Supporting Information

bmotif: a package for motif analyses of bipartite networks

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Appendix S1: Computational performance using random networks

We carried out two analyses using randomly-generated networks to examine the effects of network size (number of species) and connectance on the computational performance of individual motif and motif position calculations. For the first analysis, we generated random networks with a fixed size, varying the connectance between 0.2 and 1. We generated 1000 networks for each value of connectance. For each of these sets of 1000 networks, we recorded the mean time for our code to calculate the frequency of five motifs (motifs 1, 2, 5, 10 and 28; one from each of the five motif size classes) and the number of times each species occurred in five motif positions (positions 1, 3, 9, 23 and 85; one from each motif size class). The dimensions of the generated networks were set as the median number of rows and columns of 230 empirical ecological bipartite networks (22 rows, 13 columns) obtained from the Web of Life repository (www.web-of-life.es; Appendix 3). For the second analysis, we generated random networks of a fixed connectance, varying the size between 10 and 200 species. We generated 1000 networks for each value of size and recorded the mean time for our code to calculate the frequency of the same five motifs and positions. The connectance of the generated networks was the median connectance of the empirical network dataset (0.243) and the row:column ratio (ratio of number of species in one level, such as hosts, to the number of species in the other level, such as parasitoids) was also set as the empirical median (2). Functions were timed using the R package 'microbenchmark' (Mersmann 2015).

We found that connectance had little effect on the performance of individual motif and position calculations (Figure S1), while a polynomial of degree two explained the increase in time with network size ($R^2 > 0.99$ for all motifs and positions) (Figure S2).

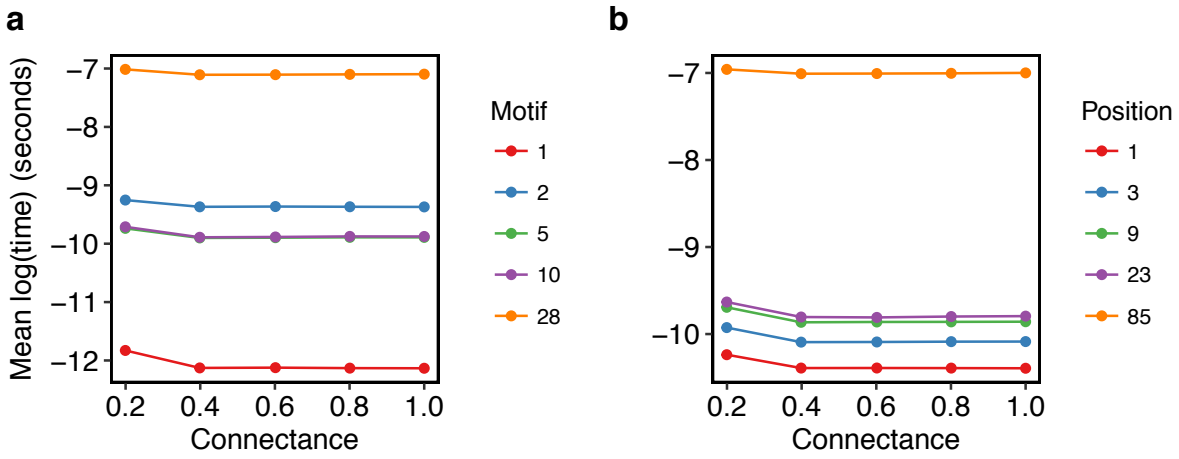

Figure S1: Relationship between connectance and computational time taken to calculate the frequency of (a) five motifs, one from each motif size class, and (b) five motif positions, one from each motif size class. Functions were run on randomly-generated networks of a given connectance. For each level of connectance, we generated 1000 random networks and record the mean time for the functions to complete. Lines connecting each point are shown for visualisation.

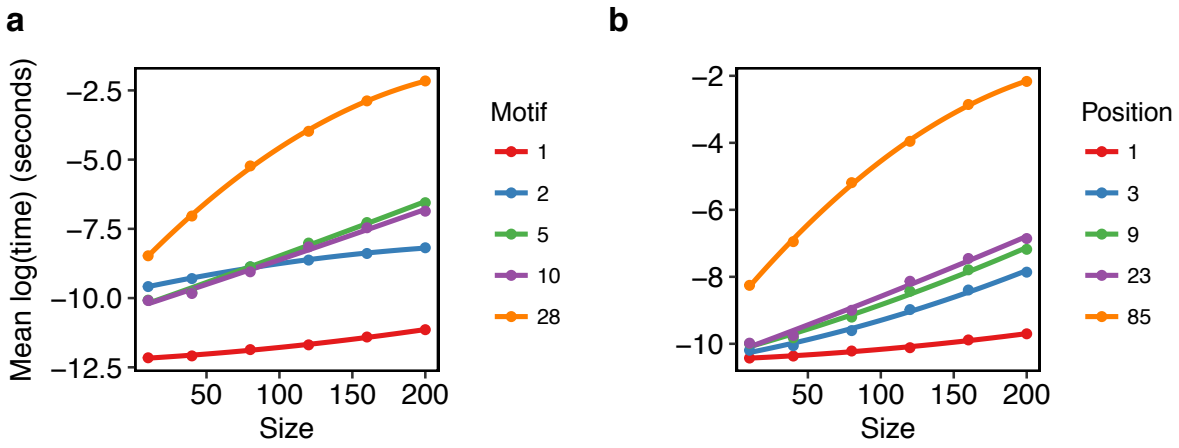

Figure S2: Relationship between size and computational time taken to calculate the frequency of (a) five motifs, one from each motif size class, and (b) five motif positions, one from each motif size class. Functions were run on randomly-generated networks of a given size. For each level of size, we generated 1000 random networks and record the mean time for the functions to complete. Lines are best fit polynomials of degree two.

References

Mersmann, O. 2015. microbenchmark: Accurate Timing Functions. R package version 1.4-2.1.

Appendix S2: Formulae

Notation

M is the incidence matrix.
Z is the number of rows of M.
P is the number of columns of M.
$N_{zp} = 1 - M_{zp}$
$d_z = \sum_{p=1}^{P} M_{zp}$
$u_p = \sum_{z=1}^{Z} M_{zp}$
$\mathcal{Z}_{zz'} = \sum_{p=1}^{P} M_{zp} M_{z'p}$
$\mathcal{Y}_{zz'} = \sum_{p=1}^{P} M_{zp} N_{z'p}$
$\mathcal{X}_{zz'} = \sum_{p=1}^{P} N_{zp} M_{z'p}$
$\mathcal{P}_{pp'} = \sum_{z=1}^{Z} M_{pz} M_{p'z}$
$\mathcal{Q}_{pp'} = \sum_{z=1}^{Z} M_{pz} N_{p'z}$
$\mathcal{R}_{pp'} = \sum_{z=1}^{Z} N_{pz} M_{p'z}$
$j_{Zi} = 1$
$j_{Pi} = 1$
$J_{Zij} = 1$
$J_{Pij} = 1$
$J_{Z3ijk} = 1$
$J_{P3ijk} = 1$
$A_{ijk} = \sum_{p=1}^{P} M_{ip} M_{jp} M_{kp} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$B_{ijk} = \sum_{p=1}^{P} M_{ip} M_{jp} N_{kp} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$C_{ijk} = \sum_{p=1}^{P} M_{ip} N_{jp} M_{kp} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$D_{ijk} = \sum_{p=1}^{P} M_{ip} N_{jp} N_{kp} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$E_{ijk} = \sum_{p=1}^{P} N_{ip} M_{jp} M_{kp} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$F_{ijk} = \sum_{p=1}^{P} N_{ip} M_{jp} N_{kp} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$G_{ijk} = \sum_{p=1}^{P} N_{ip} N_{jp} M_{kp} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$H_{ijk} = \sum_{p=1}^{P} N_{ip} N_{jp} N_{kp} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$A'_{ijk} = \sum_{z=1}^{Z} M_{zi} M_{zj} M_{zk} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$B'_{ijk} = \sum_{z=1}^{Z} M_{zi} M_{zj} N_{zk} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$C'_{ijk} = \sum_{z=1}^{Z} M_{zi} N_{zj} M_{zk} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$D'_{ijk} = \sum_{z=1}^{Z} M_{zi} N_{zj} N_{zk} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$E'_{ijk} = \sum_{z=1}^{Z} N_{zi} M_{zj} M_{zk} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$F'_{ijk} = \sum_{z=1}^{Z} N_{zi} M_{zj} N_{zk} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$G'_{ijk} = \sum_{z=1}^{Z} N_{zi} N_{zj} M_{zk} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$H'_{ijk} = \sum_{z=1}^{Z} N_{zi} N_{zj} N_{zk} (1-\delta_{ij})(1-\delta_{jk})(1-\delta_{ki})$
$f(T)_i = \sum_{j=1}^{Z} \sum_{k=1}^{Z} T_{ijk}$
$g(T)_i = \sum_{j=1}^{P} \sum_{k=1}^{P} T_{ijk}$

mcount formulae

Motif 1

$$\sum_{z=1}^{Z}\sum_{p=1}^{P} M_{zp}$$

Motif 2

$$\frac{1}{2}\sum_{z=1}^{Z} d_z(d_z-1)$$

Motif 3

$$\frac{1}{2}\sum_{p=1}^{P} u_p(u_p-1)$$

Motif 4

$$\frac{1}{6}\sum_{p=1}^{P} u_p(u_p-1)(u_p-2)$$

Motif 5

$$\sum_{z=1}^{Z}\sum_{z'=1}^{Z} \mathcal{Z}_{zz'}\mathcal{Y}_{zz'}$$

Motif 6

$$\frac{1}{4}\sum_{z=1}^{Z}\sum_{z'=1}^{Z} \mathcal{Z}_{zz'}(\mathcal{Z}_{zz'}-1) - \frac{1}{4}\sum_{z=1}^{Z} d_z(d_z-1)$$

Motif 7

$$\frac{1}{6}\sum_{z=1}^{Z} d_z(d_z-1)(d_z-2)$$

Motif 8

$$\frac{1}{24}\sum_{p=1}^{P} u_p(u_p-1)(u_p-2)(u_p-3)$$

Motif 9

$$\frac{1}{2}\sum_{p=1}^{P}\sum_{p'=1}^{P} \mathcal{P}_{pp'}\mathcal{Q}_{pp'}(\mathcal{Q}_{pp'}-1)$$

Motif 10

$$\frac{1}{2}\sum_{p=1}^{P}\sum_{p'=1}^{P}\mathcal{P}_{pp'}\mathcal{Q}_{pp'}\mathcal{R}_{pp'}$$

Motif 11

$$\frac{1}{2}\sum_{p=1}^{P}\sum_{p'=1}^{P}\mathcal{P}_{pp'}(\mathcal{P}_{pp'}-1)\mathcal{Q}_{pp'}$$

Motif 12

$$\frac{1}{12}\sum_{p=1}^{P}\sum_{p'=1}^{P}\mathcal{P}_{pp'}(\mathcal{P}_{pp'}-1)(\mathcal{P}_{pp'}-2)-\frac{1}{12}\sum_{p=1}^{P}d_p(d_p-1)(d_p-2)$$

Motif 13

$$\frac{1}{2}\sum_{z=1}^{Z}\sum_{z'=1}^{Z}\mathcal{Z}_{zz'}\mathcal{Y}_{zz'}(\mathcal{Y}_{zz'}-1)$$

Motif 14

$$\frac{1}{2}\sum_{z=1}^{Z}\sum_{z'=1}^{Z}\mathcal{Z}_{zz'}\mathcal{Y}_{zz'}\mathcal{X}_{zz'}$$

Motif 15

$$\frac{1}{2}\sum_{z=1}^{Z}\sum_{z'=1}^{Z}\mathcal{Z}_{zz'}(\mathcal{Z}_{zz'}-1)\mathcal{Y}_{zz'}$$

Motif 16

$$\frac{1}{12}\sum_{z=1}^{Z}\sum_{z'=1}^{Z}\mathcal{Z}_{zz'}(\mathcal{Z}_{zz'}-1)(\mathcal{Z}_{zz'}-2)-\frac{1}{12}\sum_{z=1}^{Z}d_z(d_z-1)(d_z-2)$$

Motif 17

$$\frac{1}{24}\sum_{z=1}^{Z}d_z(d_z-1)(d_z-2)(d_z-3)$$

Motif 18

$$\frac{1}{120}\sum_{p=1}^{P}u_p(u_p-1)(u_p-2)(u_p-3)(u_p-4)$$

Motif 19

$$\frac{1}{6}\sum_{p=1}^{P}\sum_{p'=1}^{P}\mathcal{P}_{pp'}\mathcal{Q}_{pp'}(\mathcal{Q}_{pp'}-1)(\mathcal{Q}_{pp'}-2)$$

Motif 20

$$\frac{1}{2}\sum_{p=1}^{P}\sum_{p'=1}^{P}\mathcal{P}_{pp'}\mathcal{Q}_{pp'}(\mathcal{Q}_{pp'}-1)\mathcal{R}_{pp'}$$

Motif 21

$$\frac{1}{4}\sum_{p=1}^{P}\sum_{p'=1}^{P}\mathcal{P}_{pp'}(\mathcal{P}_{pp'}-1)\mathcal{Q}_{pp'}(\mathcal{Q}_{pp'}-1)$$

Motif 22

$$\frac{1}{4}\sum_{p=1}^{P}\sum_{p'=1}^{P}\mathcal{P}_{pp'}(\mathcal{P}_{pp'}-1)\mathcal{Q}_{pp'}\mathcal{R}_{pp'}$$

Motif 23

$$\frac{1}{6}\sum_{p=1}^{P}\sum_{p'=1}^{P}\mathcal{P}_{pp'}(\mathcal{P}_{pp'}-1)(\mathcal{P}_{pp'}-2)\mathcal{Q}_{pp'}$$

Motif 24

$$\frac{1}{48}\sum_{p=1}^{P}\sum_{p'=1}^{P}\mathcal{P}_{pp'}(\mathcal{P}_{pp'}-1)(\mathcal{P}_{pp'}-2)(\mathcal{P}_{pp'}-3)-\frac{1}{48}\sum_{p=1}^{P}u_p(u_p-1)(u_p-2)(u_p-3)$$

Motif 25

$$\frac{1}{4}\sum_{i=1}^{Z}\sum_{j=1}^{Z}\sum_{k=1}^{Z}A_{ijk}F_{ijk}(F_{ijk}-1) \;=\; \frac{1}{4}\sum_{i=1}^{P}\sum_{j=1}^{P}\sum_{k=1}^{P}A'_{ijk}F'_{ijk}(F'_{ijk}-1)$$

Motif 26

$$\frac{1}{2}\sum_{i=1}^{Z}\sum_{j=1}^{Z}\sum_{k=1}^{Z}D_{ijk}B_{ijk}C_{ijk} \;=\; \frac{1}{2}\sum_{i=1}^{P}\sum_{j=1}^{P}\sum_{k=1}^{P}F'_{ijk}G'_{ijk}A'_{ijk}$$

Motif 27

$$\frac{1}{2}\sum_{i=1}^{Z}\sum_{j=1}^{Z}\sum_{k=1}^{Z}F_{ijk}G_{ijk}A_{ijk} \;=\; \frac{1}{2}\sum_{i=1}^{P}\sum_{j=1}^{P}\sum_{k=1}^{P}D'_{ijk}B'_{ijk}C'_{ijk}$$

Motif 28

$$\sum_{i=1}^{Z}\sum_{j=1}^{Z}\sum_{k=1}^{Z}B_{ijk}C_{ijk}G_{ijk} \;=\; \sum_{i=1}^{P}\sum_{j=1}^{P}\sum_{k=1}^{P}B'_{ijk}C'_{ijk}G'_{ijk}$$

Motif 29

$$\sum_{i=1}^{Z}\sum_{j=1}^{Z}\sum_{k=1}^{Z}A_{ijk}B_{ijk}D_{ijk} \;=\; \sum_{i=1}^{P}\sum_{j=1}^{P}\sum_{k=1}^{P}A'_{ijk}B'_{ijk}D'_{ijk}$$

Motif 30

$$\frac{1}{2}\sum_{i=1}^{Z}\sum_{j=1}^{Z}\sum_{k=1}^{Z}A_{ijk}B_{ijk}G_{ijk} \quad = \quad \frac{1}{2}\sum_{i=1}^{P}\sum_{j=1}^{P}\sum_{k=1}^{P}B'_{ijk}(B'_{ijk}-1)C'_{ijk}$$

Motif 31

$$\frac{1}{4}\sum_{i=1}^{Z}\sum_{j=1}^{Z}\sum_{k=1}^{Z}A_{ijk}F_{ijk}(A_{ijk}-1) \quad = \quad \frac{1}{4}\sum_{i=1}^{P}\sum_{j=1}^{P}\sum_{k=1}^{P}B'_{ijk}(B'_{ijk}-1)A'_{ijk}$$

Motif 32

$$\frac{1}{2}\sum_{i=1}^{Z}\sum_{j=1}^{Z}\sum_{k=1}^{Z}B_{ijk}(B_{ijk}-1)C_{ijk} \quad = \quad \frac{1}{2}\sum_{i=1}^{P}\sum_{j=1}^{P}\sum_{k=1}^{P}A'_{ijk}B'_{ijk}G'_{ijk}$$

Motif 33

$$\frac{1}{4}\sum_{i=1}^{Z}\sum_{j=1}^{Z}\sum_{k=1}^{Z}B_{ijk}(B_{ijk}-1)A_{ijk} \quad = \quad \frac{1}{4}\sum_{i=1}^{P}\sum_{j=1}^{P}\sum_{k=1}^{P}A'_{ijk}F'_{ijk}(A'_{ijk}-1)$$

Motif 34

$$\frac{1}{6}\sum_{i=1}^{Z}\sum_{j=1}^{Z}\sum_{k=1}^{Z}E_{ijk}B_{ijk}C_{ijk} \quad = \quad \frac{1}{6}\sum_{i=1}^{P}\sum_{j=1}^{P}\sum_{k=1}^{P}E'_{ijk}B'_{ijk}C'_{ijk}$$

Motif 35

$$\frac{1}{2}\sum_{i=1}^{Z}\sum_{j=1}^{Z}\sum_{k=1}^{Z}A_{ijk}B_{ijk}C_{ijk} \quad = \quad \frac{1}{2}\sum_{i=1}^{P}\sum_{j=1}^{P}\sum_{k=1}^{P}A'_{ijk}B'_{ijk}C'_{ijk}$$

Motif 36

$$\frac{1}{4}\sum_{i=1}^{Z}\sum_{j=1}^{Z}\sum_{k=1}^{Z}A_{ijk}(A_{ijk}-1)B_{ijk} \quad = \quad \frac{1}{4}\sum_{i=1}^{P}\sum_{j=1}^{P}\sum_{k=1}^{P}A'_{ijk}(A'_{ijk}-1)B'_{ijk}$$

Motif 37

$$\frac{1}{36}\sum_{i=1}^{Z}\sum_{j=1}^{Z}\sum_{k=1}^{Z}A_{ijk}(A_{ijk}-1)(A_{ijk}-2) \quad = \quad \frac{1}{36}\sum_{i=1}^{P}\sum_{j=1}^{P}\sum_{k=1}^{P}A'_{ijk}(A'_{ijk}-1)(A'_{ijk}-2)$$

Motif 38

$$\frac{1}{6}\sum_{z=1}^{Z}\sum_{z'=1}^{Z}\mathcal{Z}_{zz'}\mathcal{Y}_{zz'}(\mathcal{Y}_{zz'}-1)(\mathcal{Y}_{zz'}-2)$$

Motif 39

$$\frac{1}{2}\sum_{z=1}^{Z}\sum_{z'=1}^{Z}\mathcal{Z}_{zz'}\mathcal{Y}_{zz'}(\mathcal{Y}_{zz'}-1)\mathcal{X}_{zz'}$$

Motif 40

$$\frac{1}{4}\sum_{z=1}^{Z}\sum_{z'=1}^{Z}\mathcal{Z}_{zz'}(\mathcal{Z}_{zz'}-1)\mathcal{Y}_{zz'}(\mathcal{Y}_{zz'}-1)$$

Motif 41

$$\frac{1}{4}\sum_{z=1}^{Z}\sum_{z'=1}^{Z}\mathcal{Z}_{zz'}(\mathcal{Z}_{zz'}-1)\mathcal{Y}_{zz'}\mathcal{X}_{zz'}$$

Motif 42

$$\frac{1}{6}\sum_{z=1}^{Z}\sum_{z'=1}^{Z}\mathcal{Z}_{zz'}(\mathcal{Z}_{zz'}-1)(\mathcal{Z}_{zz'}-2)\mathcal{Y}_{zz'}$$

Motif 43

$$\frac{1}{48}\sum_{z=1}^{Z}\sum_{z'=1}^{Z}\mathcal{Z}_{zz'}(\mathcal{Z}_{zz'}-1)(\mathcal{Z}_{zz'}-2)(\mathcal{Z}_{zz'}-3)-\frac{1}{48}\sum_{z=1}^{Z}d_z(d_z-1)(d_z-2)(d_z-3)$$

Motif 44

$$\frac{1}{120}\sum_{z=1}^{Z}d_z(d_z-1)(d_z-2)(d_z-3)(d_z-4)$$

positions formulae

Position 1

$$\mathbf{u}$$

Position 2

$$\mathbf{d}$$

Position 3

$$\mathcal{P} \cdot \mathbf{j_P} - \mathbf{u}$$

Position 4

$$\frac{1}{2}\mathbf{d} \circ (\mathbf{d} - \mathbf{j_Z})$$

Position 5

$$\frac{1}{2}\mathbf{u} \circ (\mathbf{u} - \mathbf{j_P})$$

Position 6

$$\mathcal{Z} \cdot \mathbf{j_Z} - \mathbf{d}$$

Position 7

$$\frac{1}{6}\mathbf{u} \circ (\mathbf{u} - \mathbf{j_P}) \circ (\mathbf{u} - 2\mathbf{j_P})$$

Position 8

$$\frac{1}{2}M \cdot ((\mathbf{u} - \mathbf{j_P}) \circ (\mathbf{u} - 2\mathbf{j_P}))$$

Position 9

$$(\mathcal{P} \circ \mathcal{R}) \cdot \mathbf{j_P}$$

Position 10

$$(\mathcal{P} \circ \mathcal{Q}) \cdot \mathbf{j_P}$$

Position 11

$$(\mathcal{X} \circ \mathcal{Z}) \cdot \mathbf{j_Z}$$

Position 12

$$(\mathcal{Y} \circ \mathcal{Z}) \cdot \mathbf{j_Z}$$

Position 13

$$\frac{1}{2}(\mathcal{P} \circ (\mathcal{P} - J_P)) \cdot \mathbf{j_P} - \frac{1}{2}\mathbf{u} \circ (\mathbf{u} - \mathbf{j_P})$$

Position 14

$$\frac{1}{2}(\mathcal{Z} \circ (\mathcal{Z} - J_Z)) \cdot \mathbf{j_Z} - \frac{1}{2}\mathbf{d} \circ (\mathbf{d} - \mathbf{j_Z})$$

Position 15

$$\frac{1}{2}M^T \cdot ((\mathbf{d} - \mathbf{j_Z}) \circ (\mathbf{d} - 2\mathbf{j_Z}))$$

Position 16

$$\frac{1}{6}\mathbf{d} \circ (\mathbf{d} - \mathbf{j_Z}) \circ (\mathbf{d} - 2\mathbf{j_Z})$$

Position 17

$$\frac{1}{24}\mathbf{u} \circ (\mathbf{u} - \mathbf{j_P}) \circ (\mathbf{u} - 2\mathbf{j_P}) \circ (\mathbf{u} - 3\mathbf{j_P})$$

Position 18

$$\frac{1}{6}M \cdot ((\mathbf{u} - \mathbf{j_P}) \circ (\mathbf{u} - 2\mathbf{j_P}) \circ (\mathbf{u} - 3\mathbf{j_P}))/6$$

Position 19

$$\frac{1}{2}(\mathcal{P} \circ \mathcal{R} \circ (\mathcal{R} - J_P)) \cdot \mathbf{j_P}$$

Position 20

$$\frac{1}{2}(\mathcal{P} \circ \mathcal{Q} \circ (\mathcal{Q} - J_P)) \cdot \mathbf{j_P}$$

Position 21

$$(N \circ (M \cdot ((\mathcal{Q} - J_P) \circ \mathcal{P}))) \cdot \mathbf{j_P}$$

Position 22

$$\frac{1}{2}(M \circ (M \cdot (\mathcal{Q} \circ (\mathcal{Q} - J_P)))) \cdot \mathbf{j_P}$$

Position 23

$$(\mathcal{P} \circ \mathcal{Q} \circ \mathcal{R}) \cdot \mathbf{j_P}$$

Position 24

$$(N \circ (M \cdot (\mathcal{P} \circ \mathcal{R}))) \cdot \mathbf{j_P}$$

Position 25

$$\frac{1}{2}(M \circ (M \cdot (\mathcal{Q} \circ \mathcal{R}))) \cdot \mathbf{j_P}$$

Position 26

$$\frac{1}{2}(\mathcal{P} \circ (\mathcal{P} - J_P) \circ \mathcal{R}) \cdot \mathbf{j_P}$$

Position 27

$$\frac{1}{2}(\mathcal{P} \circ (\mathcal{P} - J_P) \circ \mathcal{Q}) \cdot \mathbf{j_P}$$

Position 28

$$\frac{1}{2}(N \circ (M \cdot (\mathcal{P} \circ (\mathcal{P} - J_P)))) \cdot \mathbf{j_P}$$

Position 29

$$(M \circ (M \cdot (\mathcal{Q} \circ (\mathcal{P} - J_P)))) \cdot \mathbf{j_P}$$

Position 30

$$\frac{1}{6}(\mathcal{P} \circ (\mathcal{P} - J_P) \circ (\mathcal{P} - 2J_P)) \cdot \mathbf{j_P} - \frac{1}{6}\mathbf{u} \circ (\mathbf{u} - \mathbf{j_P}) \circ (\mathbf{u} - 2\mathbf{j_P})$$

Position 31

$$\frac{1}{4}(M \circ M \cdot ((\mathcal{P} - J_P) \circ (\mathcal{P} - 2J_P))) \cdot \mathbf{j_P} - \frac{1}{4}M \cdot ((\mathbf{u} - \mathbf{j_P}) \circ (\mathbf{u} - 2\mathbf{j_P}))$$

Position 32

$$(N^T \circ (M^T \cdot ((\mathcal{Y} - J_Z) \circ \mathcal{Z}))) \cdot \mathbf{j_Z}$$

Position 33

$$\frac{1}{2}(M^T \circ (M^T \cdot (\mathcal{Y} \circ (\mathcal{Y} - J_Z)))) \cdot \mathbf{j_Z}$$

Position 34

$$\frac{1}{2}(\mathcal{Z} \circ \mathcal{X} \circ (\mathcal{X} - J_Z)) \cdot \mathbf{j_Z}$$

Position 35

$$\frac{1}{2}(\mathcal{Z} \circ \mathcal{Y} \circ (\mathcal{Y} - J_Z)) \cdot \mathbf{j_Z}$$

Position 36

$$(N^T \circ (M^T \cdot (Z \circ X))) \cdot \mathbf{j_Z}$$

Position 37

$$\frac{1}{2}(M^T \circ (M^T \cdot (\mathcal{Y} \circ \mathcal{X}))) \cdot \mathbf{j_Z}$$

Position 38

$$(\mathcal{Z} \circ \mathcal{Y} \circ \mathcal{X}) \cdot \mathbf{j_Z}$$

Position 39

$$\frac{1}{2}(N^T \circ (M^T \cdot (\mathcal{Z} \circ (\mathcal{Z} - J_Z)))) \cdot \mathbf{j_Z}$$

Position 40

$$(M^T \circ (M^T \cdot (\mathcal{Y} \circ (\mathcal{Z} - J_Z)))) \cdot \mathbf{j_Z}$$

Position 41

$$\frac{1}{2}(\mathcal{Z} \circ (\mathcal{Z} - J_Z) \circ \mathcal{X}) \cdot \mathbf{j_Z}$$

Position 42

$$\frac{1}{2}(\mathcal{Z} \circ (\mathcal{Z} - J_Z) \circ \mathcal{Y}) \cdot \mathbf{j_Z}$$

Position 43

$$\frac{1}{4}(M^T \circ (M^T \cdot ((\mathcal{Z} - J_Z) \circ (\mathcal{Z} - 2J_Z)))) \cdot \mathbf{j_Z} - \frac{1}{4}M^T \cdot ((\mathbf{d} - \mathbf{j_Z}) \circ (\mathbf{d} - 2\mathbf{j_Z}))$$

Position 44

$$\frac{1}{6}(\mathcal{Z} \circ (\mathcal{Z} - J_Z) \circ (\mathcal{Z} - 2J_Z)) \cdot \mathbf{j_Z} - \frac{1}{6}\mathbf{d} \circ (\mathbf{d} - \mathbf{j_Z}) \circ (\mathbf{d} - 2\mathbf{j_Z})$$

Position 45

$$\frac{1}{6}M^T \cdot ((\mathbf{d} - \mathbf{j_Z}) \circ (\mathbf{d} - 2\mathbf{j_Z}) \circ (\mathbf{d} - 3\mathbf{j_Z}))$$

Position 46

$$\frac{1}{24}\mathbf{d} \circ (\mathbf{d} - \mathbf{j_Z}) \circ (\mathbf{d} - 2\mathbf{j_Z}) \circ (\mathbf{d} - 3\mathbf{j_Z})$$

Position 47

$$\frac{1}{120}\mathbf{u} \circ (\mathbf{u} - \mathbf{j_P}) \circ (\mathbf{u} - 2\mathbf{j_P}) \circ (\mathbf{u} - 3\mathbf{j_P}) \circ (\mathbf{u} - 4 \circ \mathbf{j_P})$$

Position 48

$$\frac{1}{24}M \cdot ((\mathbf{u} - \mathbf{j_P}) \circ (\mathbf{u} - 2\mathbf{j_P}) \circ (\mathbf{u} - 3\mathbf{j_P}) \circ (\mathbf{u} - 4 \circ \mathbf{j_P}))$$

Position 49

$$\frac{1}{6}(\mathcal{P} \circ \mathcal{R} \circ (\mathcal{R} - J_P) \circ (\mathcal{R} - 2\mathbf{j_P})) \cdot \mathbf{j_P}$$

Position 50

$$\frac{1}{6}(\mathcal{P} \circ \mathcal{Q} \circ (\mathcal{Q} - J_P) \circ (\mathcal{Q} - 2\mathbf{j_P})) \cdot \mathbf{j_P}$$

Position 51

$$\frac{1}{2}(N \circ (M \cdot (\mathcal{P} \circ (\mathcal{Q} - J_P) \circ (\mathcal{Q} - 2\mathbf{j_P})))) \cdot \mathbf{j_P}$$

Position 52

$$\frac{1}{6}(M \circ (M \cdot (\mathcal{Q} \circ (\mathcal{Q} - J_P) \circ (\mathcal{Q} - 2\mathbf{j_P})))) \cdot \mathbf{j_P}$$

Position 53

$$\frac{1}{2}(\mathcal{P} \circ \mathcal{Q} \circ \mathcal{R} \circ (\mathcal{R} - J_P)) \cdot \mathbf{j_P}$$

Position 54

$$\frac{1}{2}(\mathcal{P} \circ \mathcal{Q} \circ (\mathcal{Q} - J_P) \circ \mathcal{R}) \cdot \mathbf{j_P}$$

Position 55

$$\frac{1}{2}(M \circ (N \cdot (\mathcal{P} \circ \mathcal{Q} \circ (\mathcal{Q} - J_P)))) \cdot \mathbf{j_P}$$

Position 56

$$(M \circ (N \cdot (\mathcal{P} \circ \mathcal{Q} \circ (\mathcal{R} - J_P)))) \cdot \mathbf{j_P}$$

Position 57

$$\frac{1}{2}(M \circ (M \cdot (\mathcal{Q} \circ (\mathcal{Q} - J_P) \circ \mathcal{R}))) \cdot \mathbf{j_P}$$

Position 58

$$\frac{1}{4}(\mathcal{P} \circ (\mathcal{P} - J_P) \circ \mathcal{R} \circ (\mathcal{R} - J_P)) \cdot \mathbf{j_P}$$

Position 59

$$\frac{1}{4}(\mathcal{P} \circ (\mathcal{P} - J_P) \circ \mathcal{Q} \circ (\mathcal{Q} - J_P)) \cdot \mathbf{j_P}$$

Position 60

$$\frac{1}{2}(N \circ (M \cdot ((\mathcal{Q} - J_P) \circ \mathcal{P} \circ (\mathcal{P} - J_P)))) \cdot \mathbf{j_P}$$

Position 61

$$\frac{1}{2}(M \circ (M \cdot ((\mathcal{P} - J_P) \circ \mathcal{Q} \circ (\mathcal{Q} - J_P)))) \cdot \mathbf{j_P}$$

Position 62

$$\frac{1}{2}(\mathcal{P} \circ (\mathcal{P} - J_P) \circ \mathcal{Q} \circ \mathcal{R}) \cdot \mathbf{j_P}$$

Position 63

$$\frac{1}{2}(N \circ (M \cdot (\mathcal{P} \circ (\mathcal{P} - J_P) \circ \mathcal{R}))) \cdot \mathbf{j_P}$$

Position 64

$$\frac{1}{2}(M \circ (M \cdot ((\mathcal{P} - J_P) \circ \mathcal{Q} \circ \mathcal{R}))) \cdot \mathbf{j_P}$$

Position 65

$$\frac{1}{6}(\mathcal{P} \circ (\mathcal{P} - J_P) \circ (\mathcal{P} - 2J_P) \circ \mathcal{R}) \cdot \mathbf{j_P}$$

Position 66

$$\frac{1}{6}(\mathcal{P} \circ (\mathcal{P} - J_P) \circ (\mathcal{P} - 2J_P) \circ \mathcal{Q}) \cdot \mathbf{j_P}$$

Position 67

$$\frac{1}{6}(N \circ (M \cdot (\mathcal{P} \circ (\mathcal{P} - J_P) \circ (\mathcal{P} - 2J_P)))) \cdot \mathbf{j_P}$$

Position 68

$$\frac{1}{2}(M \circ (M \cdot ((\mathcal{P} - J_P) \circ (\mathcal{P} - 2J_P) \circ \mathcal{Q}))) \cdot \mathbf{j_P}$$

Position 69

$$\frac{1}{24}(\mathcal{P} \circ (\mathcal{P} - J_P) \circ (\mathcal{P} - 2J_P) \circ (\mathcal{P} - 3J_P)) \cdot \mathbf{j_P} - \frac{1}{24}\mathbf{u} \circ (\mathbf{u} - \mathbf{j_P}) \circ (\mathbf{u} - 2\mathbf{j_P}) \circ (\mathbf{u} - 3\mathbf{j_P})$$

Position 70

$$\frac{1}{12}(M \circ (M \cdot ((\mathcal{P} - J_P) \circ (\mathcal{P} - 2J_P) \circ (\mathcal{P} - 3J_P)))) \cdot \mathbf{j_P} - \frac{1}{12}M \cdot ((\mathbf{u} - \mathbf{j_P}) \circ (\mathbf{u} - 2\mathbf{j_P}) \circ (\mathbf{u} - 3\mathbf{j_P}))$$

Position 71

$$\frac{1}{2}\mathbf{g}(F' \circ (F' - J_{P3}) \circ A')$$

Position 72

$$\frac{1}{4}\mathbf{g}(D' \circ (D' - J_{P3}) \circ A')$$

Position 73

$$\frac{1}{2}\mathbf{f}(F \circ (F - J_{Z3}) \circ A)$$

Position 74

$$\frac{1}{4}\mathbf{f}(D \circ (D - J_{Z3}) \circ A)$$

Position 75

$$\mathbf{g}(D' \circ A' \circ F')$$

Position 76

$$\frac{1}{2}\mathbf{g}(A' \circ F' \circ G')$$

Position 77

$$\frac{1}{2}\mathbf{f}(D \circ B \circ C)$$

Position 78

$$\mathbf{f}(B \circ F \circ E)$$

Position 79

$$\frac{1}{2}\mathbf{g}(B' \circ C' \circ D')$$

Position 80

$$\mathbf{g}(B' \circ F' \circ E')$$

Position 81

$$\mathbf{f}(A \circ D \circ F)$$

Position 82

$$\frac{1}{2}\mathbf{f}(A \circ F \circ G)$$

Position 83

$$\mathbf{g}(B' \circ E' \circ G')$$

Position 84

$$\mathbf{g}(E' \circ B' \circ D')$$

Position 85

$$\mathbf{g}(B' \circ C' \circ G')$$

Position 86

$$\mathbf{f}(B \circ E \circ G)$$

Position 87

$$\mathbf{f}(D \circ B \circ E)$$

Position 88

$$\mathbf{f}(F \circ B \circ C)$$

Position 89

$$\mathbf{g}(A' \circ E' \circ F')$$

Position 90

$$\mathbf{g}(A' \circ B' \circ F')$$

Position 91

$$\mathbf{g}(A' \circ B' \circ D')$$

Position 92

$$\mathbf{f}(A \circ E \circ F)$$

Position 93

$$\mathbf{f}(B \circ F \circ A)$$

Position 94

$$\mathbf{f}(A \circ B \circ D)$$

Position 95

$$\frac{1}{2}\mathbf{g}(B' \circ E' \circ (E' - J_{P3}))$$

Position 96

$$\frac{1}{2}\mathbf{g}(B' \circ (B' - J_{P3}) \circ E')$$

Position 97

$$\frac{1}{2}\mathbf{g}(B' \circ C' \circ (C' - J_{P3}))$$

Position 98

$$\frac{1}{2}\mathbf{f}(D \circ A \circ E)$$

Position 99

$$\mathbf{f}(A \circ F \circ C)$$

Position 100

$$\frac{1}{4}\mathbf{g}(A' \circ E' \circ (E' - J_{P3}))$$

Position 101

$$\frac{1}{2}\mathbf{g}(A' \circ B' \circ (B' - J_{P3}))$$

Position 102

$$\frac{1}{2}\mathbf{f}(A \circ (A - J_{Z3}) \circ F)$$

Position 103

$$\frac{1}{4}\mathbf{f}(D \circ A \circ (A - J_{Z3}))$$

Position 104

$$\frac{1}{2}\mathbf{g}(D' \circ A' \circ E')$$

Position 105

$$\mathbf{g}(B' \circ A' \circ G')$$

Position 106

$$\frac{1}{2}\mathbf{f}(B \circ E \circ (E - J_{Z3}))$$

Position 107

$$\frac{1}{2}\mathbf{f}(B \circ (B - J_{Z3}) \circ E)$$

Position 108

$$\frac{1}{2}\mathbf{f}(B \circ (B - J_{Z3}) \circ C)$$

Position 109

$$\frac{1}{2}\mathbf{g}(A' \circ (A' - J_{P3}) \circ F')$$

Position 110

$$\frac{1}{4}\mathbf{g}(A' \circ (A' - J_{P3}) \circ D')$$

Position 111

$$\frac{1}{4}\mathbf{f}(A \circ E \circ (E - J_{Z3}))$$

Position 112

$$\frac{1}{2}\mathbf{f}(A \circ B \circ (B - J_{Z3}))$$

Position 113

$$\frac{1}{2}\mathbf{g}(B' \circ C' \circ E')$$

Position 114

$$\frac{1}{2}\mathbf{f}(B \circ C \circ E)$$

Position 115

$$\mathbf{g}(A' \circ B' \circ E')$$

Position 116

$$\frac{1}{2}\mathbf{g}(A' \circ B' \circ C')$$

Position 117

$$\mathbf{f}(A \circ E \circ B)$$

Position 118

$$\frac{1}{2}\mathbf{f}(A \circ B \circ C)$$

Position 119

$$\frac{1}{4}\mathbf{g}(A' \circ (A' - J_{P3}) \circ E')$$

Position 120

$$\frac{1}{2}\mathbf{g}(A' \circ (A' - J_{P3}) \circ B')$$

Position 121

$$\frac{1}{4}\mathbf{f}(A \circ (A - J_{Z3}) \circ E)$$

Position 122

$$\frac{1}{2}\mathbf{f}(B \circ A \circ (A - J_{Z3}))$$

Position 123

$$\frac{1}{12}\mathbf{g}(A' \circ (A' - J_{P3}) \circ (A' - 2J_{P3}))$$

Position 124

$$\frac{1}{12}\mathbf{f}(A \circ (A - J_{Z3}) \circ (A - 2J_{Z3}))$$

Position 125

$$\frac{1}{2}(N^T \circ (M^T \cdot (\mathcal{Z} \circ (\mathcal{Y} - J_Z) \circ (\mathcal{Y} - 2J_Z)))) \cdot \mathbf{jz}$$

Position 126

$$\frac{1}{6}(M^T \circ (M^T \cdot (\mathcal{Y} \circ (\mathcal{Y} - J_Z) \circ (\mathcal{Y} - 2J_Z)))) \cdot \mathbf{jz}$$

Position 127

$$\frac{1}{6}(\mathcal{Z} \circ \mathcal{X} \circ (\mathcal{X} - J_Z) \circ (\mathcal{X} - 2J_Z)) \cdot \mathbf{jz}$$

Position 128

$$\frac{1}{6}(\mathcal{Z} \circ \mathcal{Y} \circ (\mathcal{Y} - J_Z) \circ (\mathcal{Y} - 2J_Z)) \cdot \mathbf{jz}$$

Position 129

$$\frac{1}{2}(M^T \circ (N^T \cdot (\mathcal{Z} \circ \mathcal{Y} \circ (\mathcal{Y} - J_Z)))) \cdot \mathbf{jz}$$

Position 130

$$(M^T \circ (N^T \cdot (\mathcal{Z} \circ \mathcal{Y} \circ (\mathcal{X} - J_Z)))) \cdot \mathbf{jz}$$

Position 131

$$\frac{1}{2}(M^T \circ (M^T \cdot (\mathcal{Y} \circ (\mathcal{Y} - J_Z) \circ \mathcal{X}))) \cdot \mathbf{jz}$$

Position 132

$$\frac{1}{2}(\mathcal{Z} \circ \mathcal{Y} \circ \mathcal{X} \circ (\mathcal{X} - J_Z)) \cdot \mathbf{jz}$$

Position 133

$$\frac{1}{2}(\mathcal{Z} \circ \mathcal{Y} \circ (\mathcal{Y} - J_Z) \circ \mathcal{X}) \cdot \mathbf{jz}$$

Position 134

$$\frac{1}{2}(N^T \circ (M^T \cdot ((\mathcal{Y} - J_Z) \circ \mathcal{Z} \circ (\mathcal{Z} - J_Z)))) \cdot \mathbf{jz}$$

Position 135

$$\frac{1}{2}(M^T \circ (M^T \cdot ((\mathcal{Z} - J_Z) \circ \mathcal{Y} \circ (\mathcal{Y} - J_Z)))) \cdot \mathbf{jz}$$

Position 136

$$\frac{1}{4}(\mathcal{Z} \circ (\mathcal{Z} - J_Z) \circ \mathcal{X} \circ (\mathcal{X} - J_Z)) \cdot \mathbf{jz}$$

Position 137

$$\frac{1}{4}(\mathcal{Z} \circ (\mathcal{Z} - J_Z) \circ \mathcal{Y} \circ (\mathcal{Y} - J_Z)) \cdot \mathbf{jz}$$

Position 138

$$\frac{1}{2}(N^T \circ (M^T \cdot (\mathcal{Z} \circ (\mathcal{Z} - J_Z) \circ \mathcal{X}))) \cdot \mathbf{jz}$$

Position 139

$$\frac{1}{2}(M^T \circ (M^T \cdot ((\mathcal{Z} - J_Z) \circ \mathcal{Y} \circ \mathcal{X}))) \cdot \mathbf{jz}$$

Position 140

$$\frac{1}{2}(\mathcal{Z} \circ (\mathcal{Z} - J_Z) \circ \mathcal{Y} \circ \mathcal{X}) \cdot \mathbf{jz}$$

Position 141

$$\frac{1}{6}(N^T \circ (M^T \cdot (\mathcal{Z} \circ (\mathcal{Z} - J_Z) \circ (\mathcal{Z} - 2J_Z)))) \cdot \mathbf{jz}$$

Position 142

$$\frac{1}{2}(M^T \circ (M^T \cdot ((\mathcal{Z} - J_Z) \circ (\mathcal{Z} - 2J_Z) \circ \mathcal{Y}))) \cdot \mathbf{jz}$$

Position 143

$$\frac{1}{6}(\mathcal{Z} \circ (\mathcal{Z} - J_Z) \circ (\mathcal{Z} - 2J_Z) \circ \mathcal{X}) \cdot \mathbf{jz}$$

Position 144

$$\frac{1}{6}(\mathcal{Z} \circ (\mathcal{Z} - J_Z) \circ (\mathcal{Z} - 2J_Z) \circ \mathcal{Y}) \cdot \mathbf{jz}$$

Position 145

$$\frac{1}{12}(M^T \circ (M^T \cdot ((\mathcal{Z} - J_Z) \circ (\mathcal{Z} - 2J_Z) \circ (\mathcal{Z} - 3J_Z)))) \cdot \mathbf{jz} - \frac{1}{12} M^T \cdot ((\mathbf{d} - \mathbf{jz}) \circ (\mathbf{d} - 2\mathbf{jz}) \circ (\mathbf{d} - 3\mathbf{jz}))$$

Position 146

$$\frac{1}{24}(\mathcal{Z} \circ (\mathcal{Z} - J_Z) \circ (\mathcal{Z} - 2J_Z) \circ (\mathcal{Z} - 3J_Z)) \cdot \mathbf{jz} - \frac{1}{24}\mathbf{d} \circ (\mathbf{d} - \mathbf{jz}) \circ (\mathbf{d} - 2\mathbf{jz}) \circ (\mathbf{d} - 3\mathbf{jz})$$

Position 147

$$\frac{1}{24}M^T \cdot ((\mathbf{d} - \mathbf{jz}) \circ (\mathbf{d} - 2\mathbf{jz}) \circ (\mathbf{d} - 3\mathbf{jz}) \circ (\mathbf{d} - 4\mathbf{jz}))$$

Position 148

$$\frac{1}{120}\mathbf{d} \circ (\mathbf{d} - \mathbf{jz}) \circ (\mathbf{d} - 2\mathbf{jz}) \circ (\mathbf{d} - 3\mathbf{jz}) \circ (\mathbf{d} - 4\mathbf{jz})$$