiple coupled heterogeneities impact transmission and the limitations of
236 general results that can be obtained in such more realistic settings. This paper
237 provides a theoretical step in that direction and aims to guide more detailed
238 studies that involve numerical exploration of specific situations, yielding further
239 insights into the epidemiological role of individual variability.

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