The evolution of genetic diversity in changing environments
Supplementary Material

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ON THE EVOLUTION OF MIGRATION IN CHANGING ENVIRONMENTS

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Consider an infinite, haploid population divided into two demes E_x and E_y, with different selection regimes. Individuals in this population are characterized by two biallelic loci, a major locus A/a and a modifier locus B/b, where the major locus controls the phenotype of an individual and the fitness of its phenotype, while the modifier locus is assumed to be selectively neutral and controls the migration rate between the two demes.

We study the evolution of the modifier locus B/b and determine the evolutionarily stable migration rate as a function of the pattern of fluctuation in selection experienced by the population. We track the frequencies of the four genotypes AB, Ab, aB and ab in the population. At each generation, there is recombination and selection in each deme separately, after which individuals may migrate between the two demes. As in general analyses of neutral modifiers (see Feldman and Liberman 1986), we frame the question in terms of the stability of the fixation equilibrium with only B present in the population, producing migration rate m_B, to invasion by allele b, which produces migration rate m_b. We assume these rates to be the same from E_x to E_y and from E_y to E_x. This symmetry assumption makes our analysis more tractable but relaxing this assumption should not change the general conclusions of the model.

We first present the case when selection is constant in time in both demes; this will serve as our reference model. We then consider regimes in which there is fluctuating selection within each of the demes. In particular, we ask how the rate of environmental volatility affects the stable migration rate.

CONSTANT SELECTION

Let the fitnesses and the frequencies of the four genotypes in each deme be

Deme	E_x				E_y				
Genotype	AB	Ab	aB	ab	AB	Ab	aB	ab	
Fitness	w_{x_1}	w_{x_2}	w_{x_3}	w_{x_4}	w_{y_1}	w_{y_2}	w_{y_3}	w_{y_4}	$(S1.1)$
Frequency	x_1	x_2	x_3	x_4	y_1	y_2	y_3	$y_4.$	

Genotypes carrying the B allele at the modifier locus migrate from one deme to the other deme at rate m_B, while the migration rate is m_b when b is present at the modifier locus. The recombination rate between the two loci is r in both demes.

After recombination, selection, and migration, in that order, the change in frequency of the genotypes (x_1', x_2', x_3', x_4' and y_1', y_2', y_3', y_4') in the two demes from one generation to the next is given by

$$M_x x_1' = (1 - m_B)\frac{w_{x_1}(x_1 - rD_x)}{w_x} + m_B\frac{w_{y_1}(y_1 - rD_y)}{w_y}$$

$$M_x x_2' = (1 - m_b)\frac{w_{x_2}(x_2 + rD_x)}{w_x} + m_b\frac{w_{y_2}(y_2 + rD_y)}{w_y}$$

$$M_x x_3' = (1 - m_B)\frac{w_{x_3}(x_3 + rD_x)}{w_x} + m_B\frac{w_{y_3}(y_3 + rD_y)}{w_y}$$

$$M_x x_4' = (1 - m_b)\frac{w_{x_4}(x_4 - rD_x)}{w_x} + m_b\frac{w_{y_4}(y_4 - rD_y)}{w_y}$$

$(S1.2a)$

and

$$M_y y_1' = (1 - m_B)\frac{w_{y_1}(y_1 - rD_y)}{w_y} + m_B\frac{w_{x_1}(x_1 - rD_x)}{w_x}$$

$$M_y y_2' = (1 - m_b)\frac{w_{y_2}(y_2 + rD_y)}{w_y} + m_b\frac{w_{x_2}(x_2 + rD_x)}{w_x}$$

$$M_y y_3' = (1 - m_B)\frac{w_{y_3}(y_3 + rD_y)}{w_y} + m_B\frac{w_{x_3}(x_3 + rD_x)}{w_x}$$

$$M_y y_4' = (1 - m_b)\frac{w_{y_4}(y_4 - rD_y)}{w_y} + m_b\frac{w_{x_4}(x_4 - rD_x)}{w_x},$$

$(S1.2b)$

with

$$D_x = x_1 x_4 - x_2 x_3 \qquad D_y = y_1 y_4 - y_2 y_3 \qquad (S1.3)$$

$$w_x = \sum_{i=1}^{4} w_{x_i} x_i \qquad w_y = \sum_{i=1}^{4} w_{y_i} y_i \qquad (S1.4)$$

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$$w_{x_1} = w_{x_2} \qquad\qquad w_{y_1} = w_{y_2}$$
$$w_{x_3} = w_{x_4} \qquad\qquad w_{y_3} = w_{y_4}. \tag{S1.5}$$

Equation (S1.5) is due to the assumption that the modifier locus is selectively neutral. M_x and M_y are normalizing factors such that $\sum_{i=1}^4 x_i' = 1$ and $\sum_{i=1}^4 y_i' = 1$.

We are able to derive closed-form mathematical results if we assume the following symmetry relations among the fitness parameters:

$$w_{x_1} = w_{x_2} = w_{y_3} = w_{y_4} = (1+s)$$
$$w_{x_3} = w_{x_4} = w_{y_1} = w_{y_2} = 1. \tag{S1.6}$$

Thus, if we assume that $s > 0$, phenotype A is preferred in deme E_x, while phenotype a is preferred in deme E_y and there is symmetry in these fitness differences between the demes.

Boundary equilibria with symmetric selection

Suppose that initially only B is present at the modifier locus ($x_2 = x_4 = y_2 = y_4 = 0$). We use the notations $x_1 = x$ and $y_3 = z$; therefore, $x_3 = 1 - x$, $y_1 = 1 - z$ The change in the frequencies on the boundary where only B is present is given by

$$x' = (1 - m_B)\frac{(1+s)x}{w_x} + m_B \frac{1-z}{w_z}$$
$$z' = (1 - m_B)\frac{(1+s)z}{w_z} + m_B \frac{1-x}{w_x}, \tag{S1.7}$$

with
$$w_x = (1+s)x + (1-x) = 1 + sx$$
$$w_y = (1+s)z + (1-z) = 1 + sz. \tag{S1.8}$$

A simple computation shows that

$$z' - x' = \frac{1+s}{(1+sx)(1+sz)}(z - x). \tag{S1.9}$$

Hence, at equilibrium, either $z = x$ or

$$\frac{1+s}{(1+sx)(1+sz)} = 1. \tag{S1.10}$$

Let
$$\mathbf{x} = (x_1, x_2, x_3, x_4) \qquad \mathbf{y} = (y_1, y_2, y_3, y_4). \tag{S1.11}$$

When $z = x$, at equilibrium, we have the following result, similar to that in Balkau and Feldman (1973):

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Result 1.

1. On the boundary where only the B allele is present, there is a unique symmetric equilibrium $(\mathbf{x}^*, \mathbf{y}^*)$

$$\mathbf{x}^* = (x^*, 0, 1 - x^*, 0) \qquad \mathbf{y}^* = (1 - x^*, 0, x^*, 0). \qquad (S1.12)$$

Here, x^* is the unique positive root of the quadratic equation

$$Q(x) = sx^2 + [(s + 2)m_B - s]\, x + m_B = 0. \qquad (S1.13)$$

2. $(\mathbf{x}^*, \mathbf{y}^*)$ is internally stable on the boundary where only the B allele is present.

3. When $s > 0$ and $0 < m_B < 1$, then $x^* > \frac{1}{s+2}$. If, in addition, $0 < m_B < \frac{1}{2}$ then $x^* > \frac{1}{2}$.

Proof of Result 1.

1. When $z = x$ at equilibrium, the mean fitnesses in the two demes are equal, $w_x = 1 + sx = 1 + sz = w_y$, and the equilibrium equation resulting from (S1.7) reduces to the quadratic equation (S1.13). Moreover, as $0 < m_B < 1$,

$$Q(0) = -m_B < 0, \qquad Q(1) = (s + 1)m_B > 0, \qquad Q(\pm\infty) > 0. \qquad (S1.14)$$

Therefore there is a unique root x^* of (S1.13) with $0 < x^* < 1$.

2. "Near" $(\mathbf{x}^*, \mathbf{y}^*)$ and on the boundary where only B is present, $(z - x)$ is small and from (S1.9) its change in magnitude, and thus the internal stability of $(\mathbf{x}^*, \mathbf{y}^*)$, are determined by the positive factor

$$C^* = \frac{1 + s}{(1 + sx^*)(1 + sz^*)}. \qquad (S1.15)$$

$(\mathbf{x}^*, \mathbf{y}^*)$ is internally stable if $C^* < 1$, which, since $x^* = z^*$ and $s > 0$, is the case if and only if $s\,(x^*)^2 + 2x^* > 1$. Using the equilibrium equation (S1.13),

$$s\,(x^*)^2 + 2x^* = (1 - m_B)(s + 2)x^* + m_B. \qquad (S1.16)$$

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Thus, as $0 < m_B < 1$, (S1.3) is greater than 1 if and only if $x^* > \frac{1}{s+2}$. Since $Q(1) > 0$ and $Q(x^*) = 0$, we have $x^* > \frac{1}{s+2}$ if and only if $Q\left(\frac{1}{s+2}\right) < 0$. Indeed, when $s > 0$,

$$Q\left(\frac{1}{s+2}\right) = -\frac{s(s+1)}{(s+2)^2} < 0, \tag{S1.17}$$

as desired. Therefore $x^* > \frac{1}{s+2}$, $C^* < 1$, and $(\mathbf{x}^*, \mathbf{y}^*)$ is internally stable.

3. When $s > 0$ and $0 < m_B < 1$, $x^* > \frac{1}{s+2}$. In addition

$$Q\left(\frac{1}{2}\right) = -\frac{s}{4}(1 - 2m_B). \tag{S1.18}$$

Hence, if $0 < m_B < \frac{1}{2}$, $Q\left(\frac{1}{2}\right) < 0$ and, as $Q(1) > 0$, we have $x^* > \frac{1}{2}$.

On the boundary when only B is present, we may also have equilibria where $z \neq x$ and (S1.10) is satisfied. These are *asymmetric equilibria*, and the following result holds:

Result 2. *On the boundary where only the B allele is present, there are two asymmetric equilibria $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ and $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$, corresponding to the fixations of either AB or aB,*

$$\hat{\mathbf{x}} = \hat{\mathbf{y}} = (1, 0, 0, 0), \qquad \tilde{\mathbf{x}} = \tilde{\mathbf{y}} = (0, 0, 1, 0). \tag{S1.19}$$

Both equilibria are internally unstable.

Proof of Result 2. If an asymmetric equilibrium with $z \neq x$ exists, then from (S1.10), we have

$$1 + sz = \frac{1 + s}{1 + sx}, \tag{S1.20}$$

and therefore

$$z = \frac{s(1 - s)}{s(1 + sx)} \qquad 1 - z = \frac{s(1 + s)x}{s(1 + sx)}. \tag{S1.21}$$

Substituting these relations into the equilibrium equations resulting from (S1.7), x satisfies the equation

$$T(x) = (1 - m)s^2 x(1 - x) = 0. \tag{S1.22}$$

Thus, $x = 0$ (and $z = 1$) or $x = 1$ (and $z = 0$), giving in the two fixations in either AB or aB. We check the internal stability of the fixation in AB, for example. "Near" this

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fixation, $w = 1 - x$ and z are small, and up to non-linear terms in w and z, their change in frequency is

$$w' = \frac{1 - m_B}{1 + s} w + m_B(1 + s)z$$
$$z' = \frac{m_B}{1 + s} w + (1 - m_B)(1 + s)z. \tag{S1.23}$$

The characteristic polynomial $P(\lambda)$ associated with the transformation (S1.23) is

$$P(\lambda) = \lambda^2 - (1 - m_B)\left[(1 + s) + \frac{1}{1 + s}\right]\lambda + (1 - m_B), \tag{S1.24}$$

with

$$P(1) = -\frac{s^2(1 - m_B)}{1 + s} < 0, \qquad P(+\infty) > 0. \tag{S1.25}$$

Thus, $P(\lambda)$ has a root larger than 1, and this fixation in AB is internally unstable. Similarly, the fixation in aB is also internally unstable.

External stability of the symmetric equilibrium

Results 1 and **2** entail that, on the boundary where only the modifier allele B is present, the unique symmetric equilibrium $(\mathbf{x}^*, \mathbf{y}^*)$ is the only internally stable equilibrium. It is therefore interesting to explore when $(\mathbf{x}^*, \mathbf{y}^*)$ is externally stable to the introduction of a modifier allele b determining the migration rate m_b in the population.

The external local stability of the symmetric equilibrium $(\mathbf{x}^*, \mathbf{y}^*)$ is determined by the linear approximation $\mathbf{L}^*$, to the transformation (S1.2) near $(\mathbf{x}^*, \mathbf{y}^*)$. Up to non-linear terms, we have

$$\begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \end{pmatrix} = \begin{pmatrix} \mathbf{x}^* \\ \mathbf{y}^* \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon}' \\ \boldsymbol{\delta}' \end{pmatrix} = \begin{pmatrix} \mathbf{x}^* \\ \mathbf{y}^* \end{pmatrix} + \mathbf{L}^* \begin{pmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\delta} \end{pmatrix}. \tag{S1.26}$$

Here $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and $\boldsymbol{\delta} = (\delta_1, \delta_2, \delta_3, \delta_4)$ are such that ε_i and δ_i are small in magnitude, for $i = 1, 2, 3, 4$, $\sum_i \varepsilon_i = \sum_i \delta_i = 0$. Here, $\boldsymbol{\varepsilon}' = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$ and $\boldsymbol{\delta}' = (\delta_1, \delta_2, \delta_3, \delta_4)$ have the same properties as $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}$, respectively. As $x_i^*, y_i^* > 0$ for $i = 1, 3$ and $x_i^* = y_i^* = 0$ for $i = 2, 4$, $\mathbf{L}^*$ has the following block structure

$$
\mathbf{L}^* = \begin{pmatrix}
\varepsilon_1 & \varepsilon_3 & \delta_1 & \delta_3 & \varepsilon_2 & \varepsilon_4 & \delta_2 & \delta_4 \\
 & & & & & & & \\
 & & \mathbf{L}_{\text{in}} & & & * & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & \mathbf{0} & & & \mathbf{L}_{\text{ex}} & & \\
 & & & & & & &
\end{pmatrix}
\begin{matrix}
\varepsilon_1 \\ \varepsilon_3 \\ \delta_1 \\ \delta_3 \\ \varepsilon_2 \\ \varepsilon_4 \\ \delta_2 \\ \delta_4
\end{matrix}
\qquad (S1.27)
$$

Here $\mathbf{0}$ is the 4×4 zero matrix and $*$ is a 4×4 matrix that does not affect the eigenvalues of $\mathbf{L}^*$, which are those of $\mathbf{L}_{\text{in}}$ and $\mathbf{L}_{\text{ex}}$. $\mathbf{L}_{\text{in}}$ determines the internal stability of $(\mathbf{x}^*, \mathbf{y}^*)$ on the boundary where only the genotypes AB and aB are present. As $(\mathbf{x}^*, \mathbf{y}^*)$ is internally stable, all eigenvalues of $\mathbf{L}_{\text{in}}$ are less than 1 in magnitude. $\mathbf{L}_{\text{ex}}$ corresponds to the linear approximation of the change in genotype frequencies for the genotypes Ab and ab "near" $(\mathbf{x}^*, \mathbf{y}^*)$. $\mathbf{L}_{\text{ex}}$ thus determines the external stability of $(\mathbf{x}^*, \mathbf{y}^*)$.

In what follows, we characterize $\mathbf{L}_{\text{ex}}$ and its largest eigenvalue. The relative relation of the largest eigenvalue to 1 gives information about the stability of the equilibrium.

A straightforward computation shows that

$$
\mathbf{L}_{\text{ex}} = \begin{pmatrix}
(1 - m_b)A^* & (1 - m_b)B^* & m_b C^* & m_b D^* \\
(1 - m_b)D^* & (1 - m_b)C^* & m_b B^* & m_b A^* \\
m_b A^* & m_b B^* & (1 - m_b)C^* & (1 - m_b)D^* \\
m_b D^* & m_b C^* & (1 - m_b)B^* & (1 - m_b)A^*
\end{pmatrix},
\qquad (S1.28)
$$

where

$$
\begin{aligned}
(1 + sx^*)A^* &= (1 + s)\left[1 - r\left(1 - x^*\right)\right] \\
(1 + sx^*)B^* &= (1 + s)rx^* \\
(1 + sx^*)C^* &= 1 - rx^* \\
(1 + sx^*)D^* &= r(1 - x^*).
\end{aligned}
\qquad (S1.29)
$$

Since $0 < x^* < 1$ and $0 < r < 1$, $0 < m_b < 1$, $\mathbf{L}_{\text{ex}}$ is a positive matrix, and by the Perron-Frobenius theorem, its largest eigenvalue is positive. Let $S(\lambda) = \det(\mathbf{L}_{\text{ex}} - \lambda \mathbf{I})$, where $\mathbf{I}$ is the 4×4 identity matrix, be the characteristic polynomial of $\mathbf{L}_{\text{ex}}$. Due to the

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structure (S1.28) of $\mathbf{L}_{\text{ex}}$ (also see **Supplementary Material 1 − Appendix A**), $S(\lambda)$ factors into the product of two quadratic polynomials $S_1(\lambda)$ and $S_2(\lambda)$:

$$S(\lambda) = S_1(\lambda) \cdot S_2(\lambda), \tag{S1.30}$$

where

$$S_1(\lambda) = \lambda^2 - \lambda\left[(1 - m_b)(A^* + C^*) + m_b(B^* + D^*)\right] + (1 - 2m_b)(A^*C^* - B^*D^*)$$

$$S_2(\lambda) = \lambda^2 - \lambda\left[(1 - m_b)(A^* + C^*) - m_b(B^* + D^*)\right] + (1 - 2m_b)(A^*C^* - B^*D^*). \tag{S1.31}$$

The discriminant of $S_1(\lambda) = 0$ is

$$\left[(1 - m_b)(A^* - C^*) + m_b(B^* - D^*)\right]^2 + 4m_b(1 - m_b)(A^*D^* + B^*C^*)$$
$$+ 4(1 - m_b)^2 B^*D^* + 4m_b^*A^*C^*, \tag{S1.32}$$

which is positive since A^*, B^*, C^*, D^* are all positive and $0 < m_b < 1$. In addition,

$$(1 + sx^*)^2(A^*C^* - B^*D^*) = (1 + s)(1 - r). \tag{S1.33}$$

Thus, when $0 < m_b < \frac{1}{2}$ and $0 < r < 1$, the two roots of $S_1(\lambda) = 0$ are positive, and they are both less than one if and only if $S_1(\lambda) > 0$ and $S_1'(1) > 0$.

It is not obvious that the roots of $S_2(\lambda) = 0$ are real. However, as the largest eigenvalue in magnitude of $\mathbf{L}_{\text{ex}}^*$ is positive, in order for it to be less than 1, it is sufficient that both positive roots of $S_2(\lambda)$ are less than 1. This is the case if $S_1'(1) > 0$ and $S_2'(1) > 0$. As $0 < m_b < 1$ and A^*, B^*, C^*, D^* are all positive, we have

$$S_2(1) > S_1(1), \qquad S_2'(1) > S_1'(1), \tag{S1.34}$$

so if $S_1(1) > 0$ and $S_1'(1) > 0$, it is also true that $S_2(1) > 0$ and $S_2'(1) > 0$. As $(\mathbf{x}^*, \mathbf{y}^*)$ is internally stable, from **Result 1** we know that $(1 + sx^*)^2 > (1 + s)$, and so, by (S1.33), the constant term of both $S_1(\lambda)$ and $S_2(\lambda)$ satisfies

$$0 < (1 - 2m_b)(A^*C^* - B^*D^*) < 1, \tag{S1.35}$$

provided $0 < r < 1$ and $0 < m_b < \frac{1}{2}$. It is therefore impossible that the two positive roots of $S_1(\lambda) = 0$ are both larger than 1. Hence, they are both less than 1 if and only if $S_1(1) > 0$, in which case also $S_1'(1) > 0$.

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To sum up, $(\mathbf{x}^*, \mathbf{y}^*)$ is externally stable if and only if $S_1(1) > 0$. Computing $S_1(1)$, we find that $S_1(1) > 0$ if and only if

$$\left(m_b - m_B \left(1 + sx^*\right)^{-2}\right)(1-r)s\left[x^*(s+2) - 1\right] > 0. \tag{S1.36}$$

As $x^* > \frac{1}{s+2}$, $0 < r < 1$ and $s > 0$, (S1.36) holds if and only if $m_b > m_B$. We have proved

Result 3. *The internally unique stable symmetric equilibrium is externally stable towards the introduction of a modifier allele b provided its associated migration rate m_b is larger than m_B.*

This result is in complete accordance with the reduction principle of Feldman and Liberman (1986) for modifiers of migration rates. It implies that the uninvadable stable migration rate is zero. We now study the case when selection pressures are no longer constant, but fluctuate in time.

PERIODICALLY FLUCTUATING SELECTION

Suppose now that, within each deme, the selection regime varies temporally. We assume two possible types of selection regimes. Under T_1 and T_2. In T_1, the fitness coefficients are assumed to be

Deme	E_x		E_y	
Allele	A	a	A	a
Fitness	$1 + s$	1	1	$1 + s$,

$$\tag{S1.37}$$

while in T_2 they are

Deme	E_x		E_y	
Allele	A	a	A	a
Fitness	1	$1 + s$	$1 + s$	1.

$$\tag{S1.38}$$

For simplicity, we assume that within each deme, phenotype A is favored in one temporal state and phenotype a is favored in the other. The environments in which allele A does better are T_1 in deme E_x and T_2 in deme E_y. The selection regimes fluctuate between T_1 and T_2. We first assume periodically fluctuating selection and explore the

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evolution of the migration rate, as a function of the length of the period of the fluctuating selection, which we also refer to as the environmental period. In the general case, an environmental cycle consists of $\tau_1 + \tau_2$ selection steps, the first τ_1 of type T_1 followed by τ_2 of type T_2 selection. We are able to derive closed-form solutions and fully explore the dynamics of the system in the case $\tau_1 = \tau_2 = 1$, i.e. the environment changes every generation. For analytical tractability, we further need to assume that all selection coefficients are equal.

Let $\overline{T}$ be transformation of population state given in (S1.2) with associated fitness parameters of T_1, and let $\widetilde{T}$ be the transformation associated with T_2 selection. The transformation of the population state after the $(\tau_1 + \tau_2)$ phases of selection is given by

$$\begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \end{pmatrix} = \underbrace{\widetilde{T} \circ \widetilde{T} \circ \cdots \circ \widetilde{T}}_{\tau_2\text{-times}} \circ \underbrace{\overline{T} \circ \overline{T} \circ \cdots \circ \overline{T}}_{\tau_1\text{-times}} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}. \tag{S1.39}$$

We start the analysis with the case $\tau_1 = \tau_2 = 1$, namely

$$\begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \end{pmatrix} = \widetilde{T} \circ \overline{T} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, \tag{S1.40}$$

which can be written as a two-phase transformation,

$$\begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \end{pmatrix} = \widetilde{T} \begin{pmatrix} \widetilde{\mathbf{x}} \\ \widetilde{\mathbf{y}} \end{pmatrix}, \qquad \begin{pmatrix} \widetilde{\mathbf{x}} \\ \widetilde{\mathbf{y}} \end{pmatrix} = \overline{T} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}. \tag{S1.41}$$

Boundary equilibria with symmetric selection

Assume that, initially, only B is present at the modifier locus, and let x be the frequency of AB in deme E_x and z the frequency of aB in deme E_y. Then $\mathbf{x} = (x, 0, 1 - x, 0)$, $\mathbf{y} = (1 - z, 0, z, 0)$, and using (S1.2), we obtain

$$\begin{aligned} \tilde{x} &= (1 - m_B)\frac{(1+s)x}{1+sx} + m_B \frac{1-z}{1+sz} \\ \tilde{z} &= (1 - m_B)\frac{(1+s)z}{1+sz} + m_B \frac{1-x}{1+sx}, \end{aligned} \tag{S1.42}$$

and

$$\begin{aligned} x' &= (1 - m_B)\frac{\tilde{x}}{1+s(1-\tilde{x})} + m_B \frac{(1+s)(1-\tilde{z})}{1+s(1-\tilde{z})} \\ z' &= (1 - m_B)\frac{\tilde{z}}{1+s(1-\tilde{z})} + m_B \frac{(1+s)(1-\tilde{x})}{1+s(1-\tilde{x})}. \end{aligned} \tag{S1.43}$$

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Then,

$$(\tilde{z} - \tilde{x}) = \frac{1+s}{(1+sx)(1+sz)}(z-x)$$

$$(z' - x') = \frac{1+s}{[1+s(1-\tilde{x})][1+s(1-\tilde{z})]}(\tilde{z}-\tilde{x}). \tag{S1.44}$$

Hence,

$$(z' - x') = \frac{(1+s)^2}{(1+sx)(1+sz)\left[1+s(1-\tilde{x})\right]\left[1+s(1-\tilde{z})\right]}(z-x). \tag{S1.45}$$

Thus, at equilibrium, we either have $z = x$ or

$$\frac{(1+s)^2}{(1+sx)(1+sz)\left[1+s(1-\tilde{x})\right]\left[1+s(1-\tilde{z})\right]} = 1. \tag{S1.46}$$

When $z = x$, we have a *symmetric equilibrium* characterized by the following result:

Result 4.

1. *On the boundary where only the modifier allele B is present, if $0 < m_B < 1$ and $s > 0$, a unique symmetric equilibrium $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ exists with*

$$\bar{\mathbf{x}} = (\bar{x}, 0, 1 - \bar{x}, 0), \qquad \bar{\mathbf{y}} = (1 - \bar{x}, 0, \bar{x}, 0). \tag{S1.47}$$

   Here, $\bar{x}$ is the unique positive root of $R(x) = 0$, where

$$R(x) = sx^2 + [2 - m_B(s+2)]x - (1 - m_B). \tag{S1.48}$$

2. *When $0 < m_B < 1$, $\bar{x} > \frac{1}{s+2}$. If, in addition, $0 < m_B < \frac{1}{2}$, then $0 < \bar{x} < \frac{1}{2}$.*

3. *$(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is internally stable on this boundary.*

Proof of Result 4.

1. At a symmetric equilibrium, $z = x$ and therefore $\tilde{z} = \tilde{x}$. From (S1.42) and (S1.43)

$$\tilde{x} = \frac{[(1 - m_B)(s+1) - m_B]x + m_B}{1 + sx} \tag{S49}$$

$$x = \frac{[(1 - m_B) - m_B(s+1)]\tilde{x} + m_B(1+s)}{1 + s(1 - \tilde{x})}. \tag{S50}$$

   Substituting (S1.49) into (S1.50) gives

$$(s+2)m_B\left\{sx^2 + [2 - m_B(s+2)]x - (1 - m_B)\right\} = 0. \tag{S1.51}$$

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As $m_B > 0$ and $s > 0$, x satisfies the quadratic equation $R(0) = 0$. As $R(\pm\infty) > 0$, $R(x) = 0$ has two real roots, one positive and one negative. But, as $0 < m_B < 1$,

$$R(1) = (s+1)(1 - m_B) > 0, \qquad (S1.52)$$

and thus $R(x) = 0$ has a unique root $\bar{x}$ with $0 < \bar{x} < 1$.

2. Observe that

$$R\left(\frac{1}{s+2}\right) = -\frac{s(s+1)}{(s+2)^2}, \qquad R\left(\frac{1}{2}\right) = \frac{s}{4}(1 - 2m_B). \qquad (S1.53)$$

As $R(0) < 0$ when $0 < m_B < 1$ and $R(1) > 0$, when $0 < m_B < 1$, we have $\bar{x} > \frac{1}{s+2}$. If, in addition, $0 < m_B < \frac{1}{2}$, we have $0 < \bar{x} < \frac{1}{2}$.

3. In view of (S1.45), the symmetric equilibrium $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is internally stable provided

$$\frac{(1+s)^2}{(1+s\bar{x})^2 \left[1 + s\left(1 - \tilde{x}\right)\right]^2} < 1. \qquad (S1.54)$$

Using (S1.49),

$$(1 + s\bar{x})[1 + s(1 - \tilde{x})] = (1 + s) + sm_B[(s+2)\bar{x} - 1]. \qquad (S1.55)$$

When $0 < m_B < 1$, $\bar{x} > \frac{1}{s+2}$ and therefore $(1 + s\bar{x})[1 + s(1 - \tilde{x})] > 1 + s$, and thus (S1.54) is satisfied.

As in the constant environment case, it is possible that asymmetric equilibria exist with $z \neq x$ and (S1.46) is satisfied. For example, the two fixations in AB or in aB, in both demes, are such equilibria. They correspond to $x = 1$, $z = 0$ or $x = 0$, $z = 1$, respectively. We can show that both fixations are internally unstable. Our computer simulations suggest that there are no stable asymmetric equilibria.

External stability of the symmetric equilibrium

We next check the external stability of the internally stable symmetric equilibrium $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ to the introduction of b at the modifier locus, changing the migration rate from m_B to m_b. As the transformation of the population state is $\widetilde{\mathbf{T}} \circ \overline{\mathbf{T}}$, following the same type

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of analysis as in the constant environment case, the matrix $\mathbf{L}_{\mathrm{ex}}$ determining the external stability of $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is given by

$$\mathbf{L}_{\mathrm{ex}} = \widetilde{\mathbf{L}}_{\mathrm{ex}} \circ \overline{\mathbf{L}}_{\mathrm{ex}}, \qquad (S1.56)$$

where, as in (S1.28),

$$\overline{\mathbf{L}}_{\mathrm{ex}} = \begin{pmatrix} (1-m_b)\overline{A} & (1-m_b)\overline{B} & m_b\overline{C} & m_b\overline{D} \\ (1-m_b)\overline{D} & (1-m_b)\overline{C} & m_b\overline{B} & m_b\overline{A} \\ m_b\overline{A} & m_b\overline{B} & (1-m_b)\overline{C} & (1-m_b)\overline{D} \\ m_b\overline{D} & m_b\overline{C} & (1-m_b)\overline{B} & (1-m_b)\overline{A} \end{pmatrix}, \qquad (S1.57)$$

$$\widetilde{\mathbf{L}}_{\mathrm{ex}} = \begin{pmatrix} (1-m_b)\widetilde{A} & (1-m_b)\widetilde{B} & m_b\widetilde{C} & m_b\widetilde{D} \\ (1-m_b)\widetilde{D} & (1-m_b)\widetilde{C} & m_b\widetilde{B} & m_b\widetilde{A} \\ m_b\widetilde{A} & m_b\widetilde{B} & (1-m_b)\widetilde{C} & (1-m_b)\widetilde{D} \\ m_b\widetilde{D} & m_b\widetilde{C} & (1-m_b)\widetilde{B} & (1-m_b)\widetilde{A} \end{pmatrix}. \qquad (S1.58)$$

As in (S1.29) we have

$$(1+s\bar{x})\overline{A} = (1+s)\left[1 - r\left(1 - \bar{x}\right)\right]$$
$$(1+s\bar{x})\overline{B} = (1+s)r\bar{x}$$
$$(1+s\bar{x})\overline{C} = 1 - r\bar{x} \qquad (S1.59)$$
$$(1+s\bar{x})\overline{D} = r(1-\bar{x}),$$

$$(1+s\tilde{x})\widetilde{A} = (1+s)\left[1 - r\left(1 - \tilde{x}\right)\right]$$
$$(1+s\tilde{x})\widetilde{B} = (1+s)r\tilde{x}$$
$$(1+s\tilde{x})\widetilde{C} = 1 - r\tilde{x} \qquad (S1.60)$$
$$(1+s\tilde{x})\widetilde{D} = r(1-\tilde{x}).$$

The relation between the equilibrium point $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ and $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is such that

$$(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \xrightarrow{\overline{\mathbf{T}}} (\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \xrightarrow{\widetilde{\mathbf{T}}} (\bar{\mathbf{x}}, \bar{\mathbf{y}}) . \qquad (S1.61)$$

We can write

$$\mathbf{L}_{\mathrm{ex}} = \begin{pmatrix} a & e & h & d \\ b & f & g & c \\ c & g & f & b \\ d & h & e & a \end{pmatrix}, \qquad (S1.62)$$

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where

$$a = (1 - m_b)^2 \, \widetilde{A}\overline{A} + (1 - m_b)^2 \, \widetilde{B}\overline{D} + m_b^2 \widetilde{C}\overline{A} + m_b^2 \widetilde{D}\overline{D}$$

$$b = (1 - m_b)^2 \, \widetilde{D}\overline{A} + (1 - m_b)^2 \, \widetilde{C}\overline{D} + m_b^2 \widetilde{B}\overline{A} + m_b^2 \widetilde{A}\overline{D}$$

$$c = m_b(1 - m_b) \left[\widetilde{A}\overline{A} + \widetilde{B}\overline{D} + \widetilde{C}\overline{A} + \widetilde{D}\overline{D} \right]$$

$$d = m_b(1 - m_b) \left[\widetilde{D}\overline{A} + \widetilde{C}\overline{D} + \widetilde{B}\overline{A} + \widetilde{A}\overline{D} \right]$$

$$e = (1 - m_b)^2 \, \widetilde{A}\overline{B} + (1 - m_b)^2 \, \widetilde{B}\overline{C} + m_b^2 \widetilde{C}\overline{B} + m_b^2 \widetilde{D}\overline{C} \qquad (S1.63)$$

$$f = (1 - m_b)^2 \, \widetilde{D}\overline{B} + (1 - m_b)^2 \, \widetilde{C}\overline{C} + m_b^2 \widetilde{B}\overline{B} + m_b^2 \widetilde{A}\overline{C}$$

$$g = m_b(1 - m_b) \left[\widetilde{A}\overline{B} + \widetilde{B}\overline{C} + \widetilde{C}\overline{B} + \widetilde{D}\overline{C} \right]$$

$$h = m_b(1 - m_b) \left[\widetilde{D}\overline{B} + \widetilde{C}\overline{C} + \widetilde{B}\overline{B} + \widetilde{A}\overline{C} \right].$$

From (S1.57) and (S1.58), the characteristic polynomial $D(\lambda) = \det \left(\mathbf{L}_{\text{ex}} - \lambda \mathbf{I} \right)$ of $\mathbf{L}_{\text{ex}}$, factors into the product of two quadratic polynomials: $D(\lambda) = D_1(\lambda)D_2(\lambda)$. In fact,

$$D(\lambda) = \begin{pmatrix} a + d - \lambda & e + h & 0 & 0 \\ b + c & f + g - \lambda & 0 & 0 \\ e & g & f - g - \lambda & b - c \\ d & h & e - h & a - d - \lambda \end{pmatrix}, \qquad (S1.64)$$

with

$$D_1(\lambda) = \lambda^2 - (a + d + f + g)\lambda + (a + d)(f + g) - (b + c)(e + h)$$
$$D_2(\lambda) = \lambda^2 - (a - d + f - g)\lambda + (a - d)(f - g) - (b - c)(e - h). \qquad (S1.65)$$

Based on **Supplementary 1 − Appendix A**, the constant terms of $D_1(\lambda)$ and $D_2(\lambda)$ are both equal to

$$(1 - 2m_b)^2 \, (1 + s\bar{x})^{-2} \, [1 + s(1 - \tilde{x})]^2 \, (1 + s)^2 \, (1 - r)^2 \, . \qquad (S1.66)$$

These constant terms are positive for $0 < r < 1$ and $0 < m_b < \frac{1}{2}$ and are less than 1 due to the internal stability of $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$. Observe that the discriminant of $D_1(\lambda) = 0$ is

$$(a + d + f + g)^2 - 4(1a + d)(f + g) - (b + c)(e + h)] \qquad (S67)$$

which reduces to

$$[(a + d) - (f + g)]^2 + 4(b + c)(e + h) > 0, \qquad (S68)$$

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since a, b, c, d, e, f, g, h are all positive. Therefore $D_1(\lambda) = 0$ has two positive roots whose product is less than 1. These positive roots are less than 1 provided both $D_1(1)$ and $D_1'(1)$ are positive. As it is impossible that both roots are larger than 1 (their product is less than 1), we only need the condition $D_1(1) > 0$. Since the matrix $\mathbf{L}_{\mathrm{ex}}$ is positive, we know that its largest eigenvalue is positive. For the external stability of $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$, we also need to prove that when the roots of $D_2(\lambda) = 0$ are real and positive (their product is positive) they are both less than 1. This is the case when $D_2(1) > 0$ and $D_2'(1) > 0$. It is clear that

$$D_2(1) > D_1(1), \qquad D_2'(1) > D_1'(1). \tag{S1.69}$$

Therefore, when $D_1(1) > 0$ and $D_1'(1) > 0$, this also implies $D_2(1) > 0$ and $D_2'(1) > 0$. To sum up, $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is externally stable provided $D_1(1) > 0$ where

$$D_1(1) = 1 - (a + d + f + g) + (a + d)(f + g) - (b + c)(e + h), \tag{S1.70}$$

and $(a + d)(f + g) - (b + c)(e + h)$ is given in (S1.66).

Let us compute $(a + d + f + g)$. Following (S1.63),

$$
\begin{aligned}
(a + d + f + g) = & \left[(1 - m_b)^2 \, \widetilde{A} + m_b^2 \widetilde{C} + m_b(1 - m_b)(\widetilde{B} + \widetilde{D}) \right] \overline{A} \\
& + \left[(1 - m_b)^2 \, \widetilde{C} + m_b^2 \widetilde{A} + m_b(1 - m_b)(\widetilde{B} + \widetilde{D}) \right] \overline{C} \\
& + \left[(1 - m_b)^2 \, \widetilde{B} + m_b^2 \widetilde{D} + m_b(1 - m_b)(\widetilde{A} + \widetilde{C}) \right] \overline{D} \\
& + \left[(1 - m_b)^2 \, \widetilde{D} + m_b^2 \widetilde{B} + m_b(1 - m_b)(\widetilde{A} + \widetilde{C}) \right] \overline{B}
\end{aligned} \tag{S1.71}
$$

which can be rewritten as

$$
\begin{aligned}
(a + d + f + g) = & (1 - m_b)^2 \left[\widetilde{A}\overline{A} + \widetilde{B}\overline{D} + \widetilde{C}\overline{C} + \widetilde{D}\overline{B} \right] \\
& + m_b(1 - m_b) \left[(\widetilde{B} + \widetilde{D})(\overline{A} + \overline{C}) + (\widetilde{A} + \widetilde{C})(\overline{B} + \overline{D}) \right] \\
& + m_b^2 \left[\widetilde{A}\overline{C} + \widetilde{B}\overline{B} + \widetilde{C}\overline{A} + \widetilde{D}\overline{D} \right].
\end{aligned} \tag{S1.72}
$$

Thus,

$$
\begin{aligned}
(a + d + f + g) = & (1 - 2m_b) \left[\widetilde{A}\overline{A} + \widetilde{B}\overline{D} + \widetilde{C}\overline{C} + \widetilde{D}\overline{B} \right] \\
& + m_b \left[(\widetilde{B} + \widetilde{D})(\overline{A} + \overline{C}) + (\widetilde{A} + \widetilde{C})(\overline{B} + \overline{D}) \right] \\
& + m_b^2 (\widetilde{A} + \widetilde{C} - \widetilde{B} - \widetilde{D})(\overline{A} + \overline{C} + \overline{B} - \overline{D}).
\end{aligned} \tag{S1.73}
$$

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By (S1.59) and (S1.60),

$$(1+s\bar{x})\big[1+s(1-\tilde{x})\big]\big[\widetilde{A}\overline{A}+\widetilde{B}\overline{D}+\widetilde{C}\overline{C}+\widetilde{D}\overline{B}\big] =$$
$$= (1+s)\big[2(1-r)+r^2\big] + r^2 s(s+1)\bar{x} - r^2 s\tilde{x} - r^2 s^2 \bar{x}\tilde{x}, \tag{S1.74}$$

$$(1+s\bar{x})\big[1+s(1-\tilde{x})\big]\Big[\big(\widetilde{B}+\widetilde{D}\big)\big(\overline{A}+\overline{C}\big)+\big(\widetilde{A}+\widetilde{C}\big)\big(\overline{B}+\overline{C}\big)\Big] =$$
$$= 2r^2(1+s\bar{x})\big[1+s(1-\tilde{x})\big] + r(1-r)(s+2)\big[(s+2)+s(\bar{x}+\tilde{x})\big], \tag{S1.75}$$

$$(1+s\bar{x})\big[1+s(1-\tilde{x})\big]\big[\widetilde{A}+\widetilde{C}-\widetilde{B}-\widetilde{D}\big]\big[\overline{A}+\overline{C}-\overline{B}-\overline{D}\big] =$$
$$= (s+2)^2(1-r)^2. \tag{S1.76}$$

Using (S1.66), $D_1(1)$ is equal to

$$1 - (1-2m_b)\frac{(1+s)\big[2(1-r)+r^2\big]+r^2 s(s+1)\bar{x}-r^2 s\tilde{x}-r^2 s^2\bar{x}\tilde{x}}{(1+s\bar{x})[1+s(1-\tilde{x})]}$$
$$- m_b\frac{2r^2(1+s\bar{x})[1+s(1-\tilde{x})]+r(1-r)(s+2)[(s+2)+s(\bar{x}-\tilde{x})]}{(1+s\bar{x})[1+s(1-\tilde{x})]}$$
$$- m_b^2\frac{(s+2)^2(1-r)^2}{(1+s\bar{x})[1+s(1-\tilde{x})]}$$
$$+ \frac{(1-2m_b)^2(s+1)^2(1-r)^2}{(1+s\bar{x})^2[1+s(1-\tilde{x})]^2}. \tag{S1.77}$$

We next show that $D_1(1)$ can be represented as

$$D_1(1) = (1-r)s(m_B-m_b)\Delta(r), \tag{S1.78}$$

where $\Delta(r)$ is a linear function of r that is positive for all $0 \le r \le 1$. Therefore the sign of $D_1(1)$ coincides with that of (m_B-m_b), and $D_1(1) > 0$ when $m_B > m_b$. We thus have the following result:

Result 5. *The unique internally stable symmetric equilibrium $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is externally stable to the introduction of the allele b at the modifier locus provided $m_B > m_b$. Thus, higher migration rates are favored and the stable uninvadable migration rate is 1.*

Computation of $D_1(1)$.

Observe that

$$r^2[(1+s)+s(s+1)\bar{x}-s\tilde{x}-s^2\bar{x}\tilde{x}] = r^2(1+s\bar{x})[1+s(1-\tilde{x})], \tag{S1.79}$$

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so (S1.77) simplifies to

$$1 - r^2 - (1 - 2m_b)\frac{2(1+s)(1-r)}{(1+s\bar{x})[1+s(1-\tilde{x})]} - m_b\frac{r(1-r)(s+2)[(s+2)+s(\bar{x}-\tilde{x})]}{(1+s\bar{x})[1+s(1-\tilde{x})]}$$

$$- m_b^2\frac{(s+2)^2(1-r)^2}{(1+s\bar{x})[1+s(1-\tilde{x})]} + \frac{(1-2m_b)^2(s+1)^2(1-r)^2}{(1+s\bar{x})^2[1+s(1-\tilde{x})]^2}.$$

$$(S1.80)$$

Clearly $D_1(1)$ of (S1.80) has a factor of $(1-r)$, and in fact

$$D_1(1) = (1-r)f(r), \qquad (S1.81)$$

where $f(r)$ is a linear function of r, for $0 \le r \le 1$. Now

$$f(1) = 2 - (1-2m_b)\frac{2(1+s)}{(1+s\bar{x})[1+s(1-x)]} - m_b\frac{(s+2)[(s+2)+s(\bar{x}-\tilde{x})]}{(1+s\bar{x})[1+s(1-\tilde{x})]}. \qquad (S1.82)$$

Following (S1.55) we have

$$(1+s\bar{x})[1+s(1-\tilde{x})] = (1+s) + sm_B[(s+2)\bar{x}-1]. \qquad (S1.83)$$

We also have an equivalent expression for (S1.83) in terms of $\tilde{x}$, namely

$$(1+s\bar{x})[1+s(1-\tilde{x})] = (1+s) + sm_B[(s+1) - (s+2)\tilde{x}]. \qquad (S1.84)$$

Also, whereas $\bar{x} > \frac{1}{s+2}$, we have $\tilde{x} < \frac{s+1}{s+2}$. Applying all of this to (S1.82) and using the fact that

$$(1+s\bar{x})[1+s(1-\tilde{x})] = (1+s) + \frac{1}{2}sm_B[s + (s+2)(\bar{x}-\tilde{x})], \qquad (S1.85)$$

we get that

$$(1+s\bar{x})[1+s(1-\tilde{x})]f(1) = 2(s+1) + sm_B[s + (s+2)(\bar{x}-\tilde{x})]$$

$$- 2(1-2m_b)(s+1) - m_b(s+2)[(s+2)+s(\bar{x}-\tilde{x})]$$

$$= s^2m_B - m_b\left[(s+2)^2 - 4(s+1)\right] + s(s+2)(m_B-m_b)(\bar{x}-\tilde{x})$$

$$= s^2(m_B - m_b) + s(s+2)(m_B - m_b)(\bar{x}-\tilde{x}).$$

$$(S1.86)$$

Thus

$$(1+s\bar{x})[1+s(1-\tilde{x})]f(1) = s(m_B - m_b)[s + (s+2)(\bar{x}-\tilde{x})]. \qquad (S1.87)$$

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But as $(s+2)\bar{x} > 1$, $(s+2)\tilde{x} < (s+1)$,

$$s + (s+2)(\bar{x} - \tilde{x}) = [(s+2)\bar{x} - 1] + [(s+1) - (s+2)\tilde{x}] > 0. \qquad (S1.88)$$

It follows that the sign of $f(1)$ is the same as the sign of $(m_B - m_b)$.

We now compute $f(0)$:

$$f(0) = 1 - (1 - 2m_b)\frac{2(1+s)}{(1+s\bar{x})[1 + s(1 - \tilde{x})]} - m_b^2 \frac{(s+2)^2}{(1+s\bar{x})[1 + s(1 - \tilde{x})]} \qquad (S1.89)$$
$$+ \frac{(1 - 2m_b)^2 (s+1)^2}{(1+s\bar{x})^2 [1 + s(1 - \tilde{x})]^2}.$$

Using the expression (S1.83) for the product of the two mean fitnesses, we get

$$(1 + s\bar{x})^2 [1 + s(1 - \tilde{x})]^2 f(0) = \{(1+s) + sm_B [(s+2)\bar{x} - 1]\}^2$$
$$- 2(1 - 2m_b)(s+1)\{(1+s) + sm_B [(s+2)\bar{x} - 1]\}$$
$$- m_b^2(s+2)\{(1+s) + sm_B [(s+2)\bar{x} - 1]\}$$
$$+ (1 - 2m_b)^2 (s+1)^2. \qquad (S1.90)$$

In (S1.90) we replace the $\bar{x}^2$ term using the equilibrium equation (S1.48) to give

$$(1 + s\bar{x})^2 [1 + s(1 - \tilde{x})]^2 f(0) = (m_B - m_b)s\Big\{m_b s(s+1)$$
$$- m_B^2 (s+2)^2 + m_B \left[(s+1)(s+4) - m_b (s+2)^2\right]$$
$$+ m_B(s+2)\bar{x} \left[m_B (s+2)^2 + m_b (s+2)^2 - 4(s+1)\right]\Big\}. \qquad (S1.91)$$

The right-hand side of (S1.91) is $(m_B - m_b)s$ multiplied by

$$m_b s(s+1) + m_B (s+2)^2 (m_B + m_b)[\bar{x}(s+2) - 1] + m_B(s+1)[(s+4) - 4\bar{x}(s+2)]. \qquad (S1.92)$$

We will show that (S1.92) is always positive. In fact, (S1.92) is equal to

$$m_b s(s+1) + m_B \cdot m_b (s+2)^2 \left[\bar{x}(s+2) - 1\right] + m_B^2 (s+2)^2 \left[\bar{x}(s+2) - 1\right] \qquad (S1.93)$$
$$+ m_B(s+1)\big[(s+4) - 4\bar{x}(s+2)\big].$$

From the equilibrium equation (S1.48) we get that

$$m_B\big[(s+2)\bar{x} - 1\big] = s\bar{x}^2 + 2\bar{x} - 1. \qquad (S1.94)$$

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Hence (S1.93) is equal to

$$m_b s(s+1) + m_B \cdot m_b (s+2)^2 \left[\bar{x}(s+2) - 1\right] + $$
$$+ m_B (s+2)^2 \left[s\bar{x}^2 + 2\bar{x} - 1\right] + m_B(s+1)\left[(s+4) - 4\bar{x}(s+2)\right]. \tag{S1.95}$$

The last two terms have a factor m_B that multiplies

$$(s+1)(s+4) - (s+2)^2 + (s+2)^2 \bar{x}(2 + s\bar{x}) - 4\bar{x}(s+1)(s+2) = $$
$$= s + (s+2)\bar{x}\left[(s+2)(2 + s\bar{x}) - 4(s+1)\right]$$
$$= s + (s+2)\bar{x}\left[(s+2)s\bar{x} - 2s\right] \tag{S1.96}$$
$$= s\left[(s+2)^2 \bar{x}^2 - 2(s+2)\bar{x} + 1\right] = s\left[(s+2)\bar{x} - 1\right]^2,$$

which is positive. To sum up, $f(0)$ also has the same sign of $(m_B - m_b)$, and so

$$D_1(1) = (1-r)s(m_B - m_b)\Delta(r), \tag{S1.97}$$

where $\Delta(r)$ is a linear function of r that is positive for all $0 \leq r \leq 1$.

We saw that in a temporally constant environment the reduction principle holds and smaller migration rates are always favored. On the other hand, with fluctuating selection of period 2, non-zero migration rates evolve and the reverse of the reduction principle holds: a higher migration rate can always invade in a population fixed on a smaller resident one and the evolutionarily stable migration rate is 1. An important result is that this evolutionarily stable switching rate maximizes mean fitness at equilibrium. This result is deeply rooted in classic population genetics theory and is connected to early work on neutral modifiers and the mean fitness principle of Karlin and McGregor (1972, 1974).

The mean fitness and external stability.

We saw that in a constant environment the symmetric equilibrium is externally stable if $m_b > m_B$ and unstable if $m_b < m_B$, so that smaller migration rates are favored. On the other hand, with fluctuating selection of period 2, non-zero migration rates evolve; the symmetric equilibrium is stable if $m_B > m_b$ and unstable if $m_B < m_b$. This phenomenon is related to the behavior of the mean fitness at equilibrium in the following way.

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Result 6. *The mean fitness at the symmetric equilibrium is*

    i. a decreasing function of m_B in a constant environment,

    ii. an increasing function of m_B in a period 2 cycling environment.

Proof of Result 6.

    i. In a constant environment, the mean fitness w^* at the symmetric equilibrium $(\bar{\mathbf{x}}^*, \bar{\mathbf{y}}^*)$ is $w^* = 1 + sx^*$, and it is a decreasing function of m_B if $\frac{\partial x^*}{\partial m_B}$ is negative. Using the equilibrium equation,

$$\frac{\partial x^*}{\partial m_B} = \frac{1 - (s+2)x^*}{2sx^* + \big[(s+2)m_B - s\big]}. \tag{S1.98}$$

As $x^* > \frac{1}{s+2}$, in order for $\frac{\partial x^*}{\partial m_B}$ to be negative, it is sufficient that

$$x^* > \frac{s - m_B(s+2)}{2s}. \tag{S1.99}$$

This follows easily from the fact that $Q(x)$ in (S1.13) satisfies $Q(0) < 0$, $Q(x^*) = 0$, and $Q\left(\frac{s - m_B(s+2)}{2s}\right) < 0$.

    ii. With a fitness cycle of period 2, the mean fitness $\bar{w}$ at the symmetric equilibrium $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is

$$\bar{w} = (1 + s) + sm_B\big[(s+2)\bar{x} - 1\big]. \tag{S1.100}$$

$\bar{w}$ is an increasing function of m_B if $\frac{\partial \bar{w}}{\partial \mu_B} > 0$. Now

$$\frac{\partial \bar{w}}{\partial m_B} = s\big[(s+2)\bar{x} - 1\big] + s(s+2)m_B\frac{\partial \bar{x}}{\partial m_B}. \tag{S1.101}$$

Thus $\frac{\partial \bar{w}}{\partial m_B} > 0$ provided $\frac{\partial \bar{x}}{\partial m_B} > 0$. Using the equilibrium equation $R(x) = 0$ for $\bar{x}$, we have

$$\frac{\partial \bar{x}}{\partial m_B} = \frac{\bar{x}(s+1) - 1}{2s\bar{x} + \big[2 - m_B(s+2)\big]}. \tag{S1.102}$$

Since $\big[\bar{x}(s+1) - 1\big] > 0$, we conclude that $\frac{\partial \bar{x}}{\partial m_B} > 0$ if

$$\bar{x} > \frac{m_B(s+2) - 2}{2s}, \tag{S1.103}$$

which follows from $R(0) < 0$, $R(\bar{x}) = 0$, and $R\left(\frac{m_B(s+2)-2}{2s}\right) < 0$.

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The family of the 4×4 matrices of the form

$$
\mathbf{L} = \begin{pmatrix} (1-m)A & (1-m)B & mC & mD \\ (1-m)D & (1-m)C & mB & mA \\ mA & mB & (1-m)C & (1-m)D \\ mD & mC & (1-m)B & (1-m)A \end{pmatrix}, \qquad \text{(S1.A1)}
$$

has the following properties:

1. If $\mathbf{P}$ is the 4×4 matrix

$$
\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{(S1.A2)}
$$

then

$$
\mathbf{PLP}^{-1} = \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ * & \mathbf{N} \end{pmatrix}, \qquad \text{(S1.A3)}
$$

where $\mathbf{0}$ is the 2×2 zero matrix and

$$
\begin{aligned}
\mathbf{M} &= \begin{pmatrix} (1-m)A + mD & (1-m)B + mC \\ mA + (1-m)D & mB + (1-m)C \end{pmatrix} \\
\mathbf{N} &= \begin{pmatrix} (1-m)C - mB & (1-m)D - mA \\ (1-m)B - mC & (1-m)A - mD \end{pmatrix}.
\end{aligned} \qquad \text{(S1.A4)}
$$

Hence $\mathbf{L}$ is similar, via $\mathbf{P}$, to the block matrix (S1.A3).

2. The determinants of $\mathbf{M}$ and $\mathbf{N}$ are equal:

$$
\det(\mathbf{M}) = \det(\mathbf{N}) = (1 - 2m)(AC - BD). \qquad \text{(S1.A5)}
$$

3. The characteristic polynomial of $\mathbf{L}$ factors into the product of the characteristic polynomials of $\mathbf{M}$ and $\mathbf{N}$, namely

$$
\det (\mathbf{L} - \mathbf{I}_4) = \det(\mathbf{M} - \lambda \mathbf{I}_2) \cdot \det(\mathbf{N} - \lambda \mathbf{I}_2), \qquad \text{(S1.A6)}
$$

where $\mathbf{I}_i$ is the identity matrix of order i for $i = 2, 4$.

4. If $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_n$ are matrices of the form (S1.A1) with corresponding matrices $\mathbf{M}_i$ and $\mathbf{N}_i$ as in (S1.A3), for $i = 1, 2, \ldots, n$, then

$$\mathbf{P}\left(\prod_{i=1}^{n} \mathbf{L}_i\right)\mathbf{P}^{-1} = \begin{pmatrix} \prod_{i=1}^{n} \mathbf{M}_i & \mathbf{0} \\ * & \prod_{i=1}^{n} \mathbf{N}_i \end{pmatrix}. \tag{S1.A7}$$

Therefore $\prod_{i=1}^{n} \mathbf{L}_i$ is similar, via $\mathbf{P}$, to the block matrix (S1.A7).

5. The characteristic polynomial of $\prod_{i=1}^{n} \mathbf{L}_i$ factors into the characteristic polynomials of $\prod_{i=1}^{n} \mathbf{M}_i$ and $\prod_{i=1}^{n} \mathbf{N}_i$.

6. The matrix $\prod_{i=1}^{n} \mathbf{L}_i$ also has the form

$$\begin{pmatrix} a & e & h & d \\ b & f & g & c \\ c & g & f & b \\ d & h & e & a \end{pmatrix}, \tag{S1.A8}$$

and the corresponding matrices $\prod_{i=1}^{n} \mathbf{M}_i$ and $\prod_{i=1}^{n} \mathbf{N}_i$ have the form

$$\begin{pmatrix} a + d & e + h \\ b + c & f + g \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f - g & b - c \\ e - h & a - d \end{pmatrix}, \tag{S1.A9}$$

respectively.

7. If $D(\lambda)$, $D_1(\lambda)$, and $D_2(\lambda)$ are the characteristic polynomials of $\prod_{i=1}^{n} \mathbf{L}_i$, $\prod_{i=1}^{n} \mathbf{M}_i$, and $\prod_{i=1}^{n} \mathbf{N}_i$, respectively, then

$$D(\lambda) = D_1(\lambda) \cdot D_2(\lambda), \tag{S1.A10}$$

where

$$\begin{aligned} D_1(\lambda) &= \lambda^2 - (a + d + f + g)\lambda + (a + d)(f + g) - (b + c)(e + h) \\ D_2(\lambda) &= \lambda^2 - (a - d + f - g)\lambda + (a - d)(f - g) - (b - c)(e - h). \end{aligned} \tag{S1.A11}$$

Moreover, the two constant terms of $D_1(\lambda)$ and $D_2(\lambda)$ are both equal

$$(a + d)(f + g) - (b + c)(e + h) = (a - d)(f - g) - (b - c)(e - h), \tag{S1.A12}$$

and (S1.A12) is equal to

$$\prod_{i=1}^{n} \det(\mathbf{M}_i) = \prod_{i=1}^{n} \det(\mathbf{N}_i) \equiv (1 - 2m)^n \prod_{i=1}^{n} (A_i C_i - B_i D_i), \tag{S1.A13}$$

where A_i, B_i, C_i, D_i, are the parameters associated with $\mathbf{L}_i$, for $i = 1, 2, \ldots, n$, as in (S1.A1).

ON THE EVOLUTION OF RECOMBINATION IN CHANGING ENVIRONMENTS

Oana Carja, Uri Liberman, and Marcus W. Feldman

Consider an infinite sexual haploid population in which each individual is defined by three biallelic loci. The two major loci A/a and B/b control the phenotype and thus the fitness of the individual. The third locus M/m is a modifier locus that controls the recombination rate between the two major loci. This modifier locus is otherwise selectively neutral.

We study the evolution of the modifier locus M/m and determine the evolutionarily stable recombination rate as a function of the pattern of fluctuation in selection experienced by the population. As in general analyses of neutral modifiers (see Feldman and Liberman 1986), we frame the question in terms of the stability of the fixation equilibrium with only M present in the population, to invasion by an allele m, introduced in the population by a rare mutation. To that end, we track the frequencies of the eight genotypes MAB, MAb, MaB, Mab, mAB, mAb, maB, mab. At each generation, the population experiences random mating, recombination, and selection.

As the modifier locus is assumed to be selectively neutral, there are four fitness parameters determined by the two major loci

$$
\begin{array}{lcccc}
\text{major loci genotype} & AB & Ab & aB & ab \\
\text{fitness} & w_1 & w_2 & w_3 & w_4.
\end{array}
\qquad (S2.1)
$$

There are three possible recombination rates depending on the mating type at the modifier locus

$$
\begin{array}{lccc}
\text{modifier locus genotype} & MM & Mm & mm \\
\text{recombination rate} & r_1 & r_2 & r_3.
\end{array}
\qquad (S2.2)
$$

Suppose that the three loci are ordered with the modifier locus located on one side of the two major loci. Let R be the recombination rate between the modifier locus and the two major loci, and we assume no interference between recombination events occurring in the two intervals separating the two major loci and between the modifier locus and the nearest major locus.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_8)$ be the frequency vector of the eight genotypes in the present generation, and let $\mathbf{x}' = (x_1', x_2', \ldots, x_8')$ be the frequency vector of the next generation. The change in genotype frequencies from one generation to the next is given by

$$\begin{aligned}
wx_1' = w_1 \Big[& x_1 + R(x_2x_5 + x_3x_5 + x_4x_5 - x_1x_7 - x_1x_8 - x_1x_6) \\
& + r_1(x_2x_3 - x_1x_4) \\
& + r_2(x_2x_5 + x_2x_7 - x_1x_8 - x_1x_6) \\
& + Rr_2(2x_1x_6 - 2x_2x_5 + x_3x_6 + x_1x_8 - x_4x_5 - x_2x_7) \Big]
\end{aligned}$$

$$\begin{aligned}
wx_2' = w_2 \Big[& x_2 + R(x_1x_6 + x_3x_6 + x_4x_6 - x_2x_5 - x_2x_7 - x_2x_8) \\
& + r_1(x_1x_4 - x_2x_3) \\
& + r_2(x_1x_6 + x_1x_8 - x_2x_5 - x_2x_7) \\
& + Rr_2(2x_2x_5 - 2x_1x_6 + x_2x_7 + x_4x_5 - x_1x_8 - x_3x_6) \Big]
\end{aligned}$$

$$\begin{aligned}
wx_3' = w_3 \Big[& x_3 + R(x_1x_7 + x_2x_7 + x_4x_7 - x_3x_6 - x_3x_8 - x_3x_5) \\
& + r_1(x_1x_4 - x_2x_3) \\
& + r_2(x_4x_7 + x_4x_5 - x_3x_8 - x_3x_6) \\
& + Rr_2(x_1x_8 + x_3x_6 + 2x_3x_8 + x_4x_5 - x_2x_7 - 2x_4x_7) \Big]
\end{aligned}$$

$$\begin{aligned}
wx_4' = w_4 \Big[& x_4 + R(x_1x_8 + x_2x_8 + x_3x_8 - x_4x_5 - x_4x_6 - x_4x_7) \\
& + r_1(x_2x_3 - x_1x_4) \\
& + r_2(x_3x_6 + x_3x_8 - x_4x_5 - x_4x_7) \\
& + Rr_2(x_2x_7 + x_4x_5 - 2x_3x_8 - x_1x_8 - x_3x_6 + 2x_4x_7) \Big]
\end{aligned}$$

$$\begin{aligned}
wx_5' = w_1 \Big[& x_5 + R(x_1x_6 + x_1x_7 + x_1x_8 - x_2x_5 - x_3x_5 - x_4x_5) \\
& + r_2(x_1x_6 + x_3x_6 - x_2x_5 - x_4x_5) \\
& + r_3(x_6x_7 - x_5x_8) \\
& + Rr_2(2x_2x_5 - 2x_1x_6 + x_2x_7 + x_4x_5 - x_1x_8 - x_3x_6) \Big]
\end{aligned}$$

$$wx_6' = w_2 \Big[x_6 + R(x_2x_5 + x_2x_7 + x_2x_8 - x_1x_6 - x_3x_6 - x_4x_6)$$
$$+ r_2(x_2x_5 + x_4x_5 - x_1x_6 - x_3x_6)$$
$$+ r_3(x_5x_8 - x_6x_7)$$
$$+ Rr_2(2x_1x_6 - 2x_2x_5 + x_1x_8 + x_3x_6 - x_2x_7 - x_4x_5) \Big]$$

$$wx_7' = w_3 \Big[x_7 + R(x_3x_5 + x_3x_6 + x_3x_8 - x_1x_7 - x_2x_7 - x_4x_7)$$
$$+ r_2(x_1x_8 + x_3x_8 - x_2x_7 - x_4x_7)$$
$$+ r_3(x_5x_8 - x_6x_7)$$
$$+ Rr_2(2x_4x_7 - 2x_3x_8 + x_4x_5 + x_2x_7 - x_1x_8 - x_3x_6) \Big]$$

$$wx_8' = w_4 \Big[x_8 + R(x_4x_5 + x_4x_6 + x_4x_7 - x_1x_8 - x_2x_8 - x_3x_8)$$
$$+ r_2(x_2x_7 + x_4x_7 - x_1x_8 - x_3x_8)$$
$$+ r_3(x_6x_7 - x_5x_8)$$
$$+ Rr_2(x_1x_8 + x_3x_6 - x_2x_7 - x_4x_5 + 2x_3x_8 - 2x_4x_7) \Big].$$

$(S2.3)$

Here w is a normalizing factor such that $\sum_{i=1}^{8} x_i' = 1$.

We first present the case when selection is constant in time in both demes; this will serve as our reference model. We then consider regimes in which there is fluctuating selection through time. In particular, we ask how the rate of environmental volatility affects the stable recombination rate.

CONSTANT SELECTION

We are able to derive closed-form mathematical results if we assume the following symmetry relations among the fitness parameters:

$$w_1 = w_4, \qquad w_2 = w_3. \tag{S2.4}$$

Also, without loss of generality, we will assume that $w_1 > w_2$; that is

$$w_1 = w_4 = 1 + s, \qquad w_2 = w_3 = 1 \tag{S2.5}$$

with $s > 0$.

Equilibria with symmetric selection

Suppose that initially only allele M is present at the modifier locus. In this case the transformation (S2.3) reduces to

$$
\begin{aligned}
wx_1' &= w_1(x_1 - r_1 D) \\
wx_2' &= w_2(x_2 + r_1 D) \\
wx_3' &= w_3(x_3 + r_1 D) \\
wx_4' &= w_4(x_4 - r_1 D),
\end{aligned}
\tag{S2.6}
$$

as $x_i = x_i' = 0$ for $i = 5, 6, 7, 8$. Here

$$
\begin{aligned}
D &= x_1 x_4 - x_2 x_3 \\
w &= \sum_{i=1}^{n} w_i x_i - r_1 D(w_1 - w_2 - w_3 + w_4).
\end{aligned}
\tag{S2.7}
$$

We now look for equilibria of (S2.6) and check their internal stability. We have the following result:

Result 1. *On the boundary where only the M modifier allele is present, the possible equilibria are the four fixations and a symmetric equilibrium $\mathbf{x}^* = \left(x^*, \frac{1}{2} - x^*, \frac{1}{2} - x^*, x^*\right)$. The fixations in AB and in ab are internally stable, whereas the other equilibria are unstable.*

Proof of Result 1.

As $w_1 = w_4$ and $w_2 = w_3$, from (S2.6), at equilibrium

$$w(x_1 - x_4) = w_1(x_1 - x_4), \quad w(x_2 - x_3) = w_2(x_2 - x_3). \tag{S2.8}$$

Hence, $x_1 = x_4$ or $w = w_1$, and $x_2 = x_3$ or $w = w_2$. If $w = w_1$, then, from (S2.6) again, $x_1 = x_1 - r_1 D$, and as $0 < r_1 < 1$, $D = 0$. Similarly, if $w = w_2$, then also $D = 0$. If $x_1 \neq x_4$ and $w = w_1$, then $x_2 = x_3$ as otherwise $w = w_2$, which is impossible. As in this case $D = 0$, we have

$$w = w_1 = w_1(x_1 + x_4) + w_2(x_2 + x_3), \tag{S2.9}$$

and as $w_1 > w_2$, (S2.9) implies that $x_2 = x_3 = 0$ and $x_1 + x_4 = 1$. Since $D = x_1 x_4 - x_2 x_3 = 0$, we then have $x_1 = 1$, $x_4 = 0$ or $x_1 = 0$, $x_4 = 1$. Therefore, the possible equilibria when either $x_1 \neq x_4$ or $x_2 \neq x_3$ are the four fixations $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ in AB, Ab, aB, ab, respectively. In addition, if $x_1 = x_4, x_2 = x_3$, there can be an equilibrium $\left(x, \frac{1}{2} - x, \frac{1}{2} - x, x\right)$ which satisfies

$$wx = w_1(x - r_1 D)$$
$$D = x^2 - \left(\frac{1}{2} - x\right)^2 = x - \frac{1}{4} \tag{S2.10}$$
$$w = 2(w_1 - w_2)(1 - r_1)x + \frac{r_1}{2}w_1 + \left(1 - \frac{r_1}{2}w_2\right).$$

Thus

$$x = \frac{w_1 x(1 - r_1) + \frac{r_1}{4}w_1}{2(w_1 - w_2)(1 - r_1)x + \frac{r_1}{2}w_1 + \left(1 - \frac{r_1}{2}\right)w_2}. \tag{S2.11}$$

Therefore $Q(x) = 0$ where

$$Q(x) = 2(w_1 - w_2)(1 - r_1)x^2 + \left[\frac{r_1}{2}w_1 + \left(1 - \frac{r_1}{2}\right)w_2 - w_1(1 - r_1)\right]x - \frac{r_1}{4}w_1. \tag{S2.12}$$

Now $Q(0) = -\frac{r_1}{4}w_1 < 0$ and $Q\left(\frac{1}{2}\right) = \frac{r_1}{4}w_2 > 0$. Hence, $Q(x) = 0$ has a unique root x^* satisfying $0 < x^* < \frac{1}{2}$ determining an equilibrium $\mathbf{x}^* = \left(x^*, \frac{1}{2} - x^*, \frac{1}{2} - x^*, x^*\right)$.

We next check the internal stability of these five possible equilibria. The fixation in AB is internally stable, as its local stability matrix is

$$\begin{pmatrix} \frac{1}{1+s} & 0 & \frac{r_1}{1+s} \\ 0 & \frac{1}{1+s} & \frac{r_1}{1+s} \\ 0 & 0 & 1 - r_1 \end{pmatrix}, \tag{S2.13}$$

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with all of its eigenvalues positive and less than 1. Similarly the fixation of ab is stable. The local stability matrix associated with the fixation of Ab is

$$\begin{pmatrix} 1+s & r_1(1+s) & 0 \\ 0 & 1-r_1 & 0 \\ 0 & r_1(1+s) & 1+s \end{pmatrix}, \qquad (S2.14)$$

which has two eigenvalues larger than 1. Similarly, the fixation in aB is also internally unstable.

We next check the stability of $\mathbf{x}^*$. At $\mathbf{x}^*$ we have $x_1 = x_4$, $x_2 = x_3$. Then "near" $\mathbf{x}^*$, up to non-linear terms,

$$x_1' - x_4' = \frac{w_1}{w^*}(x_1 - x_4), \quad x_2' - x_3' = \frac{w_2}{w^*}(x_2 - x_3). \qquad (S2.15)$$

Hence $\mathbf{x}^*$ is internally stable provided $w^* > w_1, w_2$. As $w_1 > w_2$ and

$$w^* = 2(w_1 - w_2)(1 - r_1)x^* + \frac{r}{2}w_1 + \left(1 - \frac{r}{2}\right)w_2, \qquad (S2.16)$$

$w^* > w_2$. $w^* > w_1$ if and only if

$$2(w_1 - w_2)(1 - r)x^* > \left(1 - \frac{r}{2}\right)(w_1 - w_2), \qquad (S2.17)$$

or, as $(w_1 - w_2) > 0$, if and only if

$$2(1 - r_1)x^* > 1 - \frac{r_1}{2}, \qquad (S2.18)$$

which is impossible when $0 < r_1 < 1$ and $0 < x^* < \frac{1}{2}$. Therefore $\mathbf{x}^*$ is not internally stable.

When the environment is constant, the only possible stable equilibria with M fixed are the two fixations in AB and ab. These fixations do not depend on r_1, the recombination rate determined by M We now study the case when selection pressures are no longer constant, but fluctuate in time.

PERIODICALLY FLUCTUATING SELECTION

Suppose now that the environment varies temporally. We assume two possible types of selection regimes such that the fitness parameters change as follows:

$$
\begin{array}{lcccc}
\text{phenotype} & AB & Ab & aB & ab \\
\text{environment 1} & w_1 & w_2 & w_3 & w_4 \\
\text{environment 2} & \widetilde{w}_1 & \widetilde{w}_2 & \widetilde{w}_3 & \widetilde{w}_4,
\end{array}
\tag{S2.19}
$$

where we can write

$$
\begin{aligned}
w_1 = w_4 = 1 + s \qquad & w_2 = w_3 = 1 \\
\widetilde{w}_1 = \widetilde{w}_4 = 1 \qquad & \widetilde{w}_2 = \widetilde{w}_3 = 1 + s.
\end{aligned}
\tag{S2.20}
$$

Equilibria with symmetric selection

We start with only the modifier allele M present in the population so that $x_i = x_i' = 0$ for $i = 5, 6, 7, 8$. Let the two-phase transformation of the frequencies, determined by (S2.3), be $\mathbf{x}' = \widetilde{\mathbf{T}} \cdot \mathbf{T}\mathbf{x}$. This can be represented in two stages as

$$
\mathbf{x}' = \widetilde{\mathbf{T}}\widetilde{\mathbf{x}}, \quad \widetilde{\mathbf{x}} = \mathbf{T}\mathbf{x},
\tag{S2.21}
$$

where $x_i = \widetilde{x}_i = x_i' = 0$ for $i = 5, 6, 7, 8$ and

$$
\begin{aligned}
w\widetilde{x}_i &= w_i\big(x_i \mp r_1 D\big) \\
\widetilde{w}x_i' &= \widetilde{w}_i\big(\widetilde{x}_i \mp r_1 \widetilde{D}\big)
\end{aligned}
\qquad i = 1, 2, 3, 4.
\tag{S2.22}
$$

Here

$$
w = \sum_{i=1}^{4} w_i x_i - r_1 D(w_1 - w_2 - w_3 + w_4)
$$

$$
\widetilde{w} = \sum_{i=1}^{4} \widetilde{w}_i \widetilde{x}_i - r_1 \widetilde{D}\big(\widetilde{w}_1 - \widetilde{w}_2 - \widetilde{w}_3 + \widetilde{w}_4\big)
\tag{S2.23}
$$

$$
D = x_1 x_4 - x_2 x_3
$$

$$
\widetilde{D} = \widetilde{x}_1 \widetilde{x}_4 - \widetilde{x}_2 \widetilde{x}_3.
$$

As $w_1 = w_4$, $w_2 = w_3$, and $\widetilde{w}_1 = \widetilde{w}_4$, $\widetilde{w}_2 = \widetilde{w}_3$, we have

$$
\begin{aligned}
x_1' - x_4' &= \frac{w_1 \widetilde{w}_1}{w\widetilde{w}}(x_1 - x_4) \\
x_2' - x_3' &= \frac{w_2 \widetilde{w}_2}{w\widetilde{w}}(x_2 - x_3).
\end{aligned}
\tag{S2.24}
$$

At equilibrium when $\mathbf{x}' = \mathbf{x}$, equations (S2.24) are satisfied by $x_1 = x_4$, $x_2 = x_3$, together with $\tilde{x}_1 = \tilde{x}_4$, $\tilde{x}_2 = \tilde{x}_3$. Hence, we look for symmetric equilibria of the form $(x_1, x_2, x_3, x_4) = \left(x, \frac{1}{2} - x, \frac{1}{2} - x, x\right)$. We obtain the following result:

Result 2. *On the boundary where only allele M is present, a unique symmetric equilibrium $\mathbf{x}^*$ exists where $\mathbf{x}^* = \left(x^*, \frac{1}{2} - x^*, \frac{1}{2} - x^*, x^*\right)$ and x^* is the unique positive root of $Q(x) = 0$, where $0 < x^* < \frac{1}{2}$ and*

$$Q(x) = Ax^2 + Bx + C,$$
$$A = r_1(1 - r_1)(w_1 + w_2)(w_1 - w_2)$$
$$B = \frac{1}{2}r_1(1 - r_1)(w_1 + w_2)(w_1 - w_2) + \frac{1}{2}r_1\left(1 - \frac{r_1}{2}\right)(w_1 + w_2)^2 \qquad (S2.25)$$
$$C = -\frac{1}{4}r_1\left(1 - \frac{r_1}{2}\right)w_2(w_1 + w_2).$$

Proof of Result 2.

If $\mathbf{x} = \left(x, \frac{1}{2} - x, \frac{1}{2} - x, x\right)$, then $\tilde{\mathbf{x}} = \left(\tilde{x}, \frac{1}{2} - \tilde{x}, \frac{1}{2} - \tilde{x}, \tilde{x}\right)$ and

$$D = x^2 - \left(\frac{1}{2} - x\right)^2 = x - \frac{1}{4}, \quad \tilde{D} = \tilde{x}^2 - \left(\frac{1}{2} - \tilde{x}\right)^2 = \tilde{x} - \frac{1}{4}$$
$$w = 2(w_1 - w_2)(1 - r_1)x + \frac{r_1}{2}w_1 + \left(1 - \frac{r_1}{2}\right)w_2 \qquad (S2.26)$$
$$\tilde{w} = 2(w_2 - w_1)(1 - r_1)\tilde{x} + \frac{r_1}{2}w_2 + \left(1 - \frac{r_1}{2}\right)w_1,$$

since $\tilde{w}_1 = w_2$, $\tilde{w}_2 = w_1$. From (S2.22) we have

$$w\tilde{x} = w_1(x - r_1 D) = w_1(1 - r_1)x + \frac{r_1}{4}w_1$$
$$\tilde{w}x = \tilde{w}_1\left(x - r_1\tilde{D}\right) = w_2(1 - r_1)\tilde{x} + \frac{r_1}{4}w_2. \qquad (S2.27)$$

Hence
$$\tilde{x} = \frac{w_1(1 - r_1)x + \frac{r_1}{4}w_1}{2(w_1 - w_2)(1 - r_1)x + \frac{r_1}{2}w_1 + \left(1 - \frac{r_1}{2}\right)w_2}$$
$$x = \frac{w_2(1 - r_1)\tilde{x} + \frac{r_2}{4}w_2}{2(w_2 - w_1)(1 - r_1)\tilde{x} + \frac{r_1}{2}w_2 + \left(1 - \frac{r_1}{2}\right)w_1}. \qquad (S2.28)$$

In (S2.28), substituting $\tilde{x}$ in the expression for x, gives the quadratic equation $Q(x) = 0$ given in (S2.25). Now as $0 < r_1 < 1$,

$$Q(0) = -\frac{1}{4}r_1\left(1 - \frac{r_1}{2}\right)w_2(w_1 + w_2) > 0 \qquad (S2.29)$$

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and

$$Q\left(\tfrac{1}{2}\right) = \frac{1}{2}r_1\left(1 - \frac{r_1}{2}\right)w_1(w_1 + w_2) > 0. \tag{S2.30}$$

Therefore $Q(x) = 0$ has a solution $0 < x^* < \frac{1}{2}$. It is the unique positive solution of $Q(x) = 0$ as $Q(\pm\infty) > 0$ when $w_1 > w_2$ and $r > 0$ so the other root of $Q(x) = 0$ is negative.

We next check the internal stability of $\mathbf{x}^*$.

Result 3. *The unique symmetric equilibrium $\mathbf{x}^*$ is internally stable on the boundary where only the M modifier allele is present.*

Proof of Result 3.

As $x_1 = x_4$ and $x_2 = x_3$ in $\mathbf{x}^*$, and in view of (S2.24), $\mathbf{x}^*$ is internally stable if $w\widetilde{w} > w_1\widetilde{w}_1$ and $w\widetilde{w} = w_2\widetilde{w}_2$, namely if $w\widetilde{w} > w_1 w_2$. But

$$\begin{aligned}
w &= 2(w_1 - w_2)(1 - r_1)x + w_2 + \frac{r_1}{2}(w_1 - w_2), \\
\widetilde{w} &= 2(w_2 - w_1)(1 - r_1)\widetilde{x} + w_1 + \frac{r_1}{2}(w_2 - w_1).
\end{aligned} \tag{S2.31}$$

Therefore

$$w\widetilde{w} = 2(w_2 - w_1)(1 - r_1)w\widetilde{x} + w\left[w_1 + \frac{r_1}{2}(w_2 - w_1)\right]. \tag{S2.32}$$

As $w\widetilde{x} = w_1(x - r_1 D)$, we have

$$\begin{aligned}
w\widetilde{w} = {}&2(w_2 - w_1)(1 - r_1)w_1 x - 2r_1(1 - r_1)(w_2 - w_1)w_1 D + \\
&+ \left[2(w_1 - w_2)(1 - r_1)x + w_2 + \frac{r_1}{2}(w_1 - w_2)\right]\left[w_1 + \frac{r_1}{2}(w_2 - w_1)\right].
\end{aligned} \tag{S2.33}$$

Using $D = x - \frac{1}{4}$,

$$w\widetilde{w} = r_1(w_1 - w_2)\left\{(1 - r_1)x(w_1 + w_2) + \frac{1}{2}\left[\frac{r_1}{2}(w_1 + w_2) - w_2\right]\right\} + w_1 w_2. \tag{S2.34}$$

Therefore as $0 < r_1 < 1$ and $w_1 > w_2$, $w\widetilde{w} > w_1 w_2$ if and only if

$$(1 - r_1)x(w_1 + w_2) + \frac{1}{2}\left[\frac{r_1}{2}(w_1 + w_2) - w_2\right] > 0. \tag{S2.35}$$

Treat the left side of the inequality (S2.35) as a function $f()$ of r_1, for $0 \le r_1 \le 1$, where x, w_1, w_2 are parameters. In fact,

$$f(1) = \frac{1}{2}\left[\frac{1}{2}(w_1 + w_2) - w_2\right] = \frac{1}{4}(w_1 - w_2) > 0, \tag{S2.36}$$

$$f(0) = x(w_1 + w_2) - \frac{1}{2}w_2. \tag{S2.37}$$

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So (S2.35) is satisfied, and $\mathbf{x}^*$ is internally stable, if $x^* > \frac{w_2}{2(w_1+w_2)}$. Now $w_1 = 1 + s$, $w_1 = 1$. Thus we need $x^* > \frac{1}{2(s+2)}$. From (S2.28) we can write

$$Q(x) = r_1(s+2)\left\{(1-r_1)sx^2 + \left[-\frac{1}{2}(1-r_1)s + \frac{1}{2}\left(1-\frac{r_1}{2}\right)(s+2)\right]x - \frac{1}{4}\left(1-\frac{r_1}{2}\right)\right\},$$
$$(S2.38)$$

and we have $Q(0) < 0$ and

$$Q\left(\tfrac{1}{2(s+2)}\right) = r_1(s+2)\frac{-(1-r_1)s(s+1)}{4(s+2)^2} < 0. \qquad (S2.39)$$

Since $Q(x^*) = 0$, we must have $x^* > \frac{1}{2(s+2)}$ as desired.

Remark. Using (S2.38) it is easily seen that when $s > 0$

$$Q\left(\tfrac{1}{4}\right) = r_1(s+2) \cdot \frac{s}{16} > 0. \qquad (S2.40)$$

Therefore as $Q\left(\frac{1}{2(s+2)}\right) < 0$, we conclude that when $w_1 > w_2$, $\frac{1}{2(s+2)} < x^* < \frac{1}{4}$. In addition, as $D^* = x^* - \frac{1}{4}$, at equilibrium $D^* < 0$.

External stability of the symmetric equilibrium

We next check the external stability of $\mathbf{x}^*$ to the introduction of the modifier allele m at the modifier locus. As $\mathbf{x}^*$ is internally stable, its external stability is determined by the 4×4 matrix $\mathbf{L}^*_{\text{ex}}$, which is the linear approximation of the transformation $\mathbf{x}' = \widetilde{\mathbf{T}} \circ \mathbf{T}\mathbf{x}$ near $\mathbf{x}^*$ restricted to the components x_5, x_6, x_7, x_8 of $\mathbf{x}$. As our transformation is $\widetilde{\mathbf{T}} \circ \mathbf{T}$, then $\mathbf{L}^*_{\text{ex}} = \widetilde{\mathbf{L}}^* \circ \mathbf{L}^*$, where we have

$$\mathbf{L}^* = \frac{1}{w_L}\begin{pmatrix} a & e & h & d \\ b & f & g & c \\ c & g & f & b \\ d & h & e & a \end{pmatrix} \qquad (41)$$

$$\widetilde{\mathbf{L}}^* = \frac{1}{\widetilde{w}_L}\begin{pmatrix} \tilde{a} & \tilde{e} & \tilde{h} & \tilde{d} \\ \tilde{b} & \tilde{f} & \tilde{g} & \tilde{c} \\ \tilde{c} & \tilde{g} & \tilde{f} & \tilde{b} \\ \tilde{d} & \tilde{h} & \tilde{e} & \tilde{a} \end{pmatrix}, \qquad (42)$$

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with

$$w_L = 2w_2 + (w_1 - w_2)\left[r_1 + 4(1 - r_1)x^*\right]$$

$$\widetilde{w}_L = 2w_1 + (w_2 - w_1)\left[r_1 + 4(1 - r_1)\tilde{x}\right],$$

$(S2.43)$

and

$$a = w_1[2 - r_2 - 2R(1 - r_2)(1 - x^*)]$$

$$b = w_2[R + r_2 - 2Rr_2 - 2R(1 - r_2)x^*]$$

$$c = w_2R[1 - 2x^*(1 - r_2)]$$

$$d = 2w_1x^*R(1 - r_2)$$

$$e = w_1[(1 - R)r_2 + 2R(1 - r_2)x^*]$$

$(S2.44)$

$$f = w_2[2 - r_2 - R(1 - r_2)(1 + 2x^*)]$$

$$g = w_2R(1 - r_2)(1 - 2x^*)$$

$$h = w_1R[r_2 + 2(1 - r_2)x^*]$$

$$\tilde{a} = w_2[2 - r_2 - 2R(1 - r_2)(1 - \tilde{x})]$$

$$\tilde{b} = w_1[R + r_2 - 2Rr_2 - 2R(1 - r_2)\tilde{x}]$$

$$\tilde{c} = w_1R[1 - 2\tilde{x}(1 - r_2)]$$

$$\tilde{d} = 2w_2\tilde{x}R(1 - r_2)$$

$$\tilde{e} = w_2[(1 - R)r_2 + 2R(1 - r_2)\tilde{x}]$$

$(S2.45)$

$$\tilde{f} = w_1[2 - r_2 - R(1 - r_2)(1 + 2\tilde{x})]$$

$$\tilde{g} = w_1R(1 - r_2)(1 - 2\tilde{x})$$

$$\tilde{h} = w_2R[r_2 + 2(1 - r_2)\tilde{x}].$$

Here $\tilde{\mathbf{x}} = \mathbf{T}\mathbf{x}^*$ and $\mathbf{x}^* = \widetilde{\mathbf{T}}\tilde{\mathbf{x}}$. As $\mathbf{L}^*_{\text{ex}} = \widetilde{\mathbf{L}}^* \circ \mathbf{L}^*$, due to the special form of $\mathbf{L}^*$ and $\widetilde{\mathbf{L}}^*$. $\mathbf{L}^*_{\text{ex}}$ also has the same form, and the characteristic polynomial $Q(\lambda)$ of $\mathbf{L}^*_{\text{ex}}$ factors into two quadratic polynomials, namely

$$Q(\lambda) = Q_1(\lambda) \cdot Q_2(\lambda).$$

$(S2.46)$

The analysis of $Q_1(\lambda)$ and $Q_2(\lambda)$ gives the following result.

Result 4. *The symmetric equilibrium* $\mathbf{x}^*$ *is externally stable when* $r_1 > r_2$ *and unstable when* $r_1 < r_2$. *Thus in this symmetric environment which changes every generation, larger recombination rates are favored, and the evolutionary stable recombination rate is 1.*

Proof of Result 4.

We outline here the steps of the proof.

Let $Q(\lambda)$ be the fourth degree characteristic polynomial determining the external stability, with $Q(\lambda) = Q(\lambda; r_1, r_2, R)$. Then

1. $Q(\lambda) = Q_1(\lambda) \cdot Q_2(\lambda)$.

2. $Q_i(\lambda) = \lambda^2 + p_i \lambda + q_i$ for $i = 1, 2$, where

$$p_i > 0, \quad 0 < q_i < 1, \quad \Delta_i = p_i^2 - 4q_i > 0.$$

3. $Q_i(\lambda)$, for $i = 1, 2$, has two positive roots whose product is less than 1. Both roots are less than 1 if and only if $Q_i(1) > 0$.

4. $Q_1(1) = Q_1(1; r_1, r_2, R) = (r_1 - r_2) f_1(r_1, r_2, R)$, where $f_1(r_1, r_2, R) > 0$ for all possible values of r_1, r_2, R.

5. When $r_1 > r_2$, then $Q_1(1) > 0$, and the two roots of $Q_1(\lambda) = 0$ are positive and less than 1. When $r_1 < r_2$, then $Q_1(1) < 0$, and $Q_1(\lambda) = 0$ has a root larger than 1.

6. When $R = 0$, then $Q_1(\lambda; R) = Q_2(\lambda; R)$, and also $Q_1(1; R = 0) = Q_2(1; R = 0)$.

7. $Q_2(1; R)$ is an increasing function of R.

8. When $R = 0$, we have $Q_1(1; R = 0) > 0$ when $r_1 > r_2$, therefore $Q_2(1; R) > 0$ for all R when $r_1 > r_2$.

9. When $r_1 > r_2$, both $Q_1(1)$ and $Q_2(1)$ are positive, hence all four roots of $Q(\lambda) = 0$ are positive and less than 1 making the equilibrium externally stable.

10. When $r_1 < r_2$, then $Q_1(1) < 0$, and so $Q(\lambda) = 0$ has a root that is positive and larger than 1, and our equilibrium is externally unstable.

All the steps above have been checked either analytically or using Mathematica for some computations, especially for those determining the sign of an expression, given the parameter space. Moreover, arguments involving the second polynomial $Q_2(\lambda)$ are based on its following properties:

The second polynomial $Q_2(\lambda)$ is the characteristic polynomial of

$$\mathbf{L}_{\text{ex}}^* = \begin{pmatrix} \tilde{f} - \tilde{g} & \tilde{b} - \tilde{c} \\ \tilde{e} - \tilde{h} & \tilde{a} - \tilde{d} \end{pmatrix} \begin{pmatrix} f - g & b - c \\ e - h & a - d \end{pmatrix},$$

where

$$\begin{aligned} w(a - d) &= w_1 A & \widetilde{w}(\tilde{a} - \tilde{d}) &= w_2 A \\ w(f - g) &= w_2 A & \widetilde{w}(\tilde{f} - \tilde{g}) &= w_1 A \\ w(b - c) &= w_2 B & \widetilde{w}(\tilde{b} - \tilde{c}) &= w_1 B \\ w(e - h) &= w_1 B & \widetilde{w}(\tilde{e} - \tilde{h}) &= w_2 B, \end{aligned}$$

$$A = 2 - r_2 - 2R(1 - r_1) = 2(1 - R) - r_2(1 - 2R)$$

$$B = r_2(1 - 2R).$$

In fact,

$$\mathbf{L}_{\text{ex}}^* = \frac{1}{w_L \widetilde{w}_L} \begin{pmatrix} w_1 w_2 A^2 + w_1^2 B^2 & \left(w_1 w_2 + w_1^2\right) AB \\ \left(w_1 w_2 + w_1^2\right) AB & w_1 w_2 A^2 + w_2^2 B^2 \end{pmatrix}.$$

Thus

$$Q_2(\lambda) = \lambda^2 - \frac{2w_1 w_2 A^2 + \left(w_1^2 + w_2^2\right) B^2}{w_L \widetilde{w}_L} \lambda + \frac{w_1^2 w_2^2}{(w_L \widetilde{w}_L)^2} \left(A^2 - B^2\right)^2$$

$$A^2 - B^2 = (A - B)(A + B) = 4(1 - R)[1 - r_2 - R(1 - 2r_2)]$$

$$Q_2(1) = 1 - \frac{2w_1 w_2 A^2 + \left(w_1^2 + w_2^2\right) B^2}{w_L \widetilde{w}_L} + \frac{w_1^2 w_2^2}{(w_L \widetilde{w}_L)^2} \left(A^2 - B^2\right)^2.$$

Let $Q_2(1) = Q_2(1; R)$. We compute $\frac{\partial Q_2(1)}{\partial R}$ using

$$\frac{\partial A}{\partial R} = -2(1 - r_2), \qquad \frac{\partial B}{\partial R} = -2r_2,$$

$$\frac{\partial Q_2(1)}{\partial R} = -\frac{2w_1 w_2 \cdot 2A[-2(1 - r_2)] + \left(w_1^2 + w_2^2\right) \cdot 2B[-2r_2]}{w_L \widetilde{w}_L} +$$

$$+ \frac{w_1^2 w_2^2}{(w_L \widetilde{w}_L)^2} \cdot 2(A^2 - B^2)\left\{2A[-2(1 - r_2)] - 2B[-2r_2]\right\}.$$

Hence,

$$\frac{\partial Q_2(1)}{\partial R} = \frac{8w_1 w_2 A(1 - r_2) + 4\left(w_1^2 + w_2^2\right) Br^2}{w_L \widetilde{w}_L} -$$

$$- \frac{8w_1^2 w_2^2}{(w_L \widetilde{w}_L)^2}(A^2 - B^2)[A(1 - r_2) - Br_2],$$

$$\frac{\partial Q_2(1)}{\partial R} > 0 \iff (w_L \widetilde{w}_L) \left[8w_1 w_2 A(1 - r_2) + 4\left(w_1^2 + w_2^2\right) Br_2\right]$$

$$> 8w_1^2 w_2^2 (A^2 - B^2)[A(1 - r_2) - Br_2],$$

$$A^2 - B^2 = 4(1 - r)[1 - r_2 - R(1 - 2r_2)].$$

Thus $\frac{\partial Q_2(1)}{\partial R} > 0$ if

$$w_L \widetilde{w}_L A(1 - r_2) > 4w_1 w_2 [1 - r_2 - R(1 - 2r_2)]A(1 - r_2).$$

As $A > 0$ and $0 < r_2 < 1$, this is true if

$$w_L \widetilde{w}_L > 4w_1 w_2 [1 - r_2 - R(1 - 2r_2)].$$

But $w_L \widetilde{w}_L > 4w_1 w_2$ and $1 - r_2 - R(1 - 2r_2) < 1$, and the above holds.

This sums up to the fact that $Q_2(1; R)$ is monotone increasing in R, thus proving point 7. But since $Q_2(1, 0) > 0$ when $r_1 > r_2$, therefore $Q_2(1; R) > 0$ when $r_1 > r_2$ for all $0 < R < 1$.